Lie groups and plasticity at finite strain

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Overview

1. GSM and Plasticity

2. Finite-Strain Plasticity

3. Energetic formulation

4. Existence results

Conclusions

- Continuum mechanics at finite strains leads to **geometric nonlinearities:**
  - invariance under rigid-body motions: $\mathrm{SO}(\mathbb{R}^d)$
  - invariance under previous plastic deformation, $\mathrm{SL}(\mathbb{R}^d)$.

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$\rightsquigarrow$ ***strongly dissipative geometric evolutionary system***

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GSM = generalized standard materials

(Halphen&Nguyen'75, Ziegler&Wehrli'87,...,Hackl'95,...)

$\Omega \subset \mathbb{R}^d$ body in reference configuration

$\varphi : \Omega \to \mathbb{R}^d$ deformation ($\varphi(x) = x+\varepsilon \boldsymbol{u}(x)$ with displacement $\boldsymbol{u}$)

$z : \Omega \to Z \subset \mathbb{R}^m$ internal variable(s)
(magnetization, polarization, phase, plasticity, damage, ...)

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Balance of forces $\left\{ \begin{aligned} \rho\ddot{\varphi} &= \operatorname{div} \boldsymbol{\Sigma} + \boldsymbol{f}_{\mathrm{ext}} && \text{in} && \Omega, \\ \varphi(t,x) &= \varphi_{\mathrm{Dir}}(t,x) && \text{on} && \Gamma_{\mathrm{Dir}}, \\ \boldsymbol{\Sigma}(t,x)\nu(x) &= \boldsymbol{g}_{\mathrm{ext}}(t,x) && \text{on} && \Gamma_{\mathrm{Neu}}. \end{aligned} \right.$

Constitutive law $\boldsymbol{\Sigma}(x) = \widehat{\boldsymbol{\Sigma}}(x, \nabla\varphi(x), z(x), \nabla z(x))$

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Constitutive law (hyperelasticity)
$\boldsymbol{\Sigma}(x) = \partial_{\boldsymbol{F}} W(x, \boldsymbol{F}, z, A)$, where $\boldsymbol{F} = \nabla \varphi$, $A = \nabla z$

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Flow rule $\dot{z} = H(\boldsymbol{F}, z, \nabla z, ...)$

WIAS

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Flow rule (Ziegler&Wehrli'87)

$0 = \partial_{\dot z} R(x, z, \dot z) + \partial_z W(x, \boldsymbol{F}, z, \nabla z) - \operatorname{div} \big[\partial_A W(x, \boldsymbol{F}, z, \nabla z)\big]$

dissipation potential $R : \Omega \times \mathrm{T}Z \to [0, \infty[$

$$0 = \underbrace{\partial_{\dot z} R(x,z,\dot z)}_{\text{friction force}} + \underbrace{\partial_z W(x,\boldsymbol{F},z,\nabla z) - \mathrm{div}\,\big[\partial_A W(x,\boldsymbol{F},z,\nabla z)\big]}_{-\text{thermomechanical force conjugate force to } z}$$

In general Z is a manifold, not a linear space.

Internal force balance is defined on T^*Z

$R(x,z,\cdot) : \mathrm{T}_z Z \to \mathbb{R}$ is convex $\rightsquigarrow$ $\partial_{\dot z} R(x,z,\dot z) \in \mathrm{T}^*_z Z$

Similarly, $W(x,\boldsymbol{F},\cdot,A) : Z \to \mathbb{R}$ implies $\partial_z W(x,\boldsymbol{F},z,A) \in \mathrm{T}^*_z Z$

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Example : Allen-Cahn equation

$z \in \mathbb{R}$ scalar phase-field variable (no elasticity, no $\boldsymbol{F}$)

$W(x,z,A) = \Phi(z) + \frac{\kappa^2}{2}|A|^2$

$R(x,z,\dot z) = \frac{r}{2}|\dot z|^2 \quad \rightsquigarrow \; \partial_{\dot z} R(x,z,\dot z) = r\dot z$ (viscous friction)

$$\boxed{0 = r\dot z + \Phi'(z) - \kappa^2 \Delta z}$$ Allen-Cahn equation

Overview

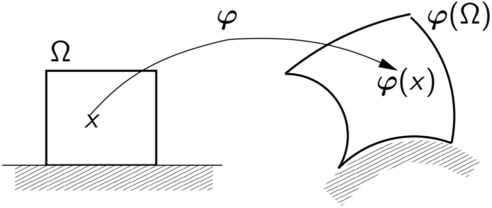
Finite-strain elasticity

$$\boldsymbol{F} = \nabla\varphi \in \mathrm{GL}^+(\mathbb{R}^d) \overset{\text{def}}{=} \{\, F \in \mathbb{R}^{d\times d} \mid \det F > 0 \,\}$$

Typical stored energy density $W : \mathbb{R}^{d\times d} \to \mathbb{R}_\infty \overset{\text{def}}{=} \mathbb{R}\cup\{\infty\}$

Polyconvex Ogden material: $(p > d,\ \gamma, c_1, c_2 > 0)$

$$W(\boldsymbol{F}) = \begin{cases} c_1|\boldsymbol{F}|^p + \frac{c_2}{(\det \boldsymbol{F})^\gamma} & \text{if } \det \boldsymbol{F} > 0, \\ \infty & \text{else.} \end{cases}$$



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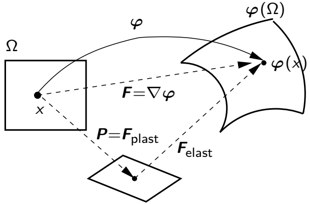
Stress-strain relation

$$\boldsymbol{\Sigma}_{\mathrm{el}}(\boldsymbol{F}) = \partial_{\boldsymbol{F}} W(\boldsymbol{F}) = \begin{cases} c_1 p|\boldsymbol{F}|^{p-2}\boldsymbol{F} - \frac{\gamma c_2}{(\det \boldsymbol{F})^\gamma}\boldsymbol{F}^{-\mathsf{T}} & \text{if } \det F > 0, \\ \text{undefined} & \text{else.} \end{cases}$$

Finite-strain elastoplasticity

$\boldsymbol{F} = \nabla \boldsymbol{\varphi} \in \mathrm{GL}^+(\mathbb{R}^d)$

$\boldsymbol{P} = \boldsymbol{F}_{\text{plast}} \in \mathrm{SL}(\mathbb{R}^d) \overset{\text{def}}{=} \{\, F \in \mathbb{R}^{d\times d} \mid \det F = 1 \,\}$



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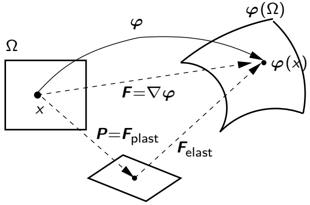
Multiplicative decomposition (Lee'69)

$$\nabla\varphi = \boldsymbol{F} = \boldsymbol{F}_{\mathrm{el}}\boldsymbol{F}_{\mathrm{plast}} = \boldsymbol{F}_{\mathrm{el}}\boldsymbol{P} \quad \rightsquigarrow \quad \boxed{\boldsymbol{F}_{\mathrm{el}} = \boldsymbol{F}\boldsymbol{P}^{-1}}$$

$$W(\boldsymbol{F},\boldsymbol{P},\boldsymbol{A}) = W_{\mathrm{el}}(\underbrace{\boldsymbol{F}\boldsymbol{P}^{-1}}_{=\boldsymbol{F}_{\mathrm{el}}}) + W_{\mathrm{hard}}(\boldsymbol{P}) + W_{\mathrm{grad}}(\boldsymbol{P},\boldsymbol{A})$$



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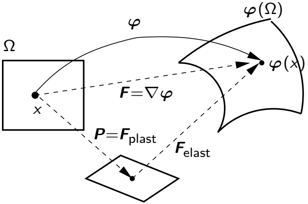

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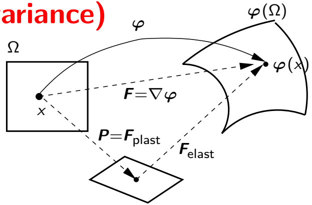

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For specialists:
Today only "kinematic hardening".
WIAS Preprint 1299 is more general: $z = (\boldsymbol{P}, p)$
Applications to isotropic hardening, crystal plasticity, ...

Plastic flow rule

(assume temporarily $W_{\mathrm{grad}} \equiv 0$)

Let $\boldsymbol{\xi} = \dot{\boldsymbol{P}}\boldsymbol{P}^{-1} \in \mathrm{T}_{\mathbf{1}}\mathrm{SL}(\mathbb{R}^d) \;=\;$ Lie algebra $\mathrm{sl}(\mathbb{R}^{\mathrm{d}})$

$$\partial_{\dot{\boldsymbol{P}}} R(\boldsymbol{P}, \dot{\boldsymbol{P}}) = \underbrace{\partial_{\boldsymbol{\xi}} \widehat{R}(\dot{\boldsymbol{P}}\boldsymbol{P}^{-1})}_{\in \mathrm{sl}^*(\mathbb{R}^d)} \boldsymbol{P}^{-\mathsf{T}} \in \mathrm{T}^*_{\boldsymbol{P}}\mathrm{SL}(\mathbb{R}^d)$$

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Internal force balance (Biot's equation)

$$0 \in \partial_{\boldsymbol{\xi}} \widehat{R}(\dot{\boldsymbol{P}}\boldsymbol{P}^{-1})\boldsymbol{P}^{-\mathsf{T}} - \boldsymbol{F}_{\text{el}}^{\mathsf{T}}\boldsymbol{\Sigma}_{\text{el}} + \partial_{\boldsymbol{P}} W_{\text{hard}}(\boldsymbol{P})$$

Plastic flow rule

$$\mathrm{sl}(\mathbb{R}^d) \ni \dot{\boldsymbol{P}}\boldsymbol{P}^{-1} \in \partial\widehat{R}^{-1}\Big(\underbrace{\boldsymbol{F}_{\text{el}}^{\mathsf{T}}\partial_{\boldsymbol{F}_{\text{el}}} W_{\text{el}}(\boldsymbol{F}_{\text{el}}) - \partial_{\boldsymbol{P}} W_{\text{hard}}(\boldsymbol{P})\boldsymbol{P}^{-\mathsf{T}}}_{\in \mathrm{sl}^*(\mathbb{R}^d)}\Big)$$

Overview

... is a special weak formulation for

rate-independent systems $(\mathcal{Q}, \mathcal{E}, \mathcal{R})$

State space $\mathcal{Q}$

contains $q = (\varphi, \boldsymbol{P})$ (specified later)

Energy storage functional

$\widehat{\mathcal{E}}(t, \varphi, \boldsymbol{P}) = \int_\Omega W(\nabla\varphi, \boldsymbol{P}, \nabla \boldsymbol{P})\,\mathrm{d}x - \langle \ell(t), \varphi \rangle$

with $\langle \ell(t), \varphi \rangle = \int_\Omega \boldsymbol{f}_{\mathrm{ext}} \cdot \varphi \,\mathrm{d}x + \int_{\Gamma_{\mathrm{Neu}}} \boldsymbol{g}_{\mathrm{ext}} \cdot \varphi \,\mathrm{d}a$

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Rate-independent dissipation potential $\mathcal{R}$

rate independence $\mathcal{R}(\boldsymbol{P}, \gamma\dot{\boldsymbol{P}}) = \gamma^1\mathcal{R}(\boldsymbol{P}, \dot{\mathcal{P}}),\ \gamma > 0$

$\rightsquigarrow$ nonsmoothness: $\partial_{\dot{\boldsymbol{P}}}\mathcal{R}(\boldsymbol{P}, \dot{\boldsymbol{P}})$ set-valued subdifferential

$\partial_{\dot{\boldsymbol{P}}}\mathcal{R}(\boldsymbol{P}, \gamma\dot{\boldsymbol{P}}) = \gamma^0\partial_{\dot{\boldsymbol{P}}}\mathcal{R}(\boldsymbol{P}, \dot{\boldsymbol{P}})$ (homog. of degree 0)

Typical approach to numerics and existence theory:

Incremental minimization problems for $0 < t_1 < \cdots < t_N = T$:

With $\tau_k = t_k - t_{k-1}$ find

$$(\varphi^k, \boldsymbol{P}^k) \text{ minimizing } (\widetilde{\varphi}, \widetilde{\boldsymbol{P}}) \mapsto \widehat{\mathcal{E}}(t_k, \widetilde{\varphi}, \widetilde{\boldsymbol{P}}) + \underbrace{\tau_k \mathcal{R}(\boldsymbol{P}_{k-1}, \frac{1}{\tau_k}(\widetilde{\boldsymbol{P}} - \boldsymbol{P}_{k-1}))}$$

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$\dot{\boldsymbol{P}} \approx \frac{1}{\tau_k}(\widetilde{\boldsymbol{P}} - \boldsymbol{P}^{k-1}) \notin \mathrm{sl}(\mathbb{R}^d)$!!

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WIAS

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Engineers: $\boldsymbol{P}^k = \exp(\boldsymbol{\xi}^k)\boldsymbol{P}^{k-1}$ with $\boldsymbol{\xi}^k \in \mathrm{sl}(\mathbb{R}^d)$:

$(\varphi^k, \boldsymbol{\xi}^k)$ minimizing $(\widetilde{\varphi}, \boldsymbol{\xi}^k) \mapsto \widehat{\mathcal{E}}(t_k, \widetilde{\varphi}, \exp(\boldsymbol{\xi}^k)\boldsymbol{P}^{k-1}) + \mathcal{R}(\boldsymbol{P}_{k-1}, \boldsymbol{\xi}^k)$

applied analysis: geometric evolution in metric spaces

rate-independent systems $(\mathcal{Q}, \mathcal{E}, \mathcal{R})$

Metric space approach for geometric evolution: Replace the infinitesimal dissipation metric $\mathcal{R}$ by a (global) distance $\mathcal{D}$

Plastic dissipation distance $\mathcal{D} : \mathcal{Z} \times \mathcal{Z} \to \mathbb{R}_\infty$:

$\mathcal{D}(\boldsymbol{P}_0, \boldsymbol{P}_1) = \int_\Omega D(x, \boldsymbol{P}_0(x), \boldsymbol{P}_1(x))\,\mathrm{d}x$

where $D(x, \cdot, \cdot) : \mathrm{SL}(\mathbb{R}^d)^2 \to [0, \infty]$ is defined via

$$D(x, P_0, P_1) = \inf\Big\{ \int_0^1 R(x, P(s), \dot{P}(s))\,\mathrm{d}s \;\Big|\; P(0) = P_0, \; P(1) = P_1,\; P \in \mathrm{C}^1([0,1]; \mathrm{SL}(\mathbb{R}^d))\Big\}$$

rate-independent systems $(\mathcal{Q}, \mathcal{E}, \mathcal{R})$

Metric space approach for geometric evolution: Replace the infinitesimal dissipation metric $\mathcal{R}$ by a (global) distance $\mathcal{D}$

Plastic dissipation distance $\mathcal{D} : \mathcal{Z} \times \mathcal{Z} \to \mathbb{R}_\infty$:

$\mathcal{D}(\boldsymbol{P}_0, \boldsymbol{P}_1) = \int_\Omega D(x, \boldsymbol{P}_0(x), \boldsymbol{P}_1(x))\,\mathrm{d}x$

where $D(x, \cdot, \cdot) : \mathrm{SL}(\mathbb{R}^d)^2 \to [0, \infty]$ is defined via

$$D(x, P_0, P_1) = \inf \Big\{ \int_0^1 R(x, P(s), \dot{P}(s))\,\mathrm{d}s \,\Big|\, P(0) = P_0, \; P(1) = P_1,\ P \in \mathrm{C}^1([0,1]; \mathrm{SL}(\mathbb{R}^d)) \Big\}$$

Plastic invariance gives $D(x, P_0, P_1) = D(x, I, P_1 P_0^{-1})$

Note that $D(x, I, \exp(\boldsymbol{\xi})) \leq \widehat{R}(\boldsymbol{\xi}) \sim |\boldsymbol{\xi}|$

Hence, D has at most logarithmic growth (not coercive in L^q)

3. Energetic formulation

$\boldsymbol{q} = (\varphi, \boldsymbol{P})$ state of the body, $\mathcal{Q} = \mathcal{F} \times \mathcal{Z}$ state space

$\mathcal{F} =$ admissible deformations, $\mathcal{Z} =$ space of internal states

$\mathcal{E} : [0, T] \times \mathcal{Q} \to \mathbb{R}_\infty$ energy storage functional

$\mathcal{D} : \mathcal{Q} \times \mathcal{Q} \to [0, \infty]$ dissipation distance

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Definition. A process $\boldsymbol{q} : [0,T] \to \mathcal{Q}$ is called an ***energetic solution for the rate-independent system*** $(\mathcal{Q}, \mathcal{E}, \mathcal{D})$, if for all $t \in [0,T]$ we have stability (S) and energy balance (E):

(S) $\forall \widetilde{\boldsymbol{q}} \in \mathcal{Q} : \ \mathcal{E}(t, \boldsymbol{q}(t)) \leq \mathcal{E}(t, \widetilde{\boldsymbol{q}}) + \mathcal{D}(\boldsymbol{q}(t), \widetilde{\boldsymbol{q}})$,

(E) $\mathcal{E}(t, \boldsymbol{q}(t)) + \mathrm{Diss}_{\mathcal{D}}(\boldsymbol{q}, [0,t]) = \mathcal{E}(0, \boldsymbol{q}(0)) + \int_0^t \partial_s \mathcal{E}(s, \boldsymbol{q}(s))\,\mathrm{d}s$.

$$\mathrm{Diss}_{\mathcal{D}}(\boldsymbol{q}, [0,t]) \overset{\text{def}}{=} \sup \left\{ \sum_1^N \mathcal{D}(\boldsymbol{q}(t_{j-1}), \boldsymbol{q}(t_j)) \,\middle|\, \text{all partit. } \right\}$$

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Smooth energetic solutions satisfy
"elastic equilibrium" and the plastic flow rule.

Incremental minimization problems: Find
$(\varphi^k, \boldsymbol{P}^k)$ minimizing $(\widetilde{\varphi}, \widetilde{\boldsymbol{P}}) \mapsto \widehat{\mathcal{E}}(t_k, \widetilde{\varphi}, \widetilde{\boldsymbol{P}}) + \mathcal{D}(\boldsymbol{P}_{k-1}, \widetilde{\boldsymbol{P}})$

Main abstract assumption for existence theory
(developed with MAINIK'05, FRANCFORT'06)

- $\mathcal{Q}$ weakly closed subset of a Banach space
- $\mathcal{D}$ extended quasi-metric on $\mathcal{Z}$ (positivity and triangle ineq.)
  no coercivity in norms needed!!
- $\mathcal{D}$ weakly lower semi-continuous
- $\mathcal{E}(t, \cdot) : \mathcal{Q} \to \mathbb{R}_\infty$ coercive and weakly lower semi-continuous
- If $\mathcal{E}(t, \boldsymbol{q}) < \infty$, then $\mathcal{E}(\cdot, \boldsymbol{q}) \in \mathrm{W}^{1,1}([0, T])$ with
  $|\partial_t \mathcal{E}(t, \boldsymbol{q})| \leq \lambda(t)\mathcal{E}(t, \boldsymbol{q})$ for fixed $\lambda \in \mathrm{L}^1([0, T])$

Overview

Choice of **admissible deformations for elastoplasticity**

Time-dependent boundary conditions:

$\varphi(t,x) = \varphi_{\mathrm{Dir}}(t,x)$ for $(t,x) \in [0,T] \times \Gamma_{\mathrm{Dir}}$

Assume that an extension $\varphi_{\mathrm{Dir}} \in \mathrm{C}^1([0,T]\times\mathbb{R}^d;\mathbb{R}^d)$ exists with

$$\nabla\varphi_{\mathrm{Dir}}, \nabla\varphi_{\mathrm{Dir}}^{-1} \in \mathrm{BC}^0([0,T]\times\mathbb{R}^d;\mathrm{Lin}(\mathbb{R}^d;\mathbb{R}^d))$$

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We search for φ in the form $\varphi(t,x) = \varphi_{\mathrm{Dir}}(t,\boldsymbol{y}(t,x))$ with $\boldsymbol{y} \in \mathcal{Y}$

$$\mathcal{Y} \stackrel{\mathrm{def}}{=} \Big\{ \boldsymbol{y} \in \mathrm{W}^{1,q_y}(\Omega;\mathbb{R}^d) \,\Big|\, \boldsymbol{y}|_{\Gamma_{\mathrm{Dir}}} = \mathrm{id},\ \text{(GI) holds} \Big\}$$

Global invertibility (GI) $\begin{cases} \det \nabla y(x) \geq 0 \text{ a.e. in } \Omega, \\ \int_\Omega \det \nabla y(x)\,\mathrm{d}x \leq \mathrm{vol}(y(\Omega)). \end{cases}$

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Ciarlet&Necas'87: $\mathcal{Y}$ is weakly closed in $\mathrm{W}^{1,q_{\boldsymbol{y}}}(\Omega;\mathbb{R}^d)$, if $q_{\boldsymbol{y}} > d$.

Final energy functional $\mathcal{E}(t,\boldsymbol{y},\boldsymbol{P}) \overset{\mathrm{def}}{=} \widehat{\mathcal{E}}(t,\varphi_{\mathrm{Dir}}(t)\circ\boldsymbol{y},\boldsymbol{P})$

Weak lower semicontinuity of $\mathcal{E}$:

$$\mathcal{E}(t,\boldsymbol{y},\boldsymbol{P}) = \int_\Omega W(\nabla\varphi_{\rm Dir}(t,\boldsymbol{y})\nabla\boldsymbol{y}\boldsymbol{P}^{-1},\boldsymbol{P},\nabla\boldsymbol{P})\,\mathrm{d}x - \underbrace{\langle \ell(t),\varphi_{\rm Dir}(t)\circ\boldsymbol{y}\rangle}_{\text{w.l.o.g. } \equiv 0}$$

- $W:\Omega\times\mathbb{R}^{d\times d}\times\mathbb{R}^{d\times d}\times\mathbb{R}^{d\times d\times d}\to\mathbb{R}_\infty$ is a normal integrand
- $W(x,\cdot,\boldsymbol{P},\boldsymbol{A}):\mathbb{R}^{d\times d}\to\mathbb{R}_\infty$ is polyconvex
- $W(x,\boldsymbol{F}_{\rm el},\boldsymbol{P},\cdot):\mathbb{R}^{d\times d\times d}\to\mathbb{R}_\infty$ is convex
- $W(x,\boldsymbol{F}_{\rm el},\boldsymbol{P},\boldsymbol{A})\geq c\big(|\boldsymbol{F}_{\rm el}|^{q_F}+|\boldsymbol{P}|^{q_P}+|\boldsymbol{A}|^r\big)-C$

Choice of internal states:

$$\mathcal{Z}\overset{\text{def}}{=}\{\,\boldsymbol{P}\in\big(\mathrm{W}^{1,r}\cap\mathrm{L}^{q_P}\big)(\Omega;\mathbb{R}^{d\times d})\mid\boldsymbol{P}(x)\in\mathrm{SL}(\mathbb{R}^d)\text{ a.e. in }\Omega\,\}$$

coercivity $W(x, \boldsymbol{F}_{\mathrm{el}}, \boldsymbol{P}, \boldsymbol{A}) \geq c\big(|\boldsymbol{F}_{\mathrm{el}}|^{q_{\boldsymbol{F}}} + |\boldsymbol{P}|^{q_{\boldsymbol{P}}} + |\boldsymbol{A}|^{r}\big) - C$

$\mathcal{Q} = \mathcal{Y} \times \mathcal{Z}$ with $\mathcal{Y} = \{\, \boldsymbol{y} \in \mathrm{W}^{1,q_{\boldsymbol{y}}}(\Omega) \mid \boldsymbol{y}|_{\Gamma_{\mathrm{Dir}}} = \mathrm{id},\ (\mathrm{GI}) \text{ holds} \,\}$

$\mathcal{Z} = \{\, \boldsymbol{P} \in \big(\mathrm{W}^{1,r} \cap \mathrm{L}^{q_{\boldsymbol{P}}}\big)(\Omega; \mathbb{R}^{d\times d}) \mid \boldsymbol{P}(x) \in \mathrm{SL}(\mathbb{R}^d) \text{ a.e. in } \Omega \,\}$

Proposition 1. Under the above assumptions with $\Gamma_{\mathrm{Dir}} \neq \emptyset$,

$$\frac{1}{q_{\boldsymbol{P}}} + \frac{1}{q_{\boldsymbol{F}}} = \frac{1}{q_{\boldsymbol{y}}} < \frac{1}{d}, \text{ and } r > 1$$

we have that

- $\mathcal{D}$ is weakly continuous on $\mathcal{Z} \times \mathcal{Z}$ and
- $\mathcal{E}(t, \cdot)$ is coercive and weakly lower semi-continuous on $\mathcal{Q}$.

Control of the power of external forces

$\partial_t \mathcal{E}(t, \boldsymbol{y}, \boldsymbol{P})$ involves $\partial_t \nabla \varphi_{\mathrm{Dir}}(t, x)$

Additional assumption (cf. Baumann&Owen&Phillips'91, Ball'02)

$W(x, \cdot, \boldsymbol{P}, \boldsymbol{A})$ is differentiable on $\mathrm{GL}^+(\mathbb{R}^d)$ and there exist $c_1 > 0$, $c_0 \in \mathbb{R}$ and a modulus of continuity ω such that

(MSC1) $|\partial_{\boldsymbol{F}} W(x, \boldsymbol{F}, \boldsymbol{P}, \boldsymbol{A}) \boldsymbol{F}^{\mathsf{T}}| \leq c_1 (W(x, \boldsymbol{F}, \boldsymbol{P}, \boldsymbol{A}) + c_0)$

(MSC2) $|\partial_{\boldsymbol{F}} W(x, \boldsymbol{F}, \boldsymbol{P}, \boldsymbol{A}) \boldsymbol{F}^{\mathsf{T}} - \partial_{\boldsymbol{F}} W(x, \boldsymbol{N}\boldsymbol{F}, \boldsymbol{P}, \boldsymbol{A}) (\boldsymbol{N}\boldsymbol{F})^{\mathsf{T}}|$
$\leq \omega(|\boldsymbol{N} - \boldsymbol{1}|)(W(x, \boldsymbol{F}, \boldsymbol{P}, \boldsymbol{A}) + c_0)$.

Both conditions hold for

$$W(\boldsymbol{F}, \boldsymbol{P}, \boldsymbol{A}) = W_{\mathrm{el}}(\boldsymbol{F}\boldsymbol{P}^{-1}) + W_{\mathrm{hard,grad}}(\boldsymbol{P}, \boldsymbol{A})$$

with $$W_{\mathrm{el}}(\boldsymbol{F}_{\mathrm{el}}) = \begin{cases} c_1 |\boldsymbol{F}_{\mathrm{el}}|^p + \frac{c_2}{(\det \boldsymbol{F}_{\mathrm{el}})^\gamma} & \text{if } \det \boldsymbol{F}_{\mathrm{el}} > 0, \\ \infty & \text{else.} \end{cases}$$

Consider $\mathrm{GL}^+(\mathbb{R}^d) \ni \boldsymbol{F} \mapsto W(F)$ (for x, $\boldsymbol{P}$, and $\boldsymbol{A}$ fixed).

$$\boldsymbol{K}(\boldsymbol{F}) = \partial_{\boldsymbol{F}} W(\boldsymbol{F})\boldsymbol{F}^{\mathsf{T}} \in \mathrm{gl}(\mathbb{R}^d)^* = \mathrm{T}^*_{\mathbf{1}}\mathrm{GL}^+(\mathbb{R}^d)$$

$$\boldsymbol{K}{:}\boldsymbol{H} = \lim_{\varepsilon\to 0} \frac{1}{\varepsilon}\big[W((\mathbf{1}+\varepsilon\boldsymbol{H})\boldsymbol{F}) - W(\boldsymbol{F})\big]$$

Multiplicative stress control:

$$\text{(MSC1)}\ \exists\, c_0, c_1\ \forall\, \boldsymbol{F} : \quad |\partial_{\boldsymbol{F}} W(\boldsymbol{F})\boldsymbol{F}^{\mathsf{T}}| \leq c_1 \big[c_0 + W(\boldsymbol{F})\big]$$

Ball'02:
- (MSC1) compatible w/ frame indiff. and polyconvexity
- (MSC1) implies $W(\boldsymbol{F}) \leq C\big[|\boldsymbol{F}|^s + |\boldsymbol{F}^{-1}|^s\big]$

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An easy nontrivial example: $W(\boldsymbol{F}) = \alpha|\boldsymbol{F}|^q + \frac{\beta}{(\det \boldsymbol{F})^r}$,

1st Piola–Kirchh. stress tensor $\boldsymbol{T} = \partial_{\boldsymbol{F}} W(\boldsymbol{F}) = \underbrace{\alpha q|\boldsymbol{F}|^{q-2}\boldsymbol{F}}_{\text{slower growth}} - \underbrace{\frac{\beta\, r}{(\det \boldsymbol{F})^{r+1}}\mathrm{cof}\,\boldsymbol{F}}_{\text{more singular}}$

Kirchhoff's stress tensor $\boldsymbol{K} = \boldsymbol{T}\boldsymbol{F}^\mathsf{T} = \alpha q|\boldsymbol{F}|^{q-2}\boldsymbol{F}\boldsymbol{F}^\mathsf{T} - \frac{\beta r}{(\det \boldsymbol{F})^r}\mathbf{1}$

A more geometric interpretation

Let d_{GL} be any left-invariant geodesic distance on $\mathrm{GL}^+(\mathbb{R}^d)$:

$$d_{\mathrm{GL}}(\boldsymbol{F}_0, \boldsymbol{F}_1) \overset{\mathrm{def}}{=} \min \Big\{ \int_0^1 \|\boldsymbol{F}(s)^{-1}\dot{\boldsymbol{F}}(s)\|\,\mathrm{d}s \;\Big|\; \boldsymbol{F}_0 = \boldsymbol{F}(0), \\ \boldsymbol{F}_1 = \boldsymbol{F}(1),\ \boldsymbol{F} \in \mathrm{C}^1([0,1], \mathrm{GL}^+(\mathbb{R}^d)) \Big\}$$

For $W \in \mathrm{C}^1(\mathrm{GL}^+(\mathbb{R}^d);\mathbb{R})$ we have
(MSC1) $\iff$

$$\exists c_0, c_1\ \forall \boldsymbol{F}, \boldsymbol{G}:\ |\log(W(\boldsymbol{F})+c_0) - \log(W(\boldsymbol{G})+c_0)| \le c_1\, d_{\mathrm{GL}}(\boldsymbol{F},\boldsymbol{G})$$

i.e., $\log(W(\cdot)+\widehat{c}_0)$ is globally Lipschitz on $(\mathrm{GL}^+(\mathbb{R}^d), d_{\mathrm{GL}})$.

W I A S

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i.e., $\log(W(\cdot)+\widehat{c}_0)$ is globally Lipschitz on $(\mathrm{GL}^+(\mathbb{R}^d),d_{\mathrm{GL}})$.

Using $|\frac12\log(\boldsymbol{F}^{\mathsf{T}}\boldsymbol{F})|\le d_{\mathrm{GL}}(\mathbf{1},\boldsymbol{F})\le d\pi+|\frac12\log(\boldsymbol{F}^{\mathsf{T}}\boldsymbol{F})|$ we obtain Ball's upper estimate:

$$W(\boldsymbol{F})-2c_0\le \exp\big(c_1\, d_{\mathrm{GL}}(\mathbf{1},\boldsymbol{F})\big)\le C\big[|\boldsymbol{F}|^s+|\boldsymbol{F}^{-1}|^s\big].$$

(MSC1+2) $\qquad |\partial_{\boldsymbol{F}} W(x,\boldsymbol{F},\boldsymbol{P},\boldsymbol{A})\boldsymbol{F}^{\mathsf{T}}| \le c_1(W(x,\boldsymbol{F},\boldsymbol{P},\boldsymbol{A})+c_0)$

$|\partial_{\boldsymbol{F}} W(x,\boldsymbol{F},\boldsymbol{P},\boldsymbol{A})\boldsymbol{F}^{\mathsf{T}} - \partial_{\boldsymbol{F}} W(x,\boldsymbol{N}\boldsymbol{F},\boldsymbol{P},\boldsymbol{A})(\boldsymbol{N}\boldsymbol{F})^{\mathsf{T}} \le \omega(|\boldsymbol{N}-\mathbf{1}|)(W(x,\boldsymbol{F},\boldsymbol{P},\boldsymbol{A})+c_0)$

Kirchhoff tensor for given $\boldsymbol{q} \in \mathcal{Q}$ and $\boldsymbol{F}$

$$\boldsymbol{K}_{\boldsymbol{q}}(x,\boldsymbol{F}) = \partial_{\boldsymbol{F}} W(\boldsymbol{F}\boldsymbol{P}(x)^{-1},\boldsymbol{P}(x),\boldsymbol{A}(x))(\boldsymbol{F}\boldsymbol{P}(x)^{-1})^{\mathsf{T}} \in \mathrm{T}^*_{\mathbf{1}}\mathrm{GL}^+(\mathbb{R}^d)$$

Proposition 2. $\mathcal{E}(t,\boldsymbol{q}) < \infty$ implies $\mathcal{E}(\cdot,\boldsymbol{q}) \in \mathrm{C}^1([0,T])$ with

$$\partial_t\mathcal{E}(t,\boldsymbol{q}) = \int_\Omega \boldsymbol{K}_{\boldsymbol{q}}(x,\nabla\varphi_{\mathrm{Dir}}(t,y(x))\nabla\boldsymbol{y}(x)) : \boldsymbol{V}(t,y(x))\,\mathrm{d}x,$$

where $\boldsymbol{V}(t,\boldsymbol{y}) = \big(\nabla\varphi_{\mathrm{Dir}}(t,\boldsymbol{y})\big)^{-1}\frac{\partial}{\partial t}\nabla\varphi_{\mathrm{Dir}}(t,\boldsymbol{y})$,

and the following estimates hold:

$$|\partial_t\mathcal{E}(t,\boldsymbol{q})| \le c_1^{\mathcal{E}}\big(\mathcal{E}(t,\boldsymbol{q})+c_0^E\big) \quad \text{and}$$

$$|\partial_t\mathcal{E}(t_1,\boldsymbol{q})-\partial_t\mathcal{E}(t_2,\boldsymbol{q})| \le \widetilde{\omega}(|t_2-t_1|)\big(\mathcal{E}(t_1,\boldsymbol{q})+c_0^E\big).$$

Main Existence Result.

Under the following assumptions (only the major ones)

- W is a normal integrand and is lower semicontinuous,
- W polyconvex in $\boldsymbol{F}_{\text{el}}$ and convex in $\boldsymbol{A} = \nabla \boldsymbol{P}$,
- $W(x, \boldsymbol{F}_{\text{el}}, \boldsymbol{P}, \boldsymbol{A}) \geq c\big(|\boldsymbol{F}_{\text{el}}|^{q_{\boldsymbol{F}}} + |\boldsymbol{P}|^{q_{\boldsymbol{P}}} + |\boldsymbol{A}|^r\big) - C$,
- $\dfrac{1}{q_{\boldsymbol{P}}} + \dfrac{1}{q_{\boldsymbol{F}}} = \dfrac{1}{q_{\boldsymbol{y}}} < \dfrac{1}{d}$, and $r > 1$,
- dissipation distance D as above,
- φ_{Dir} has extension with $\nabla\varphi_{\text{Dir}}, \nabla\varphi_{\text{Dir}}^{-1} \in \text{BC}^0$,

for each stable initial state $\boldsymbol{q}_0 \in \mathcal{Q}$ there exists at least one energetic solution $\boldsymbol{q} : [0, T] \to \mathcal{Q}$ of $(\mathcal{Q}, \mathcal{E}, \mathcal{D})$ with $\boldsymbol{q}(0) = \boldsymbol{q}_0$.

(S) $\quad \forall\, \widetilde{\boldsymbol{q}} \in \mathcal{Q} : \ \mathcal{E}(t, \boldsymbol{q}(t)) \leq \mathcal{E}(t, \widetilde{\boldsymbol{q}}) + \mathcal{D}(\boldsymbol{q}(t), \widetilde{\boldsymbol{q}})$,

(E) $\quad \mathcal{E}(t, \boldsymbol{q}(t)) + \text{Diss}_{\mathcal{D}}(\boldsymbol{q}, [0,t]) = \mathcal{E}(0, \boldsymbol{q}(0)) + \int_0^t \partial_s \mathcal{E}(s, \boldsymbol{q}(s))\, \text{d}s$.

Overview

1. GSM and Plasticity
2. Finite-Strain Plasticity
3. Energetic formulation
4. Existence results

Conclusions

Using **full regularization** and **strong coercivity** the existence of energetic solutions can be shown for many plasticity models.

Geometry and functional analysis can be combined,
if METRIC concepts for geoemtric evolution are used.

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Thank you for your attention!

Papers available under http://www.wias-berlin.de/people/mielke

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