Large-scale atmospheric circulation, semi-geostrophic motion and Lagrangian particle methods

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The compressible non-viscous Euler equations provide the starting point for modeling atmospheric and ocean dynamics [5, 6]. Given typical length and time-scales for global circulation patterns, approximations are often employed which filter non-significant flow patterns from the equations of motion. Among the most popular and useful approximations are the hydrostatic and the semi-geostrophic approximations, which reads [5]

$$\begin{aligned} \frac{D\mathbf{u}_g}{Dt} + f\mathbf{k}\times\mathbf{u} + \frac{1}{\rho}\nabla p + g\mathbf{k} &= 0,\\ \rho_t + \nabla\cdot(\rho\mathbf{u}) &= 0,\\ \theta_t + \mathbf{u}\cdot\nabla\theta &= 0, \end{aligned}$$

with the geostrophic wind approximation

$$\mathbf{u}_g = \begin{bmatrix} u_g & v_g & 0 \end{bmatrix}^T$$

and

$$f u_g = -\frac{1}{\rho}\frac{\partial p}{\partial y}, \qquad f v_g = +\frac{1}{\rho}\frac{\partial p}{\partial x}.$$

A practical implication in the northern hemisphere is that pressure increases to the right if you stand with our back to the wind.

The semi-geostrophic equations make use of the geostrophic wind approximation in a particularly clever way giving rise to many interesting underlying geometric features including links to optimal transportation, variational mechanics and constraint dynamics. One can explain these ideas by going first to the shallow water equations and then further on to a single fluid parcel approximation

$$\begin{aligned} \dot{\mathbf{p}} &= J_2\mathbf{p} - \varepsilon\,\nabla\mu(\tau,\mathbf{q}), \qquad J_2 = \begin{pmatrix} 0 & 1\\ -1 & 0\end{pmatrix},\\ \dot{\mathbf{q}} &= \mathbf{p}, \end{aligned}$$

with state variable $\mathbf{z} = (\mathbf{q}^T, \mathbf{p}^T)^T \in \mathbb{R}^4$, μ a given (time-dependent) potential, and the small parameter $\varepsilon > 0$. The associated "semi-geostrophic" equations are given by

$$\dot{\mathbf{p}}_g = J_2\mathbf{p} - \varepsilon\nabla\mu(\tau,\mathbf{q}), \tag{1}$$

$$\dot{\mathbf{q}} = \mathbf{p}, \tag{2}$$

with geostrophic "wind" $\mathbf{p}_g = -\varepsilon J_2\nabla\mu(\tau,\mathbf{q})$. For time-independent μ (which we assume from now on), the energy

$$E = \frac{1}{2}\|\mathbf{p}_g\|^2 + \varepsilon\mu(\mathbf{q})$$

is preserved.

Much insight into the semi-geostrophic approximation has been gained by the Hoskins' transformation [5]

$$\mathbf{q}_\varepsilon = \mathbf{q} + \varepsilon \nabla \mu(\mathbf{q}) = \mathbf{q} + J_2 \mathbf{p}_g,$$

which leads to the following equation in the transformed variable $\mathbf{q}_\varepsilon$:

$$\dot{\mathbf{q}}_\varepsilon = -\varepsilon J_2 \nabla \mu(\mathbf{q}). \tag{3}$$

It turns out that the Hoskin's transform is linked to an optimal transportation problem. See [5] for the fascinating details.

While the semi-geostrophic equations are well studied much less is known about its range of validity in terms of the small parameter ε. Improved semi-geostrophic models can be found in [6]. More recently, asymptotic expansions have been considered within the Lagrangian variational framework in [8, 9].

A different approach has been taken in [3], which applies Hamiltonian normal form theory to the gyroscopic particle problem (1)-(2), i.e., one finds a canonical near-identity change of coordinates $\Psi_n : \mathbf{z}_\varepsilon \to \mathbf{z}$ so that

$$H_n = H_0 \circ \Psi_n = K + \varepsilon G_n + \varepsilon^{n+1} R_n, \tag{4}$$

where

$$\{G_n, K\} = 0, \qquad K = \frac{1}{2}\|\mathbf{p}_\varepsilon\|^2, \tag{5}$$

with $\{\cdot,\cdot\}$ being the Poisson bracket for (1)-(2). Optimal truncation in the index n yields the desired exponential dependence on ε and the preservation of "geostrophic/gyroscopic" balance over exponentially long periods of time.

As a consequence of (4) and (5), we may consider the reduced equations

$$\begin{aligned} 0 = \dot{\mathbf{p}}_\varepsilon &= J_2 \nabla_{\mathbf{p}} G_n(\mathbf{q}_\varepsilon, 0) - \varepsilon G_n(\mathbf{q}_\varepsilon, 0), \\ \dot{\mathbf{q}}_\varepsilon &= \nabla_{\mathbf{p}} G_n(\mathbf{q}_\varepsilon, 0), \end{aligned}$$

for initial conditions satisfying $\mathbf{p}_0 = 0$. These equations are equivalent to

$$\dot{\mathbf{q}}_\varepsilon = -\varepsilon J_2 \nabla_{\mathbf{q}} G_n(\mathbf{q}_\varepsilon, 0)$$

and the leading order term coincide with Hoskin's transformed equation (3).

Furthermore, the normal form estimates remain valid for many particle systems of type (1)-(2), which couple through a multi-particle potential $\mu(\mathbf{q}_1, \ldots \mathbf{q}_N)$. This observation allows one to go back to the continuum limit by first considering finite dimensional particle approximations of the shallow-water equations (see, e.g., [1, 2]. The continuum limit gives rise to a set of regularized fluid equations which can be interpreted as Euler's equations subject to a regularized pressure field [7]. Similar pressure regularizations arise from semi-implicit time-stepping methods, which are widely used in numerical weather prediction. See [4] and references therein.

References

[1] J. Frank, G. Gottwald, and S. Reich, *The Hamiltonian particle-mesh method*, In M. Griebel and M.A. Schweitzer, editors, Meshfree Methods for Partial Differential Equations, volume **26** of Lect. Notes Comput. Sci. Eng., pages 131–142, Berlin Heidelberg, 2002. Springer-Verlag.

[2] J. Frank and S. Reich, *The Hamiltonian particle-mesh method for the spherical shallow water equations*, Atmos. Sci. Let., **5**, 89–95, 2004.

[3] C.J. Cotter and S. Reich, *Semi-geostrophic particle motion and exponentially accurate normal forms*, SIAM Multiscale Modelling, **5**, 476–496, 2006.

[4] A. Staniforth, N. Wood, and S. Reich, *A time-staggered semi-Lagrangian discretization of the rotating shallow-water equations*, Q.J.R. Meteorol. Soc., **132**, 3107–3116, 2006.

[5] M. Cullen, *A mathematical theory of large scale atmosphere/ocean flow*, Imperial College Press, 2006.

[6] R. Salmon, *Lectures on geophysical fluid dynamics*, Oxford University Press, 1999.

[7] R. Di Lisio,E. Grenier, and M. Pulvirenti, *On the regularization of the pressure field in compressible Euler equations*, Annali della Scuola Normale Superriore di Pisa, **24**, 1997, 227–238.

[8] M. Oliver, *Variational asymptotics for rotating shallow water near geostrophic balance: a tranformational approach*, J. Fluid Mech., **551**, 2006, 197–234.

[9] G. Gottwald, M. Oliver, and N. Tecu, *Long-time accuracy for approximate slow manifolds in a finite dimensional model of balance*, J. Nonlinear Sci., **17**, 2007, 283–307.