Dirac Cotangent Bundle Reduction

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Oberwolfach, July 20-26, 2008

Acknowledgement

- We would like to thank ***Hernán Cendra*** and ***Tudor Ratiu*** for their helpful suggestions.

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- References

  - **Lagrangian Reduction by Stages [CMR2001]**: Cendra, Marsden and Ratiu, Memoirs of AMS, 722, 2001.
  - **Hamilton-Poincaré Equations [CMPR2003]**: Cendra, Marsden, Pekarsky and Ratiu, Moscow Math. Jour. **3**, 833-867, 2003.
  - **Hamiltonian Reduction by Stages [MMOR2007]**: Marsden, Misiolek, et al, vol.1913, Lecture Notes in Mathematics. Springer.
  - **Implicit Lagrangian Systems [YoMa2006]**: Yoshimura and Marsden, J. Geom. Phys., **57**, 133–156, 209–250, 2006.
  - **Lie-Dirac Reduction [YoMa2007]**: Yoshimura and Marsden, Rep. on Math. Phy., **60**, 381–426, 2007.

Contents

Motivations

- ***Dirac structures*** (Courant and Weinstein [1988]) is an idea of synthesizing ***pre-symplectic structures*** (not necessarily closed, and possibly degenerate) and ***almost Poisson structures*** (brackets that need not satisfy Jacobi identity).

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- An ***almost*** Dirac structure on P is a subbundle

$$D_P \subset TP \oplus T^*P$$

such that $D_P = D_P^{\perp}$, where, for each $x \in P$,

$$\begin{aligned} D_P^{\perp}(x) = \{ (u_x, \beta_x) \in T_xP \times T_x^*P \mid \\ \langle\!\langle \, (v_x, \alpha_x), (u_x, \beta_x) \, \rangle\!\rangle = \alpha_x(u_x) + \beta_x(v_x) = 0, \\ \text{for all } \ (v_x, \alpha_x) \in D_P(x) \}. \end{aligned}$$

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- An ***integrable Dirac structure*** satisfies

$$\langle \pounds_{X_1} \alpha_2, X_3 \rangle + \langle \pounds_{X_2} \alpha_3, X_1 \rangle + \langle \pounds_{X_3} \alpha_1, X_2 \rangle = 0,$$

for all (X_1, α_1), (X_2, α_2), (X_3, α_3) that take values in D_P.

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where $E(q, v, p) = \langle p, v \rangle - L(q, v)$.

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$$(X, \mathbf{d}E|_{TP}) \in D_P,$$
where $E(q, v, p) = \langle p, v \rangle - L(q, v)$.
- Applications to interconnected systems of multiport networks such as ***electric circuits*** with Dirac constraints as well as ***multibody systems*** with nonholonomic constraints.

- The ***Hamilton–Pontryagin principle*** (originally developed by Livens [1919]) is given by

$$\delta \int_{t_1}^{t_2} \{L(q(t), v(t)) + p(t) \cdot (\dot{q}(t) - v(t))\}\, dt = 0$$

with the fixed endpoints $q(t)$. One can obtain the ***implicit Euler–Lagrange equations:***

$$p = \frac{\partial L}{\partial v}, \quad \dot{q} = v, \quad \dot{p} = \frac{\partial L}{\partial q}.$$

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- Namely, letting G be a Lie group and $\mathfrak{g}$ be a Lie algebra, the canonical Dirac structure on T^*G can be reduced to a Dirac structure on $\mathfrak{g}^*$ by using the ***Lie-Poisson brackets***.

Euler–Poincaré Reduction

- Let $L : TG \to \mathbb{R}$ be a left invariant Lagrangian and $l := L|\mathfrak{g}$ be the reduced Lagrangian, where we employ $TG \cong G \times \mathfrak{g}$.

  The ***reduced constrained variational principle*** is given by

$$\delta \int_{t_1}^{t_2} l(\xi(t))\, dt = 0,$$

where $\xi = g^{-1}\dot{g} \in \mathfrak{g}$ and the variations are given by

$$\delta\xi = \dot{\eta} \pm [\xi, \eta]$$

with the boundary conditions $\eta(t_1) = \eta(t_2) = 0$.

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  with the boundary conditions $\eta(t_1) = \eta(t_2) = 0$.
- It naturally induces ***Euler–Poincaré equations***

$$\frac{d}{dt}\frac{\partial l}{\partial \xi} = \pm \mathrm{ad}^*_\xi \frac{\partial l}{\partial \xi}.$$

Lie-Poisson Variational Principle

- Let H be a left invariant Hamiltonian on T^*G and let $h = H|\mathfrak{g}^*$ be a reduced Hamiltonian, where $T^*G \cong G \times \mathfrak{g}^*$. ***Reduction of Hamilton's phase space principle*** is given by

$$\delta \int_{t_1}^{t_2} \{\langle \mu(t), \xi(t) \rangle - h(\mu(t))\} \; dt = 0,$$

  where $\xi = g^{-1}\dot{g} \in \mathfrak{g}$ and $\mu = T_e^* L_g p \in \mathfrak{g}^*$ and the variation of ξ is given by $\delta\xi = \dot{\eta} + [\xi, \eta]$, with the fixed endpoint boundary conditions $\eta(t_1) = \eta(t_2) = 0$.

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  which are to be equivalent with ***Lie–Poisson equations***.

- This reduction procedure of Lie-Poisson Variational Principle is done on the ***larger space*** $V = \mathfrak{g} \oplus \mathfrak{g}^*$ ***rather than*** $\mathfrak{g}^*$***!***

Our Goals

- How can we ***reduce Hamilton-Pontryagin Principle***, namely, variational principles on the reduced Pontryagin Bundle $(TQ \oplus T^*Q)/G$?

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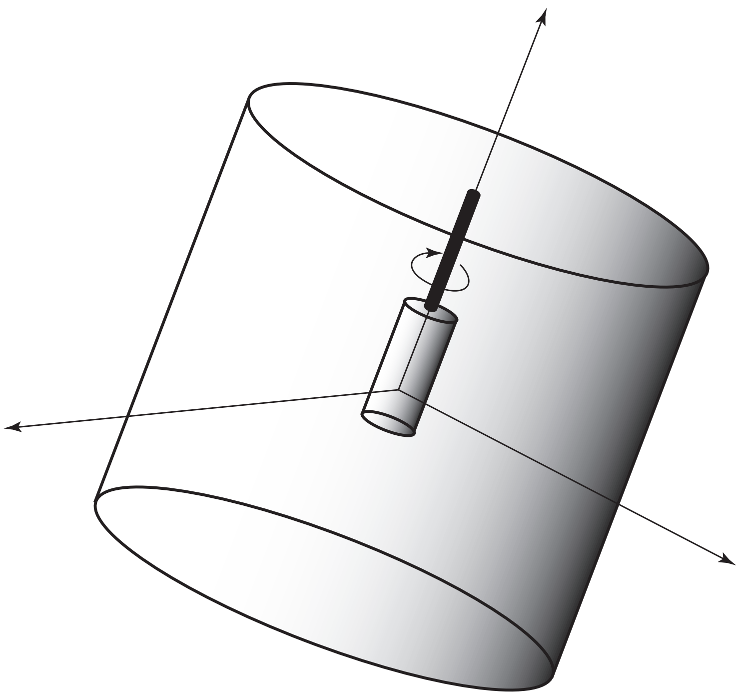
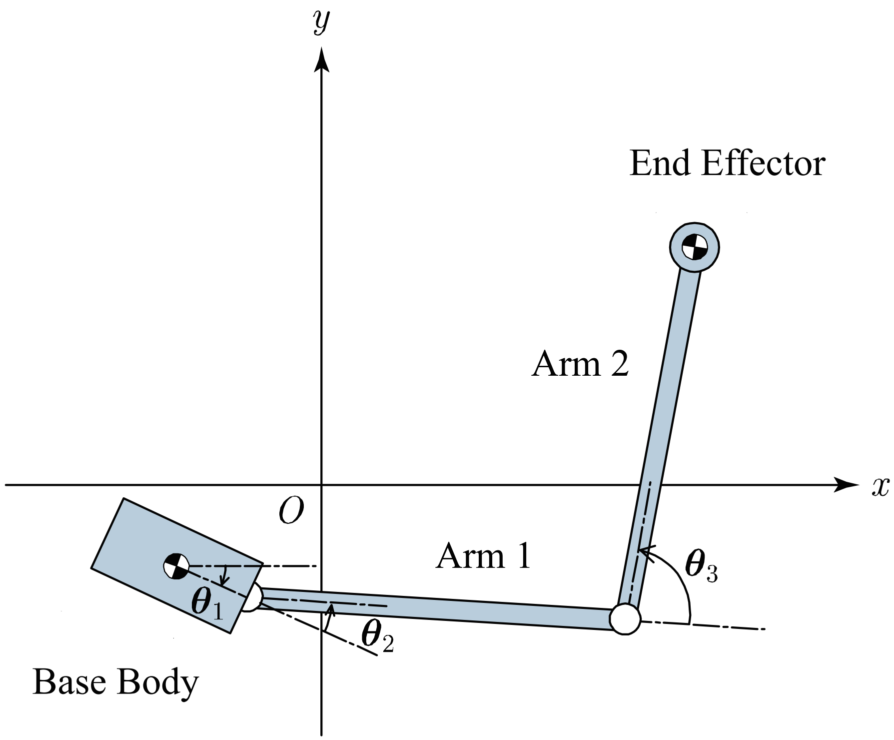
Our goals are to answer these questions !

Examples



Artificial Satellite



Space Robots

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Cotangent Bundle Reduction

- ***Gauged Poisson structures*** associated with Hamiltonian reduction on the cotangent bundle; see Montgomery, Marsden and Ratiu[1984] and Montgomery[1986].

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- Reduced Hamilton's equations on this space are called ***Hamilton Poincaré equations*** and for the case $Q = G$, it leads to ***Lie-Poisson equations*** ([CMPR2003]).

Tangent Bundle Reduction

- ***Lagrangian reduction*** on the tangent bundles: Marsden and Scheurle [1993] developed a Lagrangian analogue of the Hamiltonian reduction on the cotangent bundle by using the so-called ***mechanical connection***.

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Let us first go to see the ***special case*** $Q = G$ ***!***

Hamilton–Pontryagin Principle

- Let $L : TG \longrightarrow \mathbb{R}$ be a left Lagrangian and recall the ***Hamilton–Pontryagin principle*** is given by

$$\delta \int_{t_1}^{t_2} \left\{ L(g(t), v(t)) + p(t) \cdot (\dot{g}(t) - v(t)) \right\} dt = 0.$$

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- ***Reduction of the Hamilton–Pontryagin principle*** may be described by

$$\delta \int_{t_1}^{t_2} \{l(\eta(t)) + \mu(t) \cdot (\xi(t) - \eta(t))\} \; dt = 0$$

with $\delta\xi(t) = \dot{\zeta}(t) + [\xi(t), \zeta(t)]$ and $\zeta(t_1) = \zeta(t_2) = 0.$

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- It follows ***implicit Euler–Poincaré equations***

$$\mu = \frac{\delta l}{\delta \eta}, \quad \xi = \eta, \quad \dot{\mu} = \mathrm{ad}^*_\xi \mu.$$

Invariance of Dirac Structures

- The ***canonical Dirac structure*** on $P = T^*G$ is given by

$$D = \operatorname{graph} \Omega \subset TP \oplus T^*P.$$

In view of the ***trivialized isomorphism***

$$\bar{\lambda} : P = T^*G \to G \times \mathfrak{g}^*,$$

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- Let Φ denote the G–action on $G \times \mathfrak{g}^*$, so that

$$\Phi_h(g, \mu) = (hg, \mu).$$

The Dirac structure $\bar{D}$ is to be ***G–invariant***, since

$$(\Phi_{h^*}X, (\Phi_h^*)^{-1}\alpha) \in \bar{D}$$

holds for all $(X, \alpha) \in \bar{D}$.

Lie-Dirac Reduction

- One can obtain the ***quotient*** of $\bar{D}$ by G as

$$[\bar{D}]_G \cong D/G \subset (TP \oplus T^*P)/G,$$

where $[\bar{D}]_G$ is a Dirac structure on the bundle

$$TP/G \cong \mathfrak{g}^* \times V$$

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over $P/G \cong \mathfrak{g}^*$, which is the ***reduction of the canonical Dirac structure*** D on $P = T^*G$.

- For each $\mu \in \mathfrak{g}^*$, it follows that $[\bar{D}]_G(\mu)$ is given by

$$\begin{aligned}[\bar{D}]_G(\mu) =& \{((\xi, \kappa), (\nu, \xi)) \in V \oplus V^* \mid \\ & \qquad \nu + \kappa = \mathrm{ad}^*_\xi \mu\},\end{aligned}$$

where $V = \mathfrak{g} \oplus \mathfrak{g}^*$.

- Define the ***trivialized generalized energy*** on $G \times V$ by

$$\bar{E}(g, \eta, \mu) = \langle \mu, \eta \rangle - \bar{L}(g, \eta).$$

The quotient of $\mathbf{d}\bar{E}$ is the map

$$[\mathbf{d}\bar{E}]_G : V \longrightarrow V \times \mathfrak{g}^* \times V^*,$$

which is given by, for each $(\eta, \mu) \in V$,

$$[\mathbf{d}\bar{E}]_G(\eta, \mu) = \left(\eta, \mu, 0, \mu - \frac{\partial l}{\partial \eta}, \eta \right).$$

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$$[\mathbf{d}\bar{E}]_G(\eta, \mu) = \left(\eta, \mu, 0, \mu - \frac{\partial l}{\partial \eta}, \eta \right).$$

- Since $\partial \bar{E} / \partial \eta = 0$ naturally induces the ***reduced Legendre transform***

$$\mu = \partial l / \partial \eta \in \mathfrak{g}^*,$$

the ***restriction*** of $[\mathbf{d}\bar{E}]_G$ to $\mathfrak{g}^* \times V$ is given by

$$[\mathbf{d}\bar{E}]_G(\eta, \mu)|_{\mathfrak{g}^* \times V} = (\mu, 0, \eta) \in \mathfrak{g}^* \times V^*.$$

- The ***partial vector field***

$$X : TG \oplus T^*G \to TT^*G,$$

is given by

$$X(g, v, p) = (g, p, \dot{g}, \dot{p}),$$

where $\dot{g}$ and $\dot{p}$ are functions of (g, v, p). Notice that X is left invariant as

$$h \cdot X(g, v, p) = X(hg, T_gL_h \cdot v, T^*_{hg}L_{h^{-1}} \cdot p).$$

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- The ***reduction of the partial vector field*** is given by the quotient

$$[\bar{X}]_G : V \to \mathfrak{g}^* \times V,$$

which is denoted by

$$[\bar{X}]_G(\eta, \mu) = \Big(\mu, \xi, \dot{\mu}\Big) \in \mathfrak{g}^* \times V.$$

Here, note that ξ and $\dot{\mu}$ are functions of (η, μ).

Euler-Poincaré-Dirac Reduction

- The ***reduction of an implicit Lagrangian system*** (L, D, X) is given by a triple

$$(l,\ [\bar{D}]_G,\ [\bar{X}]_G)$$

that satisfies, for each $\eta \in \mathfrak{g}$, the condition

$$([\bar{X}]_G(\eta, \mu), [\mathbf{d}\bar{E}]_G(\eta, \mu)|_{\mathfrak{g}^* \times V}) \in [\bar{D}]_G(\mu),$$

where $\mu = \mathbb{F}l(\eta) \in \mathfrak{g}^*$ holds.

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where $\mu = \mathbb{F}l(\eta) \in \mathfrak{g}^*$ holds.

- This induces ***implicit Euler-Poincaré equations*** on $V = \mathfrak{g} \oplus \mathfrak{g}^*$ as

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- The ***reduction of an implicit Hamiltonian system*** (H, D, X) is given by a triple
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that satisfies, for each $\eta \in \mathfrak{g}$, the condition
$$([\bar{X}]_G, [\mathbf{d}\bar{H}]_G) \in [\bar{D}]_G,$$
where $[\bar{X}]_G = (\mu, \xi(\mu), \dot{\mu})$ and $[\mathbf{d}\bar{H}]_G = (\mu, 0, \partial h/\partial \mu)$.

Lie-Poisson-Dirac Reduction

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$$([\bar{X}]_G, [\mathbf{d}\bar{H}]_G) \in [\bar{D}]_G,$$

where $[\bar{X}]_G = (\mu, \xi(\mu), \dot{\mu})$ and $[\mathbf{d}\bar{H}]_G = (\mu, 0, \partial h/\partial\mu)$.

- This induces ***implicit Lie-Poisson equations*** on $V = \mathfrak{g} \oplus \mathfrak{g}^*$ as

$$\xi = \frac{\partial h}{\partial \mu}, \quad \frac{d\mu}{dt} = \mathrm{ad}^*_\xi \mu,$$

which is consistent with the results in [CMPR2003].

Principal Bundle with a Lie Group

- Let $\pi : Q \to Q/G$ be a ***principal bundle*** with a Lie group G acting freely and properly on Q and choose a ***principal connection*** on Q as $A : TQ \to \mathfrak{g}$.

Principal Bundle with a Lie Group

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- The group G acts on curves $(q(t), v(t), p(t)) \in TQ \oplus T^*Q$ as

$$
\begin{aligned}
& h \cdot (q(t), v(t), p(t)) \\
& \qquad = (hq(t), T_{q(t)}L_h \cdot v(t), T^*_{hq(t)}L_{h^{-1}} \cdot p(t)).
\end{aligned}
$$

Principal Bundle with a Lie Group

- Let $\pi : Q \to Q/G$ be a ***principal bundle*** with a Lie group G acting freely and properly on Q and choose a ***principal connection*** on Q as $A : TQ \to \mathfrak{g}$.
- The group G acts on curves $(q(t), v(t), p(t)) \in TQ \oplus T^*Q$ as

$$h \cdot (q(t), v(t), p(t)) = (hq(t), T_{q(t)}L_h \cdot v(t), T^*_{hq(t)}L_{h^{-1}} \cdot p(t)).$$

- The curves in $(TQ \oplus T^*Q)/G$ are isomorphic to the curves in $T(Q/G) \oplus T^*(Q/G) \oplus \tilde{V}$, namely,

$$[q(t), v(t), p(t)]_G \cong (x(t), u(t), y(t)) \oplus (\bar{\eta}(t), \bar{\mu}(t)),$$

where $\tilde{V} = \tilde{\mathfrak{g}} \oplus \tilde{\mathfrak{g}}^*$ and where we have employed

$$TQ/G \cong T(Q/G) \oplus \tilde{\mathfrak{g}} \quad \text{and} \quad T^*Q/G \cong T^*(Q/G) \oplus \tilde{\mathfrak{g}}^*.$$

Reduced H-P Variational Principle

- The stationarity for the ***reduced Hamilton-Pontryagin principle*** is given by

$$\delta \int_{t_0}^{t_1} \{ l(x(t), u(t), \bar{\eta}(t)) + \langle y(t), \dot{x}(t) - u(t) \rangle + \left\langle \bar{\mu}(t), \bar{\xi}(t) - \bar{\eta}(t) \right\rangle \} \, dt = 0,$$

with arbitrary variations δx, δu, $\delta \bar{\eta}$, δy, $\delta \bar{\mu}$, $\bar{\zeta}$ and the ***covariant variation***

$$\delta^A \bar{\xi} = \frac{D[q, \zeta]_G}{Dt} + [q, [\xi, \zeta]]_G + \widetilde{B}(\delta x, \dot{x}),$$

together with the boundary conditions

$$\delta x(t_0) = \delta x(t_1) = 0 \;\; \text{and} \;\; \zeta(t_0) = \zeta(t_1) = 0.$$

Implicit Lagrange-Poincaré Equations

- Corresponding to the horizontal variations $\delta x, \delta u$ and δy, it follows ***horizontal implicit Lagrange-Poincaré equations*** as

$$\frac{Dy}{Dt} = \frac{\partial l}{\partial x} - \left\langle \bar{\mu}, \widetilde{B}(\dot{x}, \cdot) \right\rangle, \qquad \dot{x} = u, \qquad y = \frac{\partial l}{\partial u}.$$

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- Corresponding to the vertical variations $\delta\bar{\eta}$, $\delta\bar{\mu}$ and $\bar{\zeta}$, it follows ***vertical implicit Lagrange-Poincaré equations*** as

$$\frac{D\bar{\mu}}{Dt} = \mathrm{ad}^*_{\bar{\xi}}\bar{\mu}, \qquad \bar{\xi} = \bar{\eta}, \qquad \bar{\mu} = \frac{\partial l}{\partial \bar{\eta}}.$$

Trivialized Expressions

- Let us ***pull back the G-principal bundle*** $\pi : Q \to Q/G$ by $\pi_{Q/G} : T^*(Q/G) \to Q/G$ to obtain the G-principal bundle

$$\widetilde{Q}^* = \Big\{ (q, \alpha_{[q]}) \mid \pi_{Q/G}(\alpha_{[q]}) = \pi(q) = [q],\ q \in Q, \\ \alpha_{[q]} \in T^*_{[q]}(Q/G) \Big\}.$$

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- Define the space $\widetilde{Q}^* \times \mathfrak{g}^*$, which is isomorphic to T^*Q by

$$\bar{\lambda} : T^*Q \to \widetilde{Q}^* \times \mathfrak{g}^*; \quad \alpha_q \mapsto (q, (\alpha_q)_q^{h^*}, \rho = \mathbf{J}(\alpha_q)),$$

The quotient map

$$[\bar{\lambda}]_G : (T^*Q)/G \to (\widetilde{Q}^* \times \mathfrak{g}^*)/G \cong T^*(Q/G) \oplus \tilde{\mathfrak{g}}^*$$

is a ***left trivialization*** as

$$[\alpha_q]_G \mapsto \big((\alpha_q)_q^{h^*}, [q, \mathbf{J}(\alpha_q)]_G \big) = \big((\alpha_q)_q^{h^*}, [q, T_e^* L_g \cdot \alpha_q]_G \big).$$

- Let us ***pull back the G-principal bundle*** $\pi : Q \to Q/G$ by the tangent bundle projection $\tau_{Q/G} : T(Q/G) \to Q/G$ to obtain the G-principal bundle

$$\widetilde{Q} = \left\{ (q, u_{[q]}) \mid \tau_{Q/G}(u_{[q]}) = \pi(q) = [q],\ q \in Q, \right.$$
$$\left. u_{[q]} \in T_{[q]}(Q/G) \right\}.$$

- Let us ***pull back the G-principal bundle*** $\pi : Q \to Q/G$ by the tangent bundle projection $\tau_{Q/G} : T(Q/G) \to Q/G$ to obtain the G-principal bundle

$$\widetilde{Q} = \left\{ (q, u_{[q]}) \mid \tau_{Q/G}(u_{[q]}) = \pi(q) = [q],\ q \in Q, \right. \\ \left. u_{[q]} \in T_{[q]}(Q/G) \right\} .$$

- Define the space $\widetilde{Q} \times \mathfrak{g}$, which is diffeomorphic to TQ by

$$\lambda : TQ \to \widetilde{Q} \times \mathfrak{g}; \quad v_q \mapsto (q, T\pi(v_q), \xi = A(v_q)).$$

The quotient

$$[\lambda]_G : (TQ)/G \to (\widetilde{Q} \times \mathfrak{g})/G \cong T(Q/G) \oplus \tilde{\mathfrak{g}}$$

is a ***left trivialization*** as

$$[v_q]_G \mapsto (T\pi(v_q), [q, A(v_q)]_G) \\ = \left(T\pi(v_q), \left[q, T_g L_{g^{-1}} \cdot v_q \right]_G \right) .$$

Dirac Structures on $\widetilde{Q}^* \times \mathfrak{g}^* \cong T^*Q$

- Given the canonical Dirac structure D on T^*Q, one can define a Dirac structure $\bar{D}$ on $\widetilde{Q}^* \times \mathfrak{g}^*$, in view of $T^*Q \cong \tilde{Q}^* \times \mathfrak{g}^*$, which is given by

$$\begin{aligned}\bar{D}(x,g,y,\rho) = \{ & ((\dot{x},\dot{g},\dot{y},\dot{\rho}),(\kappa,\nu,v,\eta)) \mid \\ & \langle \kappa, \delta x\rangle + \langle \nu, \delta g\rangle + \langle \delta y, v\rangle + \langle \delta\rho, \eta\rangle \\ & \quad = \Omega(x,g,y,\rho)((\dot{x},\dot{g},\dot{y},\dot{\rho}),(\delta x,\delta g,\delta y,\delta\rho)) \\ & \qquad \text{for all } \ (\delta x,\delta g,\delta y,\delta\rho) \in T_{(x,g,y,\rho)}(\widetilde{Q}^* \times \mathfrak{g}^*)\}.\end{aligned}$$

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- In the above, we have

$$\Omega = \gamma^* \Omega_{T^*(Q/G)} - \tilde{\pi}_Q^* B_\rho + \omega$$

is a symplectic form on $\widetilde{Q}^* \times \mathfrak{g}^*$.

Invariance of Dirac Structures

- The Dirac structure $\bar{D}$ on $\widetilde{Q}^* \times \mathfrak{g}^*$ is *G–invariant* as

$$(\Phi_{h^*} X, (\Phi_h^*)^{-1}\alpha) \in \bar{D} \quad \text{for all} \quad (X, \alpha) \in \bar{D},$$

which follows

$$\bar{D}(e, g^{-1}g, y, g^{-1}\rho) = \bar{D}(x, g, y, \rho).$$

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which follows

$$\bar{D}(e, g^{-1}g, y, g^{-1}\rho) = \bar{D}(x, g, y, \rho).$$

- By ***taking quotients*** by G, it leads to a ***reduced Dirac structure*** on the bundle

$$TT^*Q/G \cong \tilde{\mathfrak{g}}^* \times (TT^*(Q/G) \oplus \widetilde{V})$$

over $T^*(Q/G) \oplus \tilde{\mathfrak{g}}^*$ as

$$\begin{aligned}[\bar{D}]_G \subset \tilde{\mathfrak{g}}^* \times (TT^*(Q/G) \oplus \widetilde{V} \oplus T^*T^*(Q/G) \oplus \widetilde{V}^*) \\ \cong (TT^*Q/G) \oplus (T^*T^*Q/G).\end{aligned}$$

Gauged Dirac Structures

- The ***reduced Dirac structure*** $[\bar{D}]_G$ is given by, for each $(x, y, \bar{\mu}) \in T^*(Q/G) \oplus \tilde{\mathfrak{g}}^*$.

$$[\bar{D}]_G(x, y, \bar{\mu}) = [\bar{D}]_G^{\mathrm{Hor}}(x, y) \oplus [\bar{D}]_G^{\mathrm{Ver}}(\bar{\mu}),$$

where we shall call $[\bar{D}]_G = [\bar{D}]_G^{\mathrm{Hor}} \oplus [\bar{D}]_G^{\mathrm{Ver}}$ a ***gauged Dirac structure***.

Gauged Dirac Structures

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  where we shall call $[\bar{D}]_G = [\bar{D}]_G^{\mathrm{Hor}} \oplus [\bar{D}]_G^{\mathrm{Ver}}$ a ***gauged Dirac structure***.

- In the above, $[D]_G^{\mathrm{Hor}}$ is a ***horizontal Dirac structure*** on the bundle $\tilde{\mathfrak{g}}^* \times TT^*(Q/G)$ over $T^*(Q/G)$, which is given by

$$[\bar{D}]_G^{\mathrm{Hor}}(x, y) = \left\{ ((\dot{x}, \dot{y}), (\beta, \dot{x})) \mid \dot{y} + \beta = -\widetilde{B}_{\bar{\mu}}(\dot{x}, \cdot) \right\},$$

  while $[D]_G^{\mathrm{Ver}}$ is a ***vertical Dirac structure*** on the bundle $\tilde{\mathfrak{g}}^* \times \widetilde{V}$ over $\tilde{\mathfrak{g}}^*$ given by

$$[\bar{D}]_G^{\mathrm{Ver}}(\bar{\mu}) = \left\{ \left((\bar{\xi}, \dot{\bar{\mu}}), (\bar{\nu}, \bar{\xi}) \right) \mid \dot{\bar{\mu}} + \bar{\nu} = \mathrm{ad}^*_{\xi} \bar{\mu} \right\}.$$

Differential of the Generalized Energy

- Associated with the ***generalized energy*** on $TQ \oplus T^*Q$
$$E(q, v, p) = \langle p, v \rangle - L(q, v),$$
the ***quotient*** of $\mathbf{d}\bar{E}$ is given by
$$[\mathbf{d}\bar{E}]_G : T(Q/G) \oplus T^*(Q/G) \oplus \tilde{V} \\ \to \tilde{\mathfrak{g}}^* \times (T^*T(Q/G) \oplus \tilde{V} \oplus T^*T^*(Q/G) \oplus \tilde{V}^*).$$

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$$\to \tilde{\mathfrak{g}}^* \times (T^*T(Q/G) \oplus \tilde{V} \oplus T^*T^*(Q/G) \oplus \tilde{V}^*).$$

- The ***restriction*** to $\tilde{\mathfrak{g}}^* \times (TT^*(Q/G) \oplus \widetilde{V})$ is given as

$$[\mathbf{d}\bar{E}]_G|_{\tilde{\mathfrak{g}}^* \times (TT^*(Q/G) \oplus \widetilde{V})} = [\mathbf{d}\bar{E}]_G^{\mathrm{Hor}}|_{\tilde{\mathfrak{g}}^* \times TT^*(Q/G)} \oplus [\mathbf{d}\bar{E}]_G|_{\tilde{\mathfrak{g}}^* \times \widetilde{V}},$$

where

$$[\mathbf{d}\bar{E}]_G^{\mathrm{Hor}}|_{\tilde{\mathfrak{g}}^* \times TT^*(Q/G)} = \left(x, y, -\frac{\partial l}{\partial x}, u \right),$$

and

$$[\mathbf{d}\bar{E}]_G|_{\tilde{\mathfrak{g}}^* \times \widetilde{V}} = (\bar{\mu}, 0, \bar{\eta}) \,.$$

The Reduced Legendre Transform

- The ***reduced Legendre transform*** may be decomposed as

$$\mathbb{F}l = \mathbb{F}l^{\mathrm{Hor}} \oplus \mathbb{F}l^{\mathrm{Ver}} : T(Q/G) \oplus \tilde{\mathfrak{g}} \to T^*(Q/G) \oplus \tilde{\mathfrak{g}}^*.$$

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- The ***reduced Legendre transform*** may be decomposed as

$$\mathbb{F}l = \mathbb{F}l^{\mathrm{Hor}} \oplus \mathbb{F}l^{\mathrm{Ver}} : T(Q/G) \oplus \tilde{\mathfrak{g}} \to T^*(Q/G) \oplus \tilde{\mathfrak{g}}^*.$$

- In the above, the ***horizontal Legendre transformation***

$$\mathbb{F}l^{\mathrm{Hor}} : T(Q/G) \to T^*(Q/G)$$

is given by

$$(x, u) \mapsto \left(x, y = \frac{\partial l}{\partial u} \right),$$

while the ***vertical Legendre transformation***

$$\mathbb{F}l^{\mathrm{Ver}} : \tilde{\mathfrak{g}} \to \tilde{\mathfrak{g}}^*,$$

is given by

$$\bar{\eta} \mapsto \bar{\mu} = \frac{\partial l}{\partial \bar{\eta}} \in \tilde{\mathfrak{g}}^*.$$

Reduction of the Partial Vector Field

- Let $\bar{X}$ be the ***trivialized partial vector field*** associated to $X : TQ \oplus T^*Q \to TT^*Q$. Then, the reduced partial vector field $[\bar{X}]_G$ can be represented by

$$[\bar{X}]_G(x, u, y, \bar{\eta}, \bar{\mu}) = [\bar{X}]_G^{\mathrm{Hor}}(x, u, y) \oplus [\bar{X}]_G^{\mathrm{Ver}}(\bar{\eta}, \bar{\mu}).$$

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- In the above, the ***horizontal partial vector field***

$$[\bar{X}]_G^{\mathrm{Hor}} : T(Q/G) \oplus T^*(Q/G) \to \tilde{\mathfrak{g}}^* \times TT^*(Q/G)$$

is given by

$$[\bar{X}]_G^{\mathrm{Hor}}(x, u, y) = (x, y, \dot{x}, \dot{y})$$

and the ***vertical partial vector field***

$$[\bar{X}]_G^{\mathrm{Ver}} : \widetilde{V} \to \tilde{\mathfrak{g}}^* \times \widetilde{V}$$

is given by

$$[\bar{X}]_G^{\mathrm{Ver}}(\bar{\eta}, \bar{\mu}) = (\bar{\mu}, \bar{\xi}, \dot{\bar{\mu}}) \in \tilde{\mathfrak{g}}^* \times \widetilde{V}.$$

Lagrange-Poincaré-Dirac Reduction

- The ***reduction of a standard implicit Lagrangian system*** (L, D, X) that satisfies

$$(X, \mathbf{d}E|_{TT^*Q}) \in D$$

is given by a triple

$$(l, [\bar{D}]_G, [\bar{X}]_G)$$

that satisfies

$$\big([\bar{X}]_G, [\mathbf{d}\bar{E}]_G|_{\tilde{\mathfrak{g}}^* \times (TT^*(Q/G) \oplus \widetilde{V})}\big) \in [\bar{D}]_G.$$

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that satisfies

$$\left([\bar{X}]_G, [\mathbf{d}\bar{E}]_G|_{\tilde{\mathfrak{g}}^* \times (TT^*(Q/G) \oplus \widetilde{V})}\right) \in [\bar{D}]_G.$$

- The reduced implicit Lagrangian system can be decomposed into the ***horizontal and vertical*** parts such that

$$(l, [\bar{D}]_G, [\bar{X}]_G) = (l, [\bar{D}]_G^{\mathrm{Hor}}, [\bar{X}]_G^{\mathrm{Hor}}) \oplus (l, [\bar{D}]_G^{\mathrm{Ver}}, [\bar{X}]_G^{\mathrm{Ver}}).$$

Horizontal Implicit Lagrange-Poincaré Equations

- The ***horizontal implicit Lagrangian system*** is a triple

$$(l, [\bar{D}]_G^{\mathrm{Hor}}, [\bar{X}]_G^{\mathrm{Hor}})$$

that satisfies

$$([\bar{X}]_G^{\mathrm{Hor}}, [\mathbf{d}\bar{E}]_{G|_{\tilde{\mathfrak{g}}^* \times TT^*(Q/G)}}^{\mathrm{Hor}}) \in [\bar{D}]_G^{\mathrm{Hor}}$$

together with the horizontal Legendre transformation

$$\mathbb{F}l^{\mathrm{Hor}} : T(Q/G) \to T^*(Q/G).$$

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together with the horizontal Legendre transformation

$$\mathbb{F}l^{\mathrm{Hor}} : T(Q/G) \to T^*(Q/G).$$

- This induces ***horizontal implicit Lagrange-Poincaré equations***:

$$\frac{Dy}{Dt} = \frac{\partial l}{\partial x} - \left\langle \bar{\mu}, \widetilde{B}(\dot{x}, \cdot) \right\rangle, \quad \frac{dx}{dt} = u, \quad y = \frac{\partial l}{\partial u}.$$

Vertical Implicit Lagrange-Poincaré Equations

- The ***vertical implicit Lagrangian system*** is a triple

$$(l, [\bar{D}]_G^{\mathrm{Ver}}, [\bar{X}]_G^{\mathrm{Ver}})$$

that satisfies

$$([\bar{X}]_G^{\mathrm{Ver}}, [\mathbf{d}\bar{E}]_G^{\mathrm{Ver}}|_{\tilde{\mathfrak{g}}^* \times \widetilde{V}}) \in [\bar{D}]_G^{\mathrm{Ver}}$$

together with the vertical Legendre transformation

$$\mathbb{F}l^{\mathrm{Ver}} : \tilde{\mathfrak{g}} \to \tilde{\mathfrak{g}}^*.$$

Vertical Implicit Lagrange-Poincaré Equations

- The ***vertical implicit Lagrangian system*** is a triple

$$(l, [\bar{D}]_G^{\mathrm{Ver}}, [\bar{X}]_G^{\mathrm{Ver}})$$

that satisfies

$$([\bar{X}]_G^{\mathrm{Ver}}, [\mathbf{d}\bar{E}]_G^{\mathrm{Ver}}|_{\tilde{\mathfrak{g}}^* \times \widetilde{V}}) \in [\bar{D}]_G^{\mathrm{Ver}}$$

together with the vertical Legendre transformation

$$\mathbb{F}l^{\mathrm{Ver}} : \tilde{\mathfrak{g}} \to \tilde{\mathfrak{g}}^*.$$

- This induces the ***vertical implicit Lagrange-Poincaré equations***:

$$\frac{D\bar{\mu}}{Dt} = \mathrm{ad}^*_{\bar{\xi}}\bar{\mu}, \quad \bar{\xi} = \bar{\eta}, \quad \bar{\mu} = \frac{\partial l}{\partial \bar{\eta}}.$$

Hamilton-Poincaré-Dirac Reduction

- Let (H, D, X) be a standard implicit Hamiltonian system and let $h : T^*(Q/G) \oplus \tilde{\mathfrak{g}}^* \longrightarrow \mathbb{R}$ be the reduced Hamiltonian. Then, the ***reduced implicit Hamiltonian system*** of (H, D, X) is a triple

$$(h, [\bar{D}]_G, [\bar{X}]_G)$$

that satisfies the condition, for each $(x, y, \bar{\mu}) \in T^*(Q/G) \oplus \tilde{\mathfrak{g}}^*$,

$$([\bar{X}]_G\,(x, y, \bar{\mu})\,, [\mathbf{d}\bar{H}]_G(x, y, \bar{\mu})) \in [\bar{D}]_G(x, y, \bar{\mu}).$$

Hamilton-Poincaré-Dirac Reduction

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$$\left([\bar{X}]_G\,(x, y, \bar{\mu})\,, [\mathbf{d}\bar{H}]_G(x, y, \bar{\mu})\right) \in [\bar{D}]_G(x, y, \bar{\mu}).$$

- The reduced implicit Hamiltonian system is decomposed into the two parts, namely, ***horizontal*** and ***vertical implicit Hamiltonian systems*** such that
$$(h, [\bar{D}]_G, [\bar{X}]_G) = (h, [\bar{D}]_G^{\mathrm{Hor}}, [\bar{X}]_G^{\mathrm{Hor}}) \oplus (h, [\bar{D}]_G^{\mathrm{Ver}}, [\bar{X}]_G^{\mathrm{Ver}}).$$

- In the above, $(h, [\bar{D}]_G^{\mathrm{Hor}}, [\bar{X}]_G^{\mathrm{Hor}})$ is the horizontal implicit Hamiltonian system that satisfies, for $(x, y) \in T^*(Q/G)$,

$$\left([\bar{X}]_G^{\mathrm{Hor}}(x,y), [\mathbf{d}\bar{H}]_G^{\mathrm{Hor}}(x,y)\right) \in [\bar{D}]_G^{\mathrm{Hor}}(x,y),$$

which induces ***horizontal implicit Hamilton-Poincaré equations***:

$$\frac{Dy}{Dt} = -\frac{\partial h}{\delta x} - \left\langle \bar{\mu}, \widetilde{B}(\dot{x}, \cdot) \right\rangle, \quad \frac{dx}{dt} = \frac{\partial h}{\partial y}.$$

- In the above, $(h, [\bar{D}]_G^{\mathrm{Hor}}, [\bar{X}]_G^{\mathrm{Hor}})$ is the horizontal implicit Hamiltonian system that satisfies, for $(x, y) \in T^*(Q/G)$,

$$([\bar{X}]_G^{\mathrm{Hor}}(x, y), [\mathbf{d}\bar{H}]_G^{\mathrm{Hor}}(x, y)) \in [\bar{D}]_G^{\mathrm{Hor}}(x, y),$$

which induces ***horizontal implicit Hamilton-Poincaré equations***:

$$\frac{Dy}{Dt} = -\frac{\partial h}{\delta x} - \left\langle \bar{\mu}, \widetilde{B}(\dot{x}, \cdot) \right\rangle, \quad \frac{dx}{dt} = \frac{\partial h}{\partial y}.$$

- On the other hand, $(h, [\bar{D}]_G^{\mathrm{Ver}}, [\bar{X}]_G^{\mathrm{Ver}})$ is the vertical implicit Hamiltonian system that satisfies, for $\bar{\mu} \in \tilde{\mathfrak{g}}^*$,

$$([\bar{X}]_G^{\mathrm{Ver}}(\bar{\mu}), [\mathbf{d}\bar{H}]_G^{\mathrm{Ver}}(\bar{\mu})) \in [\bar{D}]_G^{\mathrm{Ver}}(\bar{\mu}),$$

which induces ***vertical implicit Hamilton-Poincaré equations***:

$$\frac{D\bar{\mu}}{Dt} = \mathrm{ad}_{\bar{\xi}}^* \bar{\mu}, \quad \xi = \frac{\partial h}{\partial \bar{\mu}}.$$

Summary

- We have shown a reduction procedure for the Hamilton-Pontryagin principle, which yields ***horizontal*** and ***vertical implicit Lagrange-Poincaré equations*** as the reduced implicit Euler-Lagrange equations.
- Using a chosen principal connection, we have developed a reduction procedure for the canonical Dirac structure on the cotangent bundle, which we call ***Dirac cotangent bundle reduction***. It induces a ***gauged Dirac structure***, which is the direct sum of horizontal and vertical Dirac structures.
- We have constructed ***Lagrange-Poincaré-Dirac reduction*** that induces horizontal and vertical implicit Lagrange-Poincaré equations as well as ***Hamilton-Poincaré-Dirac reduction*** that yields horizontal and vertical implicit Hamilton-Poincaré equations.

Current and Future Works

- A general class of ***Dirac anchored vector bundles*** and its associated reduction (with Cendra, Marsden and Tudor).
- Dirac cotangent bundle reduction for ***nonholonomic mechanical systems with symmetry*** together with variational structures.
- Construction of ***Dirac structures for Field theory***; to bridge with multisymplectic structures and Stokes-Dirac structures.
- Construction of Dirac structures and implicit Lagrangian systems for ***time dependent systems***, which might include the stochastic systems.
- Reduction for ***Implicit Controlled Lagrangian systems***
- ***Dirac integrators*** for constrained mechanical systems.