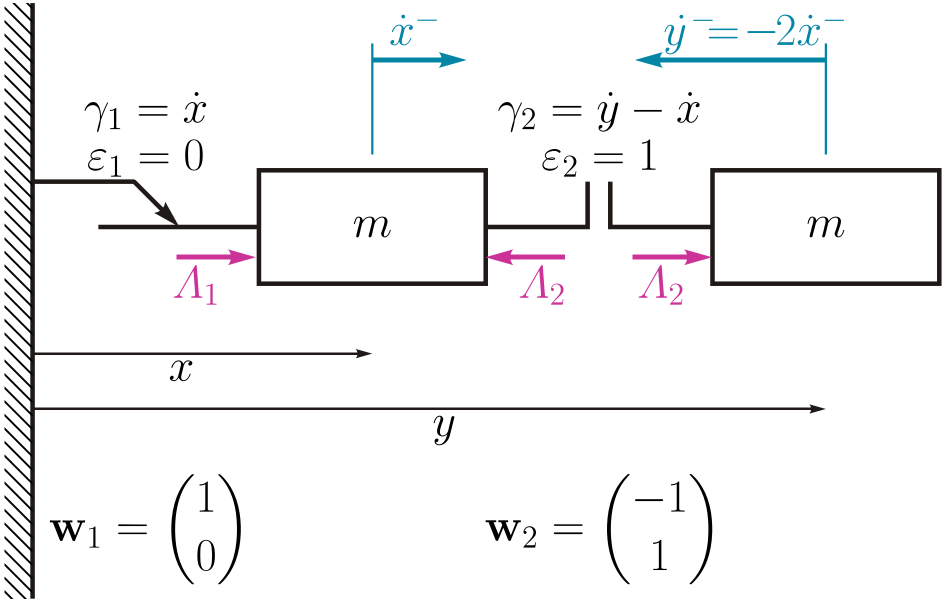
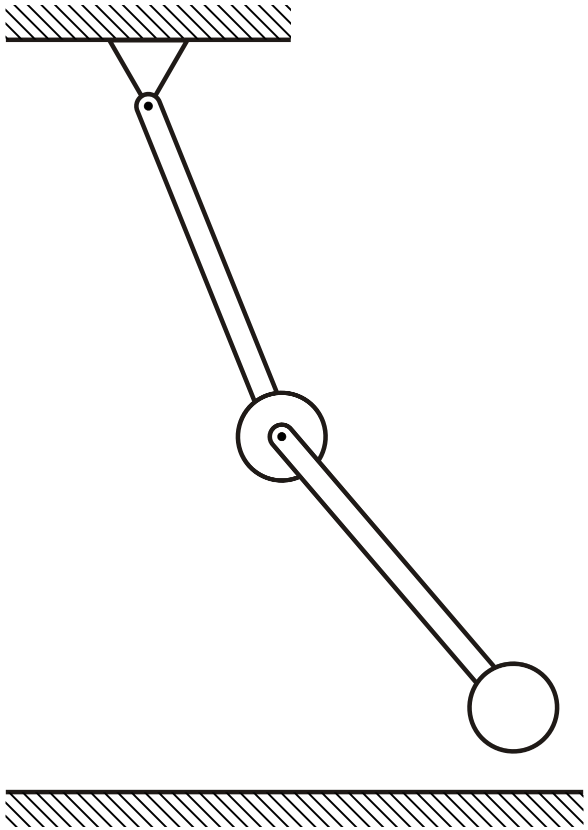
Discussion of the Standard Inequality Impact Law of Newton Type

Ch. Glocker





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I. The Normal Cone

- Let $\mathcal{L}$ be a linear space, — velocities

  $\mathcal{L}^*$ its dual and — forces

  $\mathcal{C} \subset \mathcal{L}^*$ a convex set. — force reservoir

- Normal cone $\mathcal{N}_{\mathcal{C}}(\vec{x}) \subset \mathcal{L}$ to $\mathcal{C}$ at $\vec{x} \in \mathcal{C}$:

$$\mathcal{N}_{\mathcal{C}}(\vec{x}) = \{\vec{y} \mid \vec{y}^{\,T}(\vec{x}^* - \vec{x}) \leq 0 \quad \forall \vec{x}^* \in \mathcal{C}\}$$

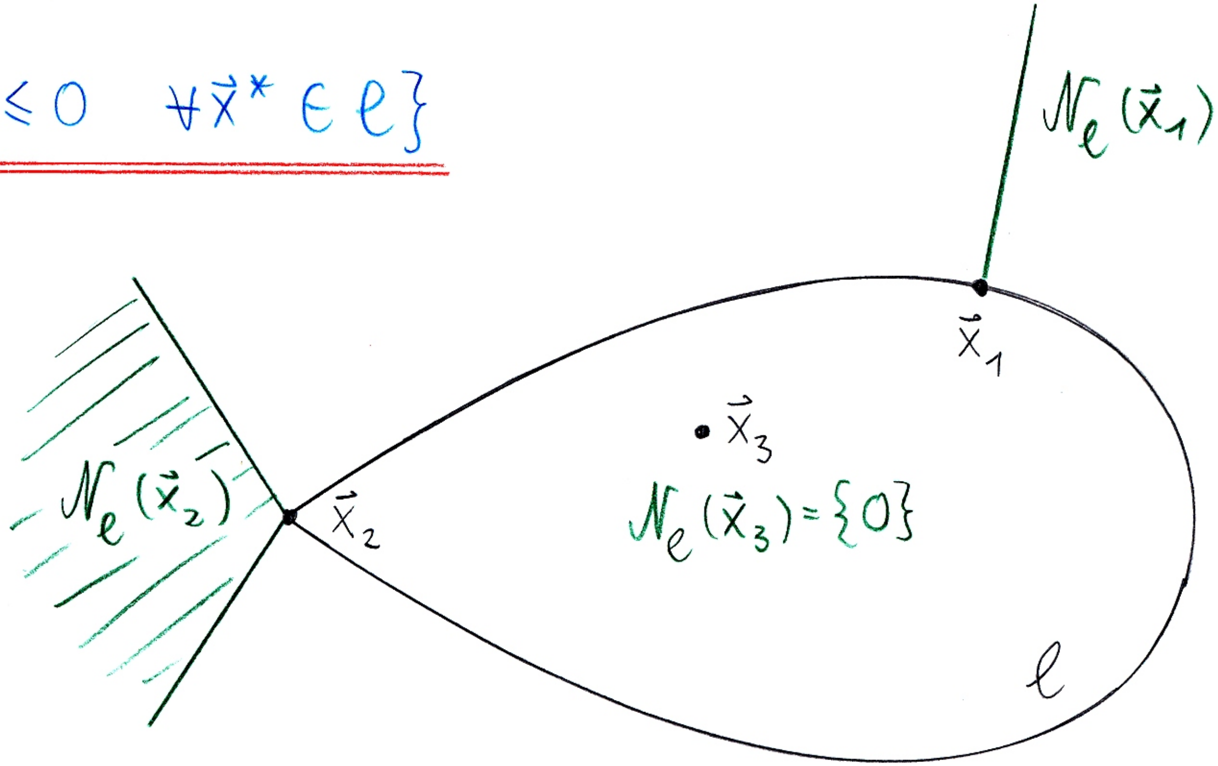
Note:

If $0 \in \mathcal{C}$,

then $\vec{y}^{\,T}\vec{x} \geq 0$

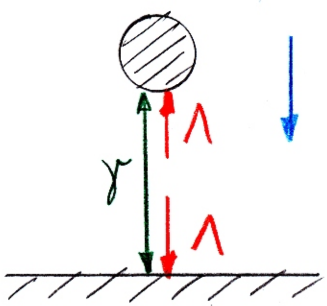
$\forall \vec{y} \in \mathcal{N}_{\mathcal{C}}(\vec{x})$

"dissipativity"



II. Impact Laws

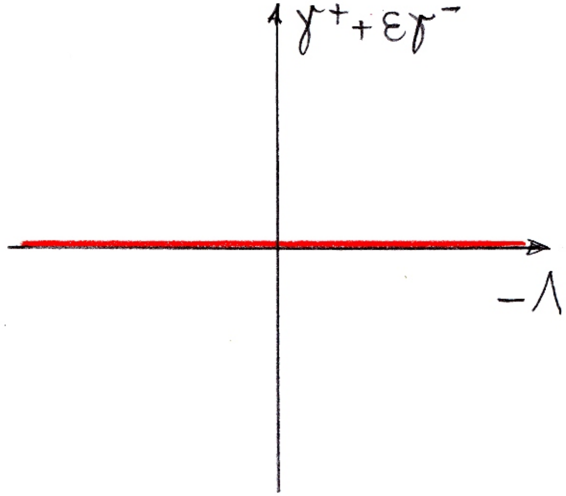
1. Newton's law for collisions

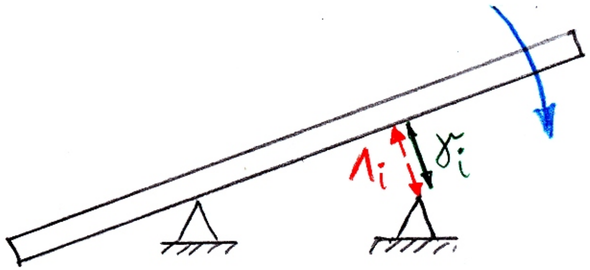


$\gamma^+ = -\varepsilon \gamma^-$

(Λ arbitrary)

$(\gamma^+ + \varepsilon \gamma^-) \in \mathcal{N}_{\mathbb{R}}(-\Lambda)$

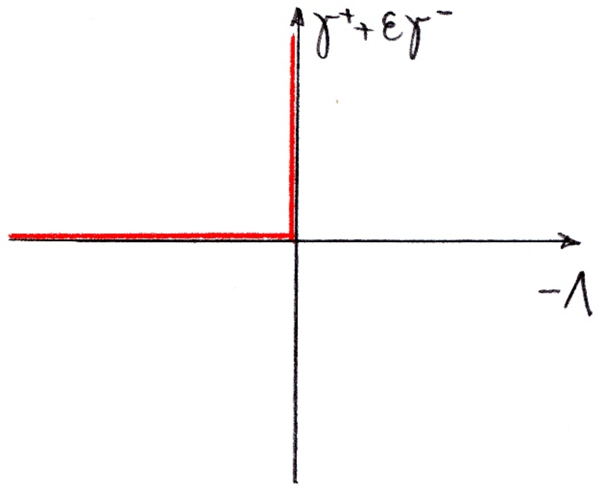


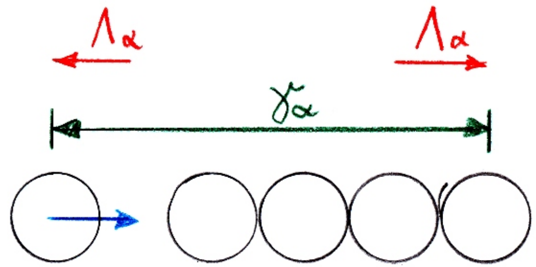


$\Lambda_i > 0 \Rightarrow \gamma_i^+ = -\varepsilon_i \gamma_i^-$

$\Lambda_i = 0 \Rightarrow \gamma_i^+ \geq -\varepsilon_i \gamma_i^-$

$\underline{\underline{(\gamma_i^+ + \varepsilon_i \gamma_i^-) \in \mathcal{N}_{\mathbb{R}^-}(-\Lambda_i)}}$





$\Lambda_i > 0 \Rightarrow \gamma_i^+ = -\sum_j \varepsilon_{ij} \gamma_j^-$

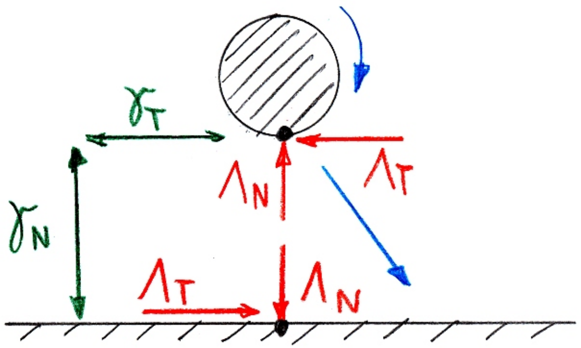
$\Lambda_i = 0 \Rightarrow \gamma_i^+ \geq -\sum_j \varepsilon_{ij} \gamma_j^-$

$(\gamma_\alpha^+ + \varepsilon_\alpha \gamma_\alpha^-) \in \mathcal{N}_{\mathbb{R}^-}(-\Lambda_\alpha)$

"abstract" contact α between first and last ball

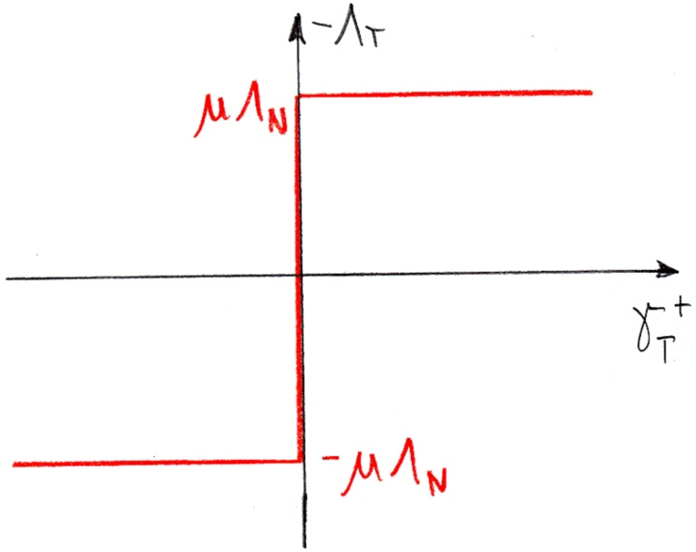
2. Coulomb type friction (1-dim)

- standard friction

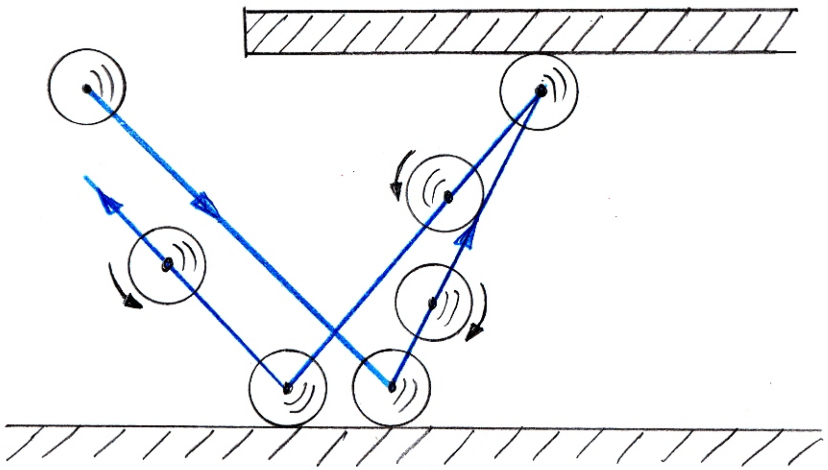


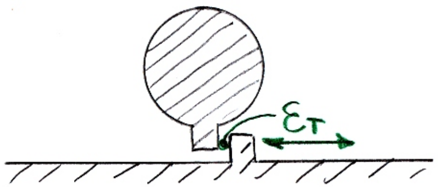
$$\Lambda_T = +\mu\Lambda_N \Rightarrow \gamma_T^+ \leq 0$$
$$|\Lambda_T| < \mu\Lambda_N \Rightarrow \gamma_T^+ = 0$$
$$\Lambda_T = -\mu\Lambda_N \Rightarrow \gamma_T^+ \geq 0$$

$$\underline{\underline{-\Lambda_T \in \mu\Lambda_N \cdot \mathrm{Sgn}(\gamma_T^+)}}$$



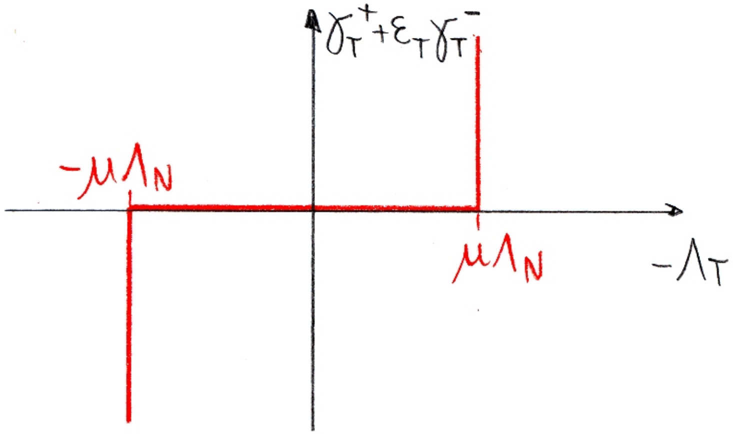
- with tangential restitution





$$-\Lambda_T \in \mu\Lambda_N \cdot \mathrm{Sgn}(\gamma_T^+ + \varepsilon_T\gamma_T^-)$$

$$\Leftrightarrow \underline{\underline{(\gamma_T^+ + \varepsilon_T\gamma_T^-) \in \mathcal{N}_{[-1,1]\mu\Lambda_N}(-\Lambda_T)}}$$

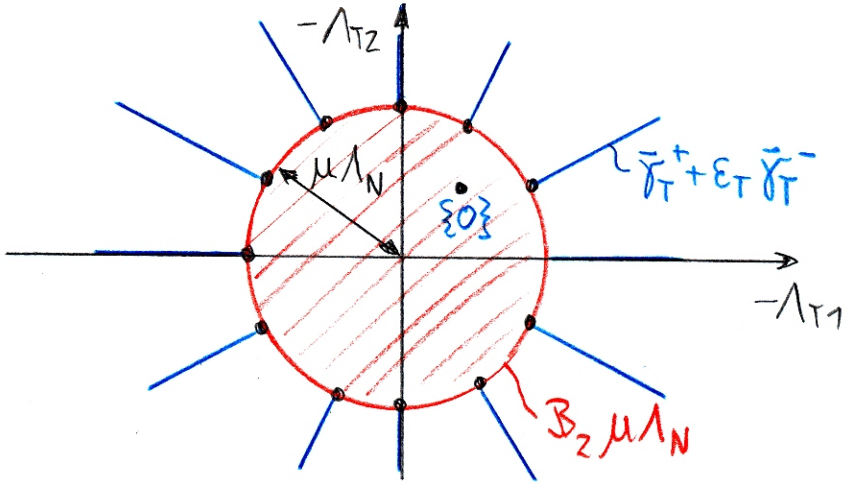


3. Coulomb type friction (2-dim)

$$(\vec{\gamma}_T^+ + \varepsilon_T \vec{\gamma}_T^-) \in \mathcal{N}_{B_2 \mu \Lambda_N}(-\vec{\Lambda}_T)$$

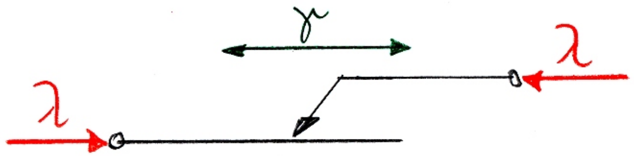
with: $\vec{\gamma}_T = (\gamma_{T1}, \gamma_{T2})^T$; $\vec{\Lambda}_T = (\Lambda_{T1}, \Lambda_{T2})^T$

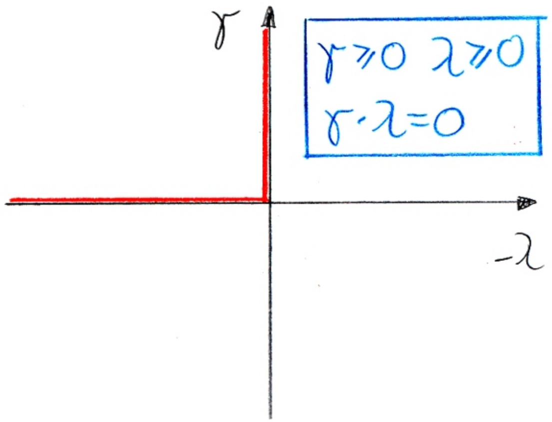
B_2 unit disk in $\mathbb{R}^2$



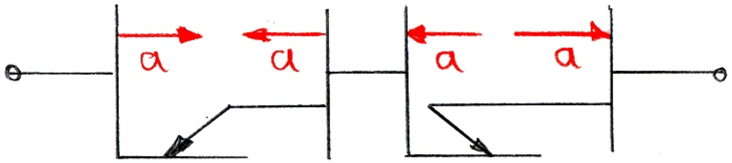
4. Sprag Clutches (unilateral kinematic constraint)

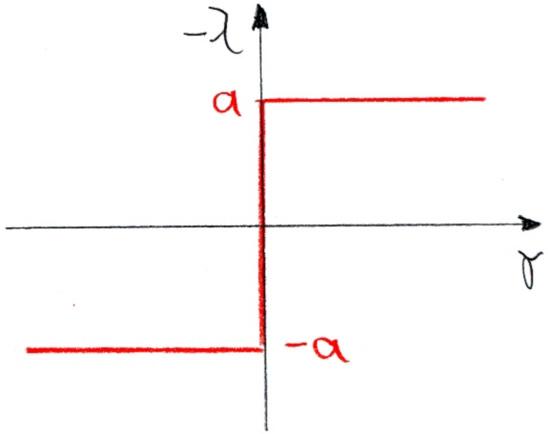
- impact-free motion





- relation to Coulomb





- impact law

$$(\gamma^+ + \varepsilon \gamma^-) \in \mathcal{N}_{\mathbb{R}^-}(-\Lambda)$$

5. The Impact Problem

- determine post-impact velocities $\vec{u}^+$ from

$$M(\vec{u}^+ - \vec{u}^-) = \sum_i W_i \vec{\Lambda}_i \qquad (1)$$

$$\vec{\gamma}_i = W_i^T \vec{u} \qquad (2)$$

$$(\vec{\gamma}_i^+ + \varepsilon_i \vec{\gamma}_i^-) \in \mathcal{N}_{\ell_i}(-\vec{\Lambda}_i) \qquad (3)$$

(1) impact equations

(2) relative velocities

(3) impact laws

(only scleronomic case without impulsive forcing)

- impact laws for particular elements

(i) geometric unilateral constraint

$$\ell = \mathbb{R}^-$$

(ii) kinematic unilateral constraint

$$\ell = \mathbb{R}^-$$

(iii) planar Coulomb friction

$$\ell = [-1, 1]\,\mu \Lambda_N$$

(iv) spatial Coulomb friction

$$\ell = \mathcal{B}_2\, \mu \Lambda_N$$

III. Consistency Conditions

1. Kinetic Consistency

Impact equations

$$M(\vec{u}^+ - \vec{u}^-) = \sum_i W_i \vec{\Lambda}_i$$

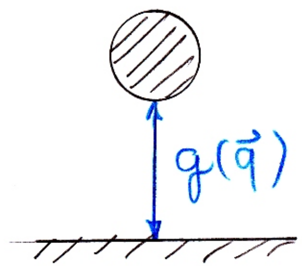
and force restrictions

$$-\vec{\Lambda}_i \in \mathcal{C}_i$$

always fulfilled ok.

2. Kinematic Consistency

(i) geometric unilateral constraint



- geometry

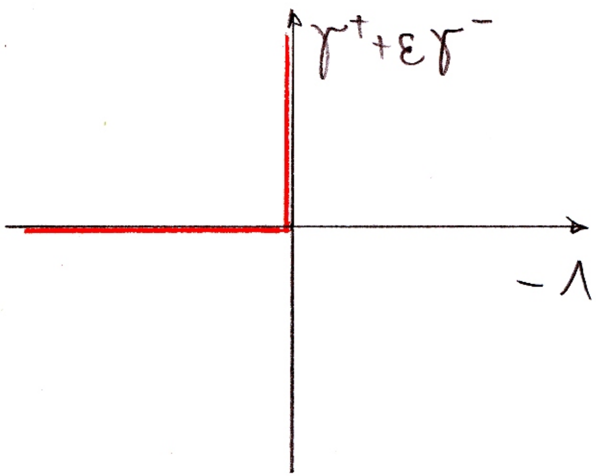
$$g(\vec{q}) \geq 0 \quad ; \quad \gamma = \frac{\partial g}{\partial \vec{q}} \vec{u}$$

$$\Rightarrow \gamma^- \leq 0 \qquad \gamma^+ \geq 0$$

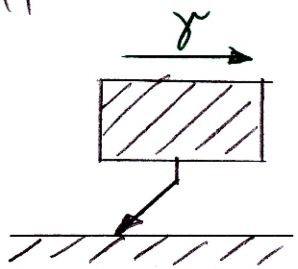
- impact law

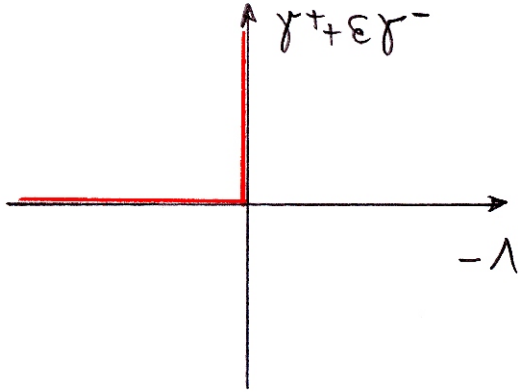
$$\gamma^+ + \varepsilon \gamma^- \geq 0$$

$$\Rightarrow \text{if } \gamma^- \leq 0 \text{ then } \gamma^+ \geq 0 \quad \text{ok} \quad (\varepsilon \geq 0)$$



(ii) kinematic unilateral constraint





- geometry

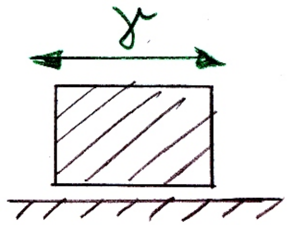
  $\gamma \geq 0 \Rightarrow \gamma^- \geq 0, \; \gamma^+ \geq 0$

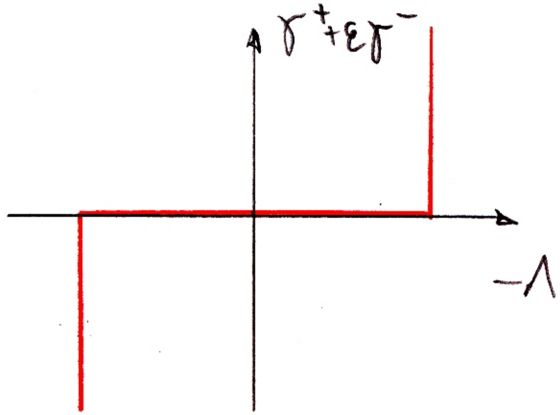
- impact law

  $\gamma^+ + \varepsilon\gamma^- \geq 0$ (holds as an equality if $\Lambda > 0$)

$\Rightarrow$ if $\gamma^- \geq 0$ then $\gamma^+ \geq 0$ <u>requires</u> $\varepsilon = 0$ to give $\gamma^+ = 0$

(iii) planar and spatial Coulomb friction





- geometry

  no restriction

- impact law

  no restriction

$\Rightarrow$ <u>ok.</u>

3. Energetic Consistency

- Equations

$$M(\vec{u}^+ - \vec{u}^-) = \sum_i W_i \vec{\Lambda}_i$$

$$\vec{\gamma}_i = W_i^T \vec{u}; \quad \vec{\xi}_i := \vec{\gamma}_i^+ + \varepsilon_i \vec{\gamma}_i^-$$

$$\vec{\xi}_i \in \mathcal{N}_{\mathcal{C}_i}(-\vec{\Lambda}_i)$$

- Difference in Kinetic Energy

$$\underline{\underline{T^+ - T^-}} = \frac{1}{2} \vec{u}^{+T} M \vec{u}^+ - \frac{1}{2} \vec{u}^{-T} M \vec{u}^-$$

$$= \frac{1}{2} (\vec{u}^+ + \vec{u}^-)^T M (\vec{u}^+ - \vec{u}^-)$$

$$= \frac{1}{2} (\vec{u}^+ + \vec{u}^-)^T W \vec{\Lambda}$$

$$= \underline{\underline{\frac{1}{2} \vec{\Lambda}^T (\vec{\gamma}^+ + \vec{\gamma}^-)}}$$

A) First Form of Energy Difference

$$T^+ - T^- = \frac{1}{2} \vec{\Lambda}^T (\vec{\gamma}^+ + \bar{\bar{\varepsilon}} \vec{\gamma}^-) + \frac{1}{2} \vec{\Lambda}^T (\vec{\gamma}^- - \bar{\bar{\varepsilon}} \vec{\gamma}^-)$$

$$= \underbrace{\frac{1}{2} \vec{\Lambda}^T \vec{\xi}}_{\substack{\leq 0 \\ \text{normal cone}}} + \frac{1}{2} \vec{\Lambda}^T \underbrace{(E - \bar{\bar{\varepsilon}})}_{\text{diag } \delta_i} \vec{\gamma}^-$$

- Sufficient for Consistency

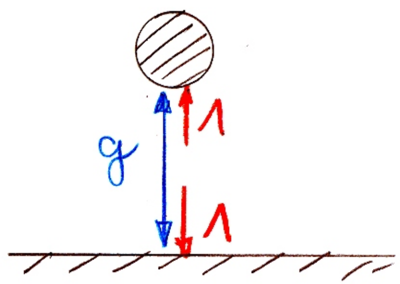
$$\underline{\underline{\sum_i \delta_i \vec{\Lambda}_i^T \vec{\gamma}_i^- \overset{!}{\leq} 0}}$$

- Frictionless Unilat. Constraints

Choose $0 \leq \varepsilon_i \leq 1 \Rightarrow 0 \leq \delta_i \leq 1$

$\Lambda_i \geq 0 \qquad \gamma_i^- \leq 0$

$\Rightarrow$ <u>consistent</u>



B) Second Form of Energy Difference

- Equations $G := W^T M^{-1} W$

$$\left.\begin{aligned} M(\vec{u}^+ - \vec{u}^-) &= W\vec{\Lambda} \\ \vec{\gamma} &= W^T \vec{u} \\ \vec{\xi} &= \vec{\gamma}^+ + \bar{\bar{\varepsilon}}\vec{\gamma}^- \end{aligned}\right\} \Rightarrow \begin{aligned} \vec{\gamma}^+ - \vec{\gamma}^- &= G\vec{\Lambda} \\ \vec{\xi} &= \vec{\gamma}^+ + \bar{\bar{\varepsilon}}\vec{\gamma}^- \end{aligned}$$

- Solve for $\vec{\gamma}^+$ and $\vec{\gamma}^-$

$$\left.\begin{aligned} \vec{\gamma}^- &= (E + \bar{\bar{\varepsilon}})^{-1}(\vec{\xi} - G\vec{\Lambda}) \\ \vec{\gamma}^+ &= (E + \bar{\bar{\varepsilon}})^{-1}(\vec{\xi} + \bar{\bar{\varepsilon}}G\vec{\Lambda}) \end{aligned}\right\} \gamma^+ + \gamma^- = (E + \bar{\bar{\varepsilon}})^{-1}(2\vec{\xi} - (E - \bar{\bar{\varepsilon}})G\vec{\Lambda})$$

- Kinetic Energy

$$T^+ - T^- = \frac{1}{2}\vec{\Lambda}^T(\vec{\gamma}^+ + \vec{\gamma}^-)$$

$$\Rightarrow \underline{\underline{T^+ - T^- = \underbrace{\vec{\Lambda}^T(E + \bar{\bar{\varepsilon}})^{-1}\vec{\xi}}_{\substack{\leq 0 \\ \text{normal cone}}} - \frac{1}{2}\vec{\Lambda}^T \underbrace{(E + \bar{\bar{\varepsilon}})^{-1}(E - \bar{\bar{\varepsilon}})}_{=:\Delta} G\vec{\Lambda}}}$$

$\rightsquigarrow$ work on term

$$\underline{\underline{\frac{1}{2}\vec{\Lambda}^T(\Delta G)\vec{\Lambda}}}$$

- Energetic consistency holds when

(i) standard inelastic shock (~ Moreau)

$$\varepsilon_1 = \varepsilon_2 = \ldots = 0$$

(ii) equal restitution coeff. (~ Moreau's half line)

$$\varepsilon_1 = \varepsilon_2 = \ldots =: \varepsilon \quad (0 \leq \varepsilon \leq 1)$$

(iii) small restitution coeff. (Leine, 2006)

$$\frac{2\,\varepsilon_{max}}{1+\varepsilon_{max}} \leq \frac{1}{\mathrm{cond}\, G}$$

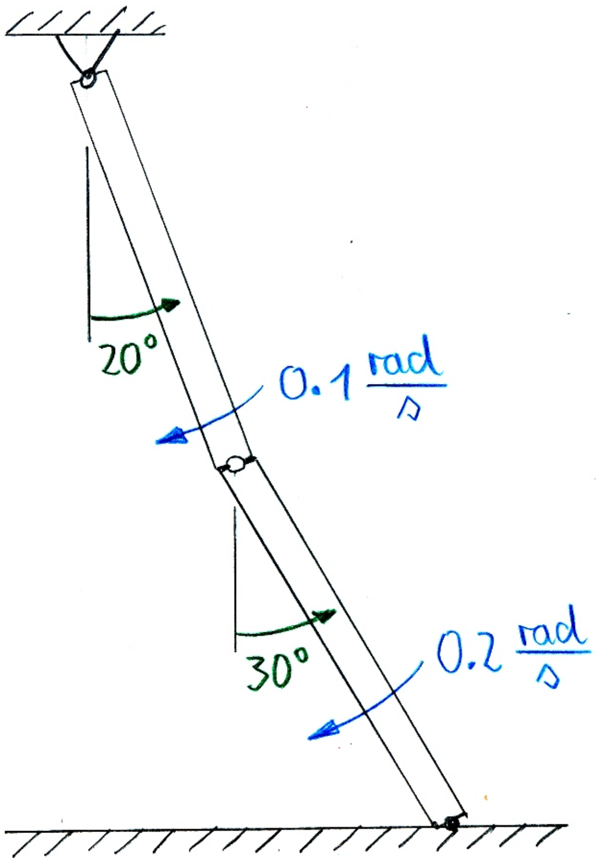
(iv) similar restitution coeff. (Leine, 2006)

$$\frac{\mathrm{cond}\,\Delta - 1}{\mathrm{cond}\,\Delta + 1} \leq \frac{1}{\mathrm{cond}\, G}$$

$$\mathrm{cond}\, A = \frac{\lambda_{max}}{\lambda_{min}} \;;\; A = A^T \text{ PSD} \;;\; G = W^T M^{-1} W \;;\; \Delta = (E + \bar{\bar{\varepsilon}})^{-1} (E - \bar{\bar{\varepsilon}})$$

IV. Counter Examples

1. Double Pendulum (Kane/Levinson 1985)



$l_1 = l_2 = 2m$

$m_1 = m_2 = 3kg$

a) slip at end of impact

$\varepsilon_N = 0.7 \quad \mu = 0.5$ energy increase

b) stick at end of impact

$\varepsilon_N = 0.5 \quad \mu = 0.5$ energy increase

! no tangential restitution used,

$\varepsilon_T = 0$;

$(\varepsilon_N, \varepsilon_T)$ too different

2. Contact and Sprag Clutch (Möller ~2006)

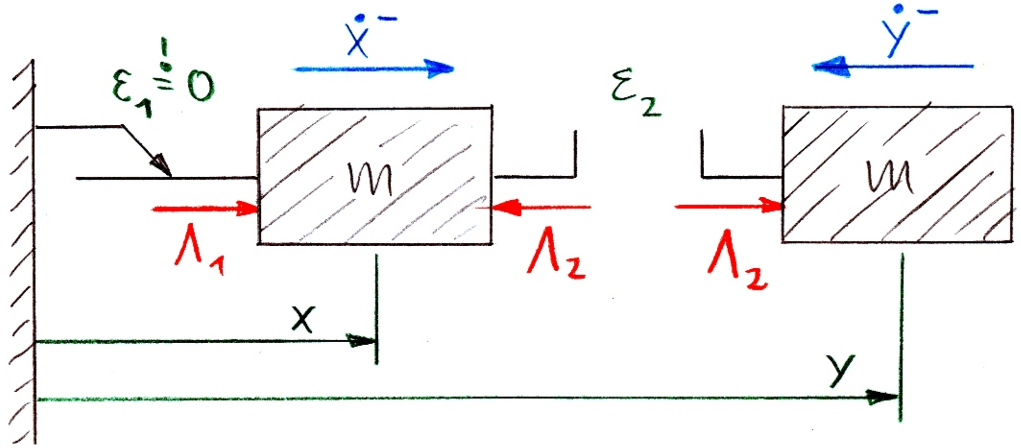
- Equations

$$m(\dot{x}^+ - \dot{x}^-) = \Lambda_1 - \Lambda_2$$

$$m(\dot{y}^+ - \dot{y}^-) = \Lambda_2$$

$$\gamma_1 = \dot{x} \qquad 0 \leq \Lambda_1 \perp (\gamma_1^+ + \varepsilon_1 \gamma_1^-) \geq 0$$

$$\gamma_2 = \dot{y} - \dot{x} \qquad 0 \leq \Lambda_2 \perp (\gamma_2^+ + \varepsilon_2 \gamma_2^-) \geq 0$$



$$\underline{\underline{(T^+ - T^-)(\varepsilon_2 = 1) = -m \cdot \dot{x}^- \dot{y}^- > 0}} \ \text{!!}$$

- Pre-Impact State

$$\dot{x}^- > 0, \qquad \dot{y}^- < 0$$

- Post-Impact State $(\Lambda_1, \Lambda_2 > 0)$

$$\dot{x}^+ = 0, \qquad \dot{y}^+ = -\varepsilon_2(\dot{y}^- - \dot{x}^-)$$

- Energy Difference

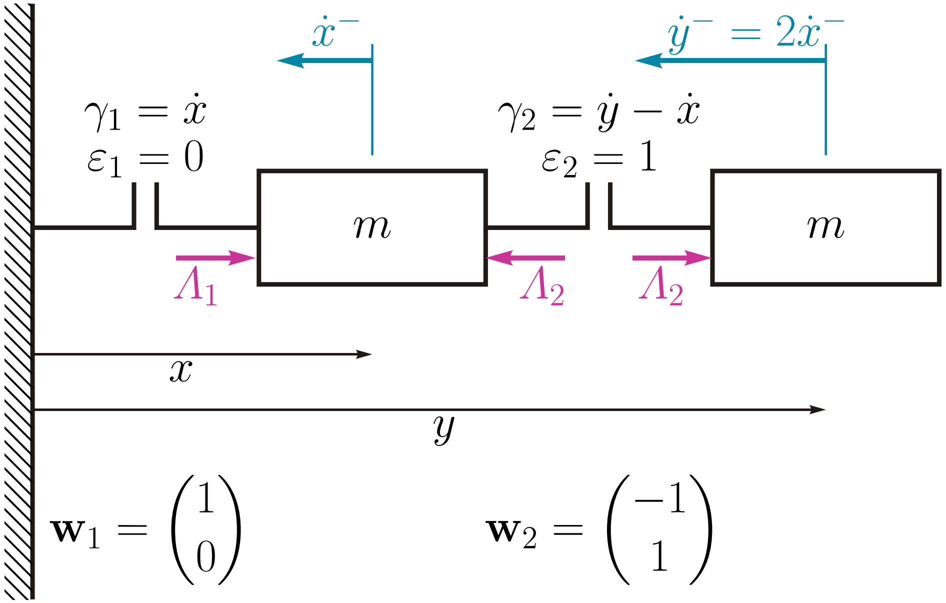
$$T^+ - T^- = -\tfrac{1}{2} m \left[(1-\varepsilon_2^2)(\dot{x}^{-2} + \dot{y}^{-2}) + 2\varepsilon_2^2 \dot{x}^- \dot{y}^- \right]$$

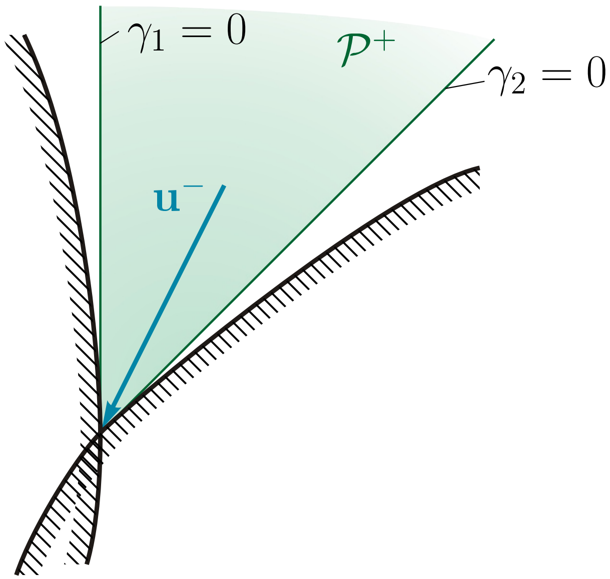
Unilateral Geometric Constraints

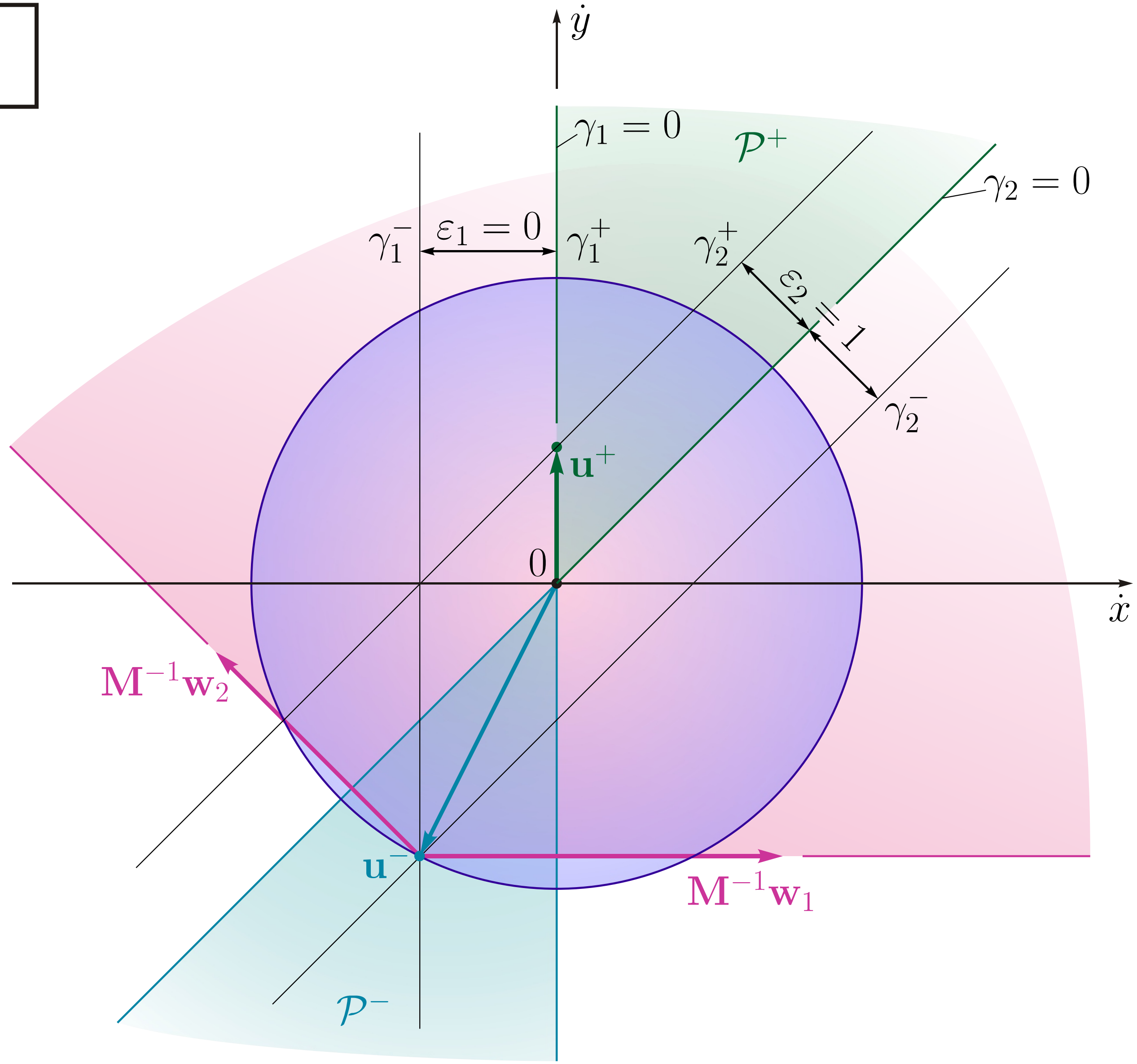
$$\boldsymbol{\gamma} = \mathbf{W}^{\mathsf{T}}\mathbf{u} \qquad \mathbf{u}^- \in \mathcal{P}^- \qquad \mathbf{u}^+ \in \mathcal{P}^+$$

$$\mathbf{u}^+ = \mathbf{u}^- + \mathbf{M}^{-1}\mathbf{W}\,\boldsymbol{\Lambda} \quad (\boldsymbol{\Lambda} \geq 0)$$

$$\|\mathbf{u}^+\|_{\mathbf{M}} \leq \|\mathbf{u}^-\|_{\mathbf{M}}$$





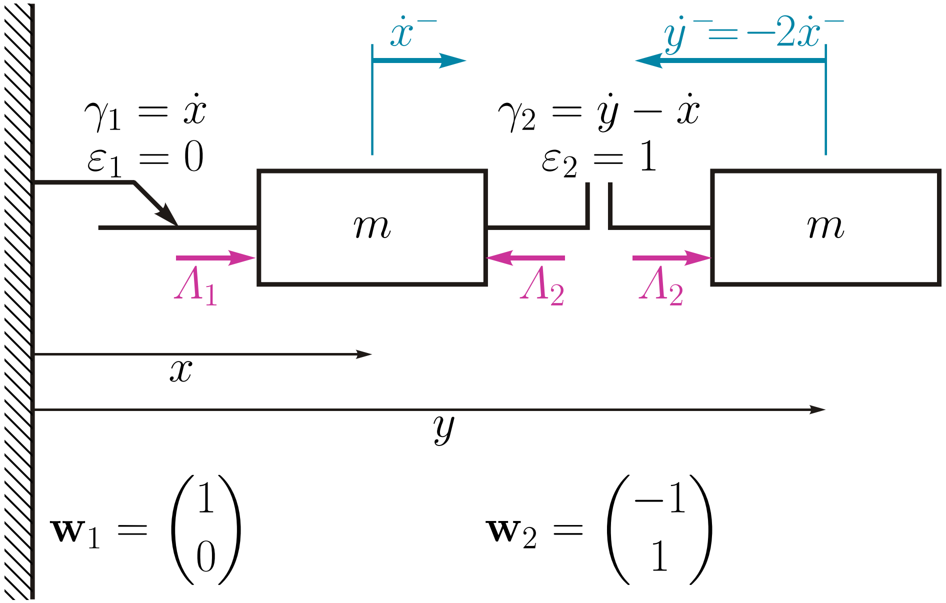


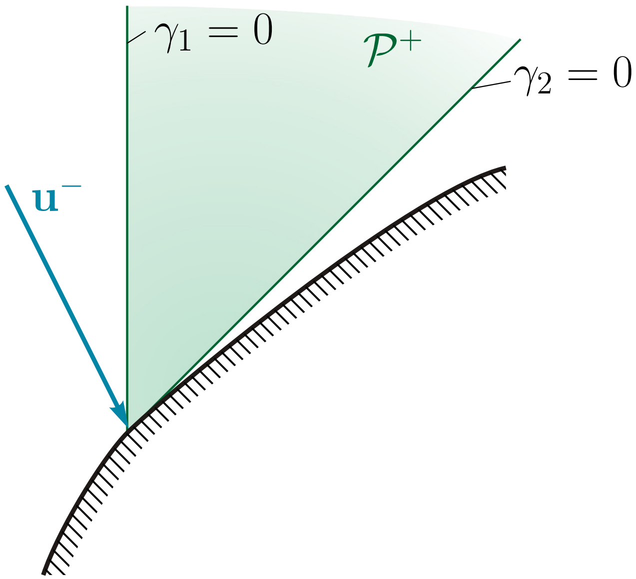
Unilat. Geom. & Kinemat. Constraints

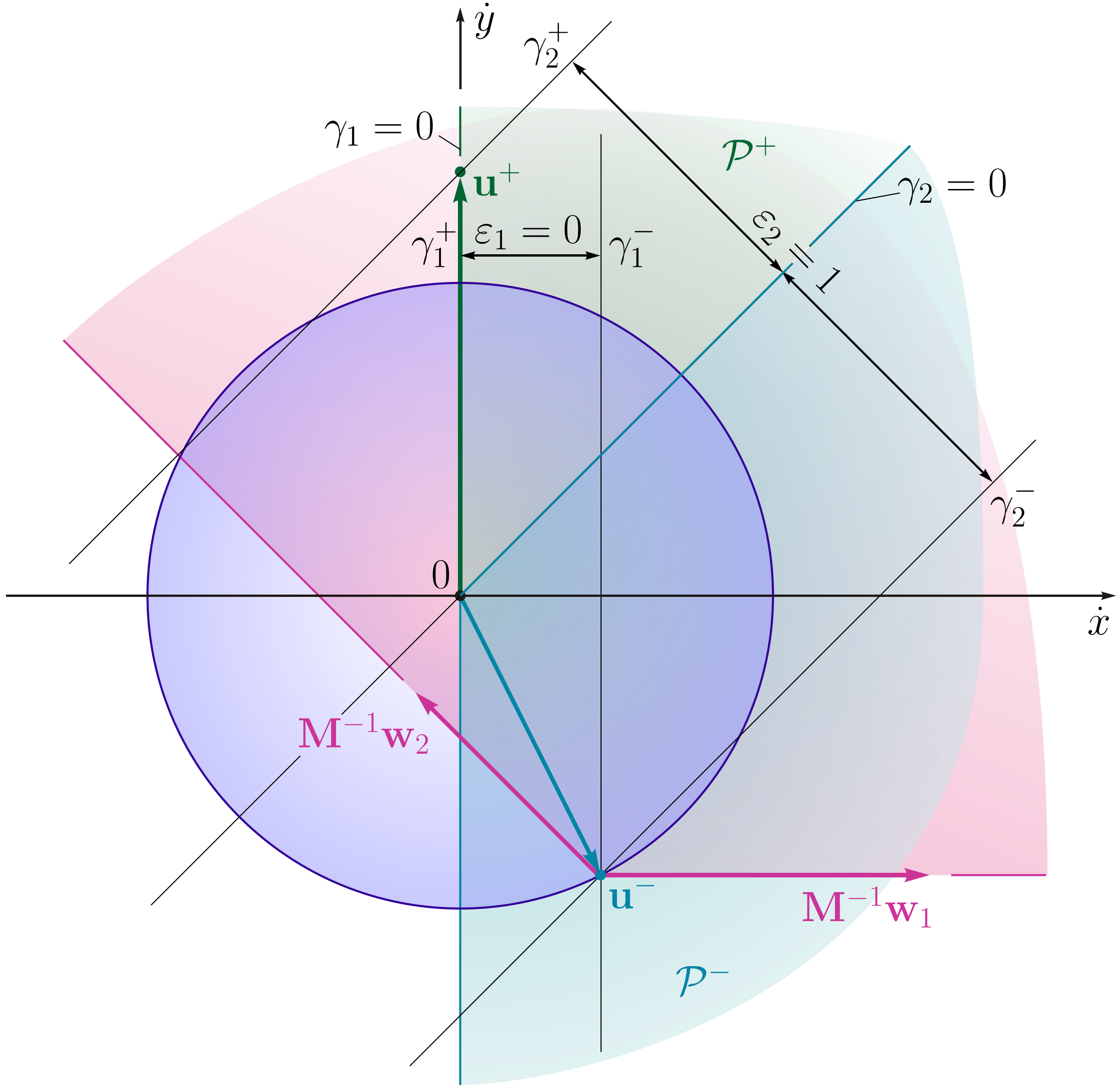
$$\boldsymbol{\gamma} = \mathbf{W}^{\mathsf{T}}\mathbf{u} \qquad \mathbf{u}^- \in \mathcal{P}^- \qquad \mathbf{u}^+ \in \mathcal{P}^+$$

$$\mathbf{u}^+ = \mathbf{u}^- + \mathbf{M}^{-1}\mathbf{W}\,\boldsymbol{\Lambda} \quad (\boldsymbol{\Lambda} \geq 0)$$

$$\|\mathbf{u}^+\|_{\mathbf{M}} \leq \|\mathbf{u}^-\|_{\mathbf{M}}$$





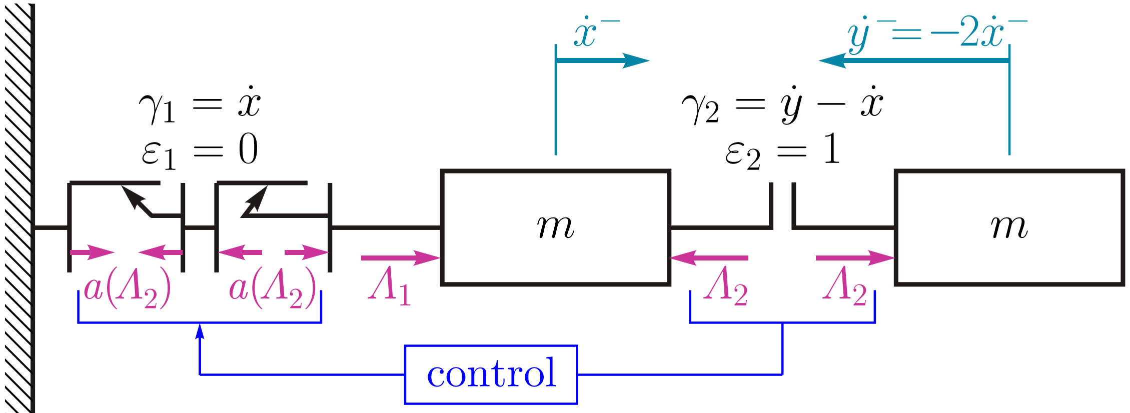


Unilateral Spike Constraint

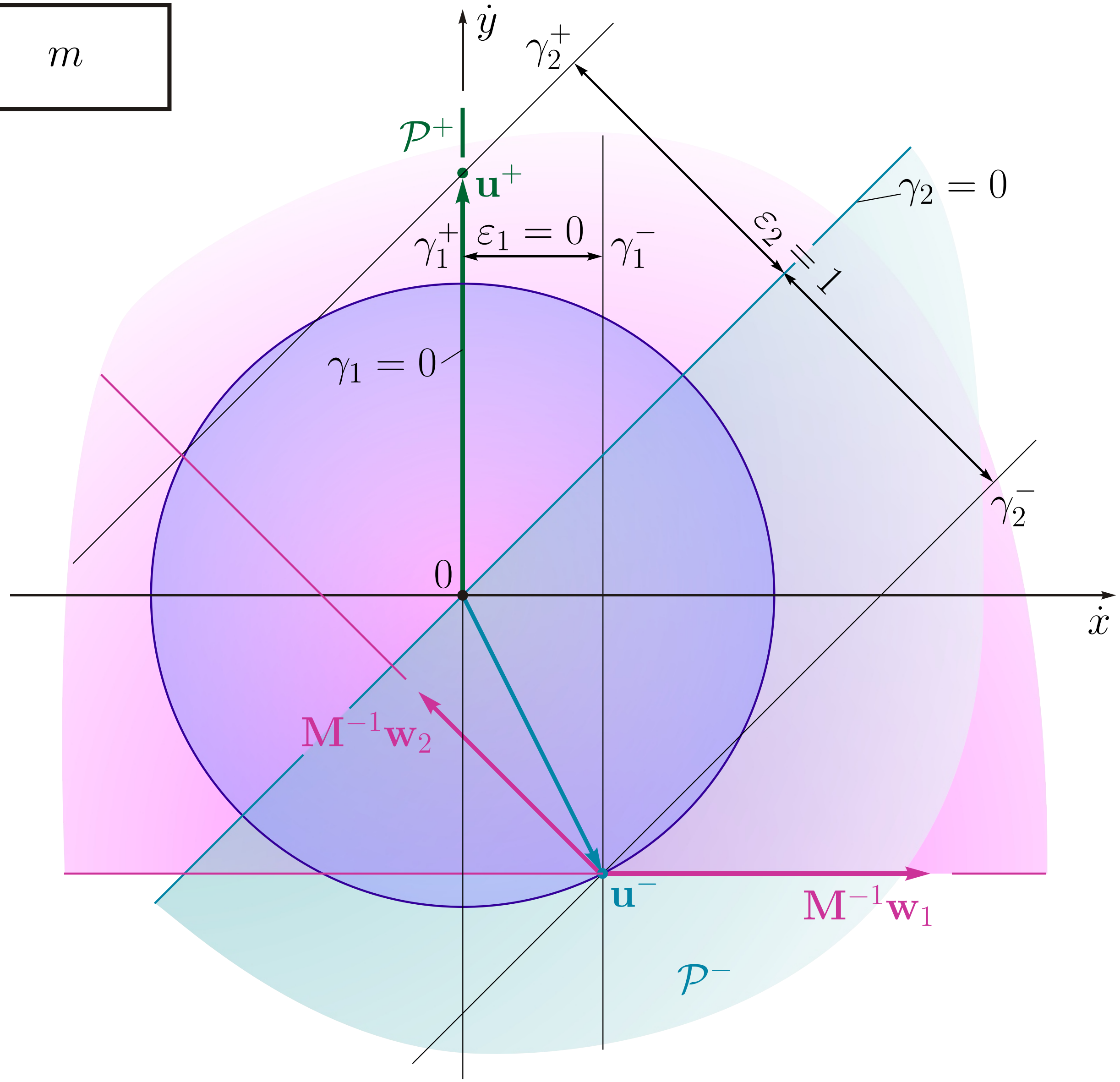
$$\boldsymbol{\gamma} = \mathbf{W}^\mathsf{T}\mathbf{u} \qquad \mathbf{u}^- \in \mathcal{P}^- \qquad \mathbf{u}^+ \in \mathcal{P}^+$$

$$\mathbf{u}^+ = \mathbf{u}^- + \mathbf{M}^{-1}\mathbf{W}\,\boldsymbol{\Lambda}$$

$$\|\mathbf{u}^+\|_\mathbf{M} \leq \|\mathbf{u}^-\|_\mathbf{M}$$



$$\mathbf{w}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \mathbf{w}_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$



Heavy Friction:

$$a(\Lambda_2) = \begin{cases} 0 & \text{for } \Lambda_2 = 0 \\ +\infty & \text{for } \Lambda_2 > 0 \end{cases}$$