Bilinear discretization of quadratic vector fields, or The strange case of Prof. Kahan and Hirota-sensei

Yuri B. Suris

(Technische Universität München)

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The problem of integrable discretization. Hamiltonian approach (Birkhäuser, 2003)

Consider a completely integrable flow

$$\dot{x} = f(x) = \{H, x\} \tag{1}$$

with a Hamilton function H on a Poisson manifold $\mathcal{P}$ with a Poisson bracket $\{\cdot,\cdot\}$. Thus, the flow (1) possesses many functionally independent integrals $I_k(x)$ in involution.

The *problem of integrable discretization*: find a family of diffeomorphisms $\mathcal{P} \to \mathcal{P}$,

$$\widetilde{x} = \Phi(x; \epsilon), \tag{2}$$

depending smoothly on a small parameter $\epsilon > 0$, with the following properties:

1. The maps (2) *approximate* the flow (1):

$$\Phi(x;\epsilon) = x + \epsilon f(x) + O(\epsilon^2).$$

2. The maps (2) are *Poisson* w. r. t. the bracket $\{\cdot,\cdot\}$ or some its deformation $\{\cdot,\cdot\}_\epsilon = \{\cdot,\cdot\} + O(\epsilon)$.
3. The maps (2) are *integrable*, i.e. possess the necessary number of independent integrals in involution, $I_k(x;\epsilon) = I_k(x) + O(\epsilon)$.

Missing in the book: Hirota-Kimura discretizations

- R.Hirota, K.Kimura. *Discretization of the Euler top.* J. Phys. Soc. Japan **69** (2000) 627–630,
- K.Kimura, R.Hirota. *Discretization of the Lagrange top.* J. Phys. Soc. Japan **69** (2000) 3193–3199.

Reasons for this omission: discretization of the Euler top seemed to be an isolated curiosity; discretization of the Lagrange top seemed to be completely incomprehensible, if not even wrong.

Hirota-Kimura's discrete time Euler top

$$\begin{cases} \dot{x}_1 = \alpha_1 x_2 x_3, \\ \dot{x}_2 = \alpha_2 x_3 x_1, \\ \dot{x}_3 = \alpha_3 x_1 x_2, \end{cases} \quad \rightsquigarrow \quad \begin{cases} \widetilde{x}_1 - x_1 = \epsilon\alpha_1(\widetilde{x}_2 x_3 + x_2 \widetilde{x}_3), \\ \widetilde{x}_2 - x_2 = \epsilon\alpha_2(\widetilde{x}_3 x_1 + x_3 \widetilde{x}_1), \\ \widetilde{x}_3 - x_3 = \epsilon\alpha_3(\widetilde{x}_1 x_2 + x_1 \widetilde{x}_2). \end{cases}$$

Features:

- Equations are linear w.r.t. $\widetilde{x} = (\widetilde{x}_1, \widetilde{x}_2, \widetilde{x}_3)^{\mathrm{T}}$, result in an *explicit* (rational) map: $\widetilde{x} = f(x, \epsilon) = A^{-1}(x, \epsilon)x$,

$$A(x, \epsilon) = \begin{pmatrix} 1 & -\epsilon\alpha_1 x_3 & -\epsilon\alpha_1 x_2 \\ -\epsilon\alpha_2 x_3 & 1 & -\epsilon\alpha_2 x_1 \\ -\epsilon\alpha_3 x_2 & -\epsilon\alpha_3 x_1 & 1 \end{pmatrix}.$$

- The map is *reversible* (therefore birational):

$$f^{-1}(x, \epsilon) = f(x, -\epsilon).$$

- Explicit formulas rather messy:

$$\begin{cases} \widetilde{x}_1 = \dfrac{x_1 + 2\epsilon\alpha_1 x_2 x_3 + \epsilon^2 x_1(-\alpha_2\alpha_3 x_1^2 + \alpha_3\alpha_1 x_2^2 + \alpha_1\alpha_2 x_3^2)}{\Delta(x,\epsilon)}, \\ \widetilde{x}_2 = \dfrac{x_2 + 2\epsilon\alpha_2 x_3 x_1 + \epsilon^2 x_2(\alpha_2\alpha_3 x_1^2 - \alpha_3\alpha_1 x_2^2 + \alpha_1\alpha_2 x_3^2)}{\Delta(x,\epsilon)}, \\ \widetilde{x}_3 = \dfrac{x_3 + 2\epsilon\alpha_3 x_1 x_2 + \epsilon^2 x_3(\alpha_2\alpha_3 x_1^2 + \alpha_3\alpha_1 x_2^2 - \alpha_1\alpha_2 x_3^2)}{\Delta(x,\epsilon)}, \end{cases}$$

where

$$\begin{aligned} \Delta(x,\epsilon) = \det A(x,\epsilon) \quad &= \quad 1 - \epsilon^2(\alpha_2\alpha_3 x_1^2 + \alpha_3\alpha_1 x_2^2 + \alpha_1\alpha_2 x_3^2) \\ &\quad -2\epsilon^3\alpha_1\alpha_2\alpha_3 x_1 x_2 x_3. \end{aligned}$$

(Try to see reversibility directly from these formulas!)

- Two independent integrals:

$$I_1(x,\epsilon) = \frac{1-\epsilon^2\alpha_2\alpha_3 x_1^2}{1-\epsilon^2\alpha_3\alpha_1 x_2^2}, \quad I_2(x,\epsilon) = \frac{1-\epsilon^2\alpha_3\alpha_1 x_2^2}{1-\epsilon^2\alpha_1\alpha_2 x_3^2}.$$

- Invariant volume measure and bi-Hamiltonian structure found in: M.Petrera, Yu.Suris. *On the Hamiltonian structure of the Hirota-Kimura discretization of the Euler top.* Math. Nachr. 2008 (to appear).

Hirota-Kimura or Kahan?

- W. Kahan. *Unconventional numerical methods for trajectory calculations* (Unpublished lecture notes, 1993).

$$\dot{x} = Q(x) + Bx \quad \rightsquigarrow \quad (\widetilde{x} - x)/\epsilon = Q(x, \widetilde{x}) + B(x + \widetilde{x}),$$

where $B \in \mathbb{R}^{n \times n}$, $Q : \mathbb{R}^n \to \mathbb{R}^n$ is a *quadratic* function, and

$$Q(x, \widetilde{x}) = Q(x + \widetilde{x}) - Q(x) - Q(\widetilde{x})$$

is the corresponding symmetric *bilinear* function.

Note: equations for $\widetilde{x}$ always linear, $\widetilde{x} = f(x, \epsilon) = A^{-1}(x, \epsilon)x$, the map is always reversible and birational, $f^{-1}(x, \epsilon) = f(x, -\epsilon)$.

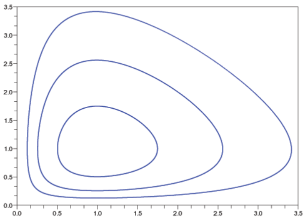
Illustration: Lotka-Volterra system

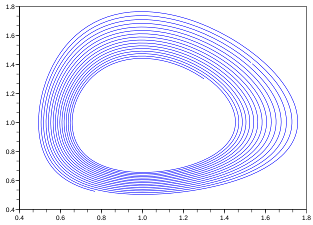
Kahan's integrator for the Lotka-Volterra system:

$$\left\{\begin{array}{l}\dot{x}=x(1-y),\\ \dot{y}=y(x-1),\end{array}\right. \quad\rightsquigarrow\quad \left\{\begin{array}{l}\widetilde{x}-x=\epsilon(\widetilde{x}+x)-\epsilon(\widetilde{x}y+x\widetilde{y}),\\ \widetilde{y}-y=\epsilon(\widetilde{x}y+x\widetilde{y})-\epsilon(\widetilde{y}+y).\end{array}\right.$$

Explicitly:

$$\left\{\begin{array}{l}\widetilde{x}=x\,\dfrac{(1+\epsilon)^2-\epsilon(1+\epsilon)x-\epsilon(1-\epsilon)y}{1-\epsilon^2-\epsilon(1-\epsilon)x+\epsilon(1+\epsilon)y},\\ \widetilde{y}=y\,\dfrac{(1-\epsilon)^2+\epsilon(1+\epsilon)x+\epsilon(1-\epsilon)y}{1-\epsilon^2-\epsilon(1-\epsilon)x+\epsilon(1+\epsilon)y}.\end{array}\right.$$





Left: three orbits of Kahan's discretization with $\epsilon = 0.1$, right: one orbit of the explicit Euler with $\epsilon = 0.01$.

- J.M.Sanz-Serna. *An unconventional symplectic integrator of W.Kahan.* Applied Numer. Math. **16** (1994) 245–250.

A sort of an explanation of a non-spiralling behavior: Kahan's integrator for the Lotka-Volterra system in Poisson.

Hirota-Kimura's discrete time Lagrange top

$$\begin{cases} \dot{\omega}_1 = (1-\alpha)\omega_2\omega_3 + z_0\gamma_2, \\ \dot{\omega}_2 = -(1-\alpha)\omega_3\omega_1 - z_0\gamma_1, \\ \dot{\omega}_3 = 0, \\ \dot{\gamma}_1 = \omega_3\gamma_2 - \omega_2\gamma_3, \\ \dot{\gamma}_2 = \omega_1\gamma_3 - \omega_3\gamma_1, \\ \dot{\gamma}_3 = \omega_2\gamma_1 - \omega_1\gamma_2, \end{cases} \quad \rightsquigarrow$$

$$\rightsquigarrow \quad \begin{cases} \widetilde{\omega}_1 - \omega_1 = \epsilon(1-\alpha)(\widetilde{\omega}_2\omega_3 + \omega_2\widetilde{\omega}_3) + \epsilon z_0(\widetilde{\gamma}_2 + \gamma_2), \\ \widetilde{\omega}_2 - \omega_2 = -\epsilon(1-\alpha)(\widetilde{\omega}_3\omega_1 + \omega_3\widetilde{\omega}_1) - \epsilon z_0(\widetilde{\gamma}_1 + \gamma_1), \\ \widetilde{\omega}_3 - \omega_3 = 0, \\ \widetilde{\gamma}_1 - \gamma_1 = \epsilon(\widetilde{\omega}_3\gamma_2 + \omega_3\widetilde{\gamma}_2) - \epsilon(\widetilde{\omega}_2\gamma_3 + \omega_2\widetilde{\gamma}_3), \\ \widetilde{\gamma}_2 - \gamma_2 = \epsilon(\widetilde{\omega}_1\gamma_3 + \omega_1\widetilde{\gamma}_3) - \epsilon(\widetilde{\omega}_3\gamma_1 + \omega_3\widetilde{\gamma}_1), \\ \widetilde{\gamma}_3 - \gamma_3 = \epsilon(\widetilde{\omega}_2\gamma_1 + \omega_2\widetilde{\gamma}_1) - \epsilon(\widetilde{\omega}_1\gamma_2 + \omega_1\widetilde{\gamma}_2), \end{cases}$$

which gives a birational map $(\widetilde{\omega}, \widetilde{\gamma}) = f(\omega, \gamma, \epsilon)$.

Hirota-Kimura's "method" for finding integrals

Consider $$A = \omega_1^2 + \omega_2^2 - B\gamma_3 - C\gamma_3^2,$$

$$D = \omega_1\gamma_1 + \omega_2\gamma_2 - E\gamma_3 - F\gamma_3^2, \qquad K = \gamma_1^2 + \gamma_2^2 - L\gamma_3 - M\gamma_3^2.$$

Determine $A, \ldots, M$ by requiring that they are conserved quantities. For instance, for A, B, C, solve the system of three equations for these three unknowns:

$$\begin{cases} A + B\widetilde{\gamma}_3 + C\widetilde{\gamma}_3^2 = \widetilde{\omega}_1^2 + \widetilde{\omega}_2^2, \\ A + B\gamma_3 + C\gamma_3^2 = \omega_1^2 + \omega_2^2, \\ A + B\underset{\sim}{\gamma}_3 + C\underset{\sim}{\gamma}_3^2 = \underset{\sim}{\omega}_1^2 + \underset{\sim}{\omega}_2^2 \end{cases}$$

with $(\widetilde{\omega}, \widetilde{\gamma}) = f(\omega, \gamma, \epsilon)$ and $(\underset{\sim}{\omega}, \underset{\sim}{\gamma}) = f^{-1}(\omega, \gamma, \epsilon)$. Then check that $A, B, C = A, B, C(\omega, \gamma, \epsilon)$ are conserved quantities, indeed.

Does this make any sense for you???

Nevertheless, this turns out to be not only true but also remarkably deep (as everything by R. Hirota...).

Clebsch system

Clebsch system describes the motion of a rigid body in an ideal fluid:

$$
\begin{aligned}
\dot{m}_1 &= (\omega_3 - \omega_2) p_2 p_3, \\
\dot{m}_2 &= (\omega_1 - \omega_3) p_3 p_1, \\
\dot{m}_3 &= (\omega_2 - \omega_1) p_1 p_2, \\
\dot{p}_1 &= m_3 p_2 - m_2 p_3, \\
\dot{p}_2 &= m_1 p_3 - m_3 p_1, \\
\dot{p}_3 &= m_2 p_1 - m_1 p_2.
\end{aligned}
$$

It is Hamiltonian w.r.t. Lie-Poisson bracket of $e(3)$, has four functionally independent integrals in involution:

$$I_i = p_i^2 + \frac{m_j^2}{\omega_k - \omega_i} + \frac{m_k^2}{\omega_j - \omega_i}, \quad (i,j,k) = c.p.(1,2,3),$$

and $H_4 = m_1 p_1 + m_2 p_2 + m_3 p_3$.

Hirota-Kimura discretization of the Clebsch system

Hirota-Kimura-type discretization (proposed by T. Ratiu on Oberwolfach Meeting “Geometric Integration”, March 2006):

$$
\begin{aligned}
\widetilde{m}_1 - m_1 &= \epsilon(\omega_3 - \omega_2)(\widetilde{p}_2 p_3 + p_2 \widetilde{p}_3), \\
\widetilde{m}_2 - m_2 &= \epsilon(\omega_1 - \omega_3)(\widetilde{p}_3 p_1 + p_3 \widetilde{p}_1), \\
\widetilde{m}_3 - m_3 &= \epsilon(\omega_2 - \omega_1)(\widetilde{p}_1 p_2 + p_1 \widetilde{p}_2), \\
\widetilde{p}_1 - p_1 &= \epsilon(\widetilde{m}_3 p_2 + m_3 \widetilde{p}_2) - \epsilon(\widetilde{m}_2 p_3 + m_2 \widetilde{p}_3), \\
\widetilde{p}_2 - p_2 &= \epsilon(\widetilde{m}_1 p_3 + m_1 \widetilde{p}_3) - \epsilon(\widetilde{m}_3 p_1 + m_3 \widetilde{p}_1), \\
\widetilde{p}_3 - p_3 &= \epsilon(\widetilde{m}_2 p_1 + m_2 \widetilde{p}_1) - \epsilon(\widetilde{m}_1 p_2 + m_1 \widetilde{p}_2).
\end{aligned}
$$

What follows is based on a joint work with Matteo Petrera (Rome) and Andreas Pfadler (München).

A birational map

$$\begin{pmatrix} \widetilde{m} \\ \widetilde{p} \end{pmatrix} = f(m,p,\epsilon) = M^{-1}(m,p,\epsilon) \begin{pmatrix} m \\ p \end{pmatrix},$$

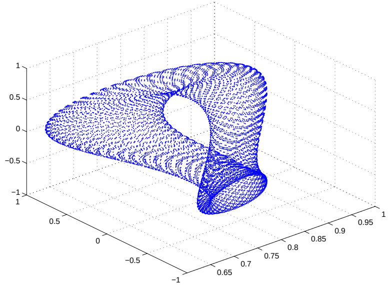
$$M(m,p,\epsilon) = \begin{pmatrix} 1 & 0 & 0 & 0 & \epsilon\omega_{23}p_3 & \epsilon\omega_{23}p_2 \\ 0 & 1 & 0 & \epsilon\omega_{31}p_3 & 0 & \epsilon\omega_{31}p_1 \\ 0 & 0 & 1 & \epsilon\omega_{12}p_2 & \epsilon\omega_{12}p_1 & 0 \\ 0 & \epsilon p_3 & -\epsilon p_2 & 1 & -\epsilon m_3 & \epsilon m_2 \\ -\epsilon p_3 & 0 & \epsilon p_1 & \epsilon m_3 & 1 & -\epsilon m_1 \\ \epsilon p_2 & -\epsilon p_1 & 0 & -\epsilon m_2 & \epsilon m_1 & 1 \end{pmatrix},$$

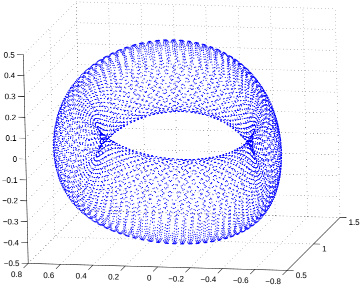
with $\omega_{ij} = \omega_i - \omega_j$. The usual reversibility:

$$f^{-1}(m,p,\epsilon) = f(m,p,-\epsilon).$$

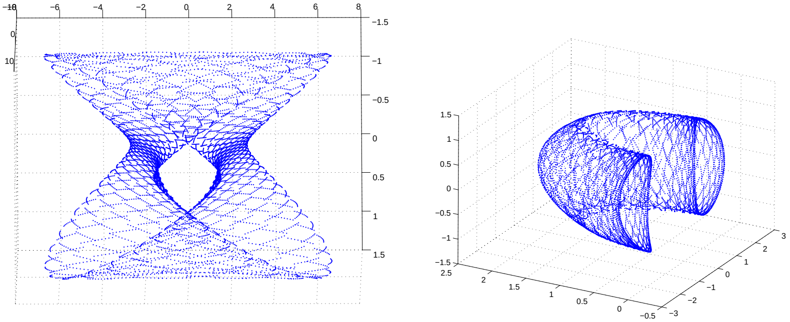
Numerators and denominators of components of $\widetilde{m}, \widetilde{p}$ are polynomials of degree 6, the numerators of $\widetilde{p}_i$ consist of 31 monomials, the numerators of $\widetilde{m}_i$ consist of 41 monomials, the common denominator consists of 28 monomials.

Phase portraits





An orbit of the discrete Clebsch system with $\omega_1 = 0.1$, $\omega_2 = 0.2$, $\omega_3 = 0.3$ and $\epsilon = 1$; projections to (m_1, m_2, m_3) and to (p_1, p_2, p_3); initial point $(m_0, p_0) = (1, 1, 1, 1, 1, 1)$.



An orbit of the discrete Clebsch system with $\omega_1 = 1$, $\omega_2 = 0.2$, $\omega_3 = 30$ and $\epsilon = 1$; projections to (m_1, m_2, m_3) and to (p_1, p_2, p_3); initial point $(m_0, p_0) = (1, 1, 1, 1, 1, 1)$.

Hirota-Kimura bases

Definition. *For a given birational map $f : \mathbb{R}^n \to \mathbb{R}^n$, a set of functions $\Phi = (\varphi_1, \ldots, \varphi_l)$, linearly independent over $\mathbb{R}$, is called a* **HK-basis**, *if for every $x_0 \in \mathbb{R}^n$ there exists a vector $c = (c_1, \ldots, c_l) \neq 0$ such that*

$$c_1\varphi_1(f^i(x_0)) + \ldots + c_l\varphi_l(f^i(x_0)) = 0 \quad \forall i \in \mathbb{Z}.$$

For a given $x_0 \in \mathbb{R}^n$, the set of all vectors $c \in \mathbb{R}^l$ with this property will be denoted by $K_\Phi(x_0)$ and called the null-space of the basis Φ (at the point x_0). This set clearly is a vector space.

Note: we cannot claim that $h = c_1\varphi_1 + ... + c_l\varphi_l$ is an integral of motion, since vectors $c \in K_\Phi(x_0)$ vary from one initial point x_0 to another.

However: existence of a HK-basis Φ with $\dim K_\Phi(x_0) = d$ confines the orbits of f to $(n-d)$-dimensional invariant sets.

From HK-bases to integrals

Proposition. *If Φ is a HK-basis for a map f, then* $K_\Phi(f(x_0)) = K_\Phi(x_0)$.

Thus, the d-dimensional null-space $K_\Phi(x_0)$ is a $Gr(d,l)$-valued integral. Its Plücker coordinates are real-valued integrals:

Corollary. *Let Φ be a HK-basis for f with $\dim K_\Phi(x_0) = d$ for all $x_0 \in \mathbb{R}^n$. Take a basis of $K_\Phi(x_0)$ consisting of d vectors $c^{(i)} \in \mathbb{R}^l$ and put them into the columns of a $l \times d$ matrix $C(x_0)$. For any d-index $\alpha = (\alpha_1, \ldots, \alpha_d) \subset \{1, 2, \ldots, n\}$ let $C_\alpha = C_{\alpha_1 \ldots \alpha_d}$ denote the $d \times d$ minor of the matrix C built from the rows $\alpha_1, \ldots, \alpha_d$. Then for any two d-indices α, β the function C_α / C_β is an integral of f.*

Especially simple is the situation when the null-space of a HK-basis has dimension $d = 1$.

Corollary. *Let* Φ *be a HK-basis for* f *with* $\dim K_\Phi(x_0) = 1$ *for all* $x_0 \in \mathbb{R}^n$*. Let* $K_\Phi(x_0) = [c_1(x_0) : \ldots : c_l(x_0)] \in \mathbb{RP}^{l-1}$*. Then the functions* c_j/c_k *are integrals of motion for* f*.*

In other words, normalizing $c_l(x_0) = 1$ (say), we find that all other c_j ($j = 1, \ldots, l-1$) are integrals of motion. It is not clear whether one can say something general about the number of functionally independent integrals among them. It varies in examples (sometimes just $= 1$ and sometimes > 1).

Finding HK-bases

Theorem. *Let, for all* $x_0 \in \mathbb{R}^n$*, the dimension of the solution space of the homogeneous system for* $c_1, \ldots, c_l$*,*

$$c_1\varphi_1(f^i(x_0)) + \ldots + c_l\varphi_l(f^i(x_0)) = 0, \quad i = 0, \ldots, s-1,$$

be equal to $l - s$ *for* $1 \le s \le l - d$ *and to* d *for* $s = l - d + 1$*. Then* $K_\Phi(x_0)$ *coincides with the solution space for* $s = l - d$*, and, in particular,* $\dim K_\Phi(x_0) = d$*.*

Numerical algorithm:

(N) For several randomly chosen initial points $x_0 \in \mathbb{R}^n$, compute the dimension of the solution space of the above system for $1 \le s \le l$. If for every x_0 the dimension fails to drop after $s = l - d$ with one and the same $d \ge 1$, then Φ is likely to be a HK-basis for f, with $\dim K_\Phi(x_0) = d$.

Especially important case: $d = 1$.

To *prove* that some Φ is a HK-basis, have to check the conditions of the theorem *symbolically*. Some possible tricks (for $d = 1$):

(A) Consider the non-homogeneous system of $l - 1$ equations

$$c_1 \varphi_1(f^i(x_0)) + \ldots + c_{l-1} \varphi_{l-1}(f^i(x_0)) = \varphi_l(f^i(x_0))$$

for two different but overlapping ranges $i \in [i_0, i_0 + l - 2]$ and $i \in [i_1, i_1 + l - 2]$. If the solutions coincide, then Φ is a HK-basis with $d = 1$.

(B) Consider the above system for the index range $i \in [i_0, i_0 + l - 2]$ which contains 0 but is non-symmetric. If the solution functions $c_1(x_0, \epsilon), \ldots, c_{l-1}(x_0, \epsilon)$ are even w.r.t. ϵ, then Φ is a HK-basis with $d = 1$.

Results for the discrete Clebsch system

Theorem. *a) The set of functions*

$$\Phi = (p_1^2, p_2^2, p_3^2, m_1^2, m_2^2, m_3^2, m_1 p_1, m_2 p_2, m_3 p_3, 1)$$

is a HK-basis for f, with $\dim K_\Phi(m,p) = 4$. Thus, any orbit of f lies on an intersection of four quadrics in $\mathbb{R}^6$.
b) The following four sets of functions are HK-bases for f with one-dimensional null-spaces:

$$\begin{aligned} \Phi_0 &= (p_1^2, p_2^2, p_3^2, 1), \\ \Phi_1 &= (p_1^2, p_2^2, p_3^2, m_1^2, m_2^2, m_3^2, m_1 p_1), \\ \Phi_2 &= (p_1^2, p_2^2, p_3^2, m_1^2, m_2^2, m_3^2, m_2 p_2), \\ \Phi_3 &= (p_1^2, p_2^2, p_3^2, m_1^2, m_2^2, m_3^2, m_3 p_3). \end{aligned}$$

There holds: $K_\Phi = K_{\Phi_0} \oplus K_{\Phi_1} \oplus K_{\Phi_2} \oplus K_{\Phi_3}$.

Complexity issues

The claims in part b) of the above theorem refer to the solutions of the following systems:

$$(c_1 p_1^2 + c_2 p_2^2 + c_3 p_3^2) \circ f^i = 1,$$

$$(\alpha_1 p_1^2 + \alpha_2 p_2^2 + \alpha_3 p_3^2 + \alpha_4 m_1^2 + \alpha_5 m_2^2 + \alpha_6 m_3^2) \circ f^i = m_1 p_1 \circ f^i,$$

$$(\beta_1 p_1^2 + \beta_2 p_2^2 + \beta_3 p_3^2 + \beta_4 m_1^2 + \beta_5 m_2^2 + \beta_6 m_3^2) \circ f^i = m_2 p_2 \circ f^i,$$

$$(\gamma_1 p_1^2 + \gamma_2 p_2^2 + \gamma_3 p_3^2 + \gamma_4 m_1^2 + \gamma_5 m_2^2 + \gamma_6 m_3^2) \circ f^i = m_3 p_3 \circ f^i.$$

The first one has to be solved for one non-symmetric range of $l - 1 = 3$ values of i, or for two different such ranges. The last three systems have to be solved for a non-symmetric range of $l - 1 = 6$ values of i. This can be done numerically (in rational arithmetic) without any difficulties, but becomes (nearly) impossible for a symbolic computation, due to complexity of f^2.

Complexity of f^2

Degrees of numerators and denominators of f^2:

	deg	$\deg_{p_1}$	$\deg_{p_2}$	$\deg_{p_3}$	$\deg_{m_1}$	$\deg_{m_2}$	$\deg_{m_3}$
Denom. of f^2	27	24	24	24	12	12	12
Num. of $p_1 \circ f^2$	27	25	24	24	12	12	12
Num. of $p_2 \circ f^2$	27	24	25	24	12	12	12
Num. of $p_3 \circ f^2$	27	24	24	25	12	12	12
Num. of $m_1 \circ f^2$	33	28	28	28	15	14	14
Num. of $m_2 \circ f^2$	33	28	28	28	14	15	14
Num. of $m_3 \circ f^2$	33	28	28	28	14	14	15

The numerator of the p_1-component of $f^2(m, p)$, as a polynomial of m_k, p_k, contains 64 056 monomials; as a polynomial of m_k, p_k, and ω_k, it contains 1 647 595 terms.

Need new ideas! The main one: find (observe numerically) linear relations between the components of $K_\Phi(x_0)$, and then use them to replace the dynamical relations.

HK-basis Φ_0

Theorem. *At each point* $(m,p)\in\mathbb{R}^6$ *there holds:*

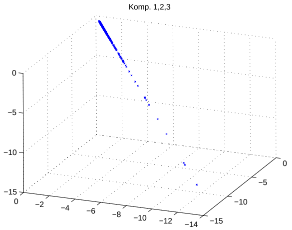
$$K_{\Phi_0}(m,p)=\Big[\,\frac{1}{J_0}+\epsilon^2\omega_1:\frac{1}{J_0}+\epsilon^2\omega_2:\frac{1}{J_0}+\epsilon^2\omega_3:-1\,\Big],$$

where

$$J_0(m,p,\epsilon)=\frac{p_1^2+p_2^2+p_3^2}{1-\epsilon^2(\omega_1p_1^2+\omega_2p_2^2+\omega_3p_3^2)}.$$

This function is an integral of motion of the map f.

This is the only “simple” integral of f!



Plot of solutions (c_1,c_2,c_3) of

$(c_1p_1^2+c_2p_2^2+c_3p_3^2)\circ f^i=1,\ \ i=0,1,2.$

Straight line (two linear relations)!

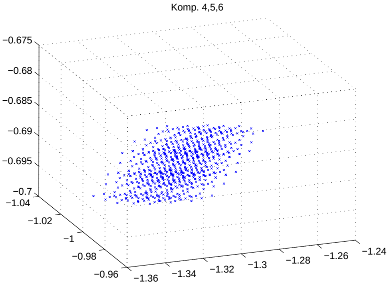
Additional HK-basis $\Psi = (p_1^2, p_2^2, p_3^2, m_1p_1, m_2p_2, m_3p_3)$

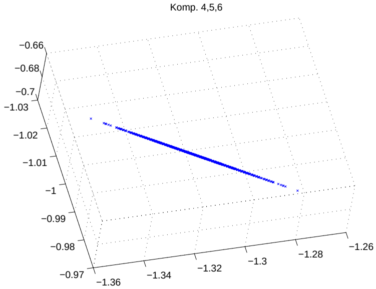
Important (numerical) observation: the homogeneous system

$$(d_1p_1^2 + d_2p_2^2 + d_3p_3^2 + d_7m_1p_1 + d_8m_2p_2 + d_9m_3p_3) \circ f^i = 0$$

has a 1-dim space of solutions with $d_1 = d_2 = d_3$. Normalizing this to -1, consider the non-homogeneous system

$$(d_7m_1p_1 + d_8m_2p_2 + d_9m_3p_3) \circ f^i = (p_1^2 + p_2^2 + p_3^2) \circ f^i, \quad i = 0, 1, 2.$$





HK-basis Ψ (continued)

Solutions (d_7, d_8, d_9) lie (visually) on a plane in $\mathbb{R}^3$. Equation of this plane can be determined (numerically) with the PSLQ algorithm:

$$(\omega_2 - \omega_3)d_7 + (\omega_3 - \omega_1)d_8 + (\omega_1 - \omega_2)d_9 = 0.$$

This equation replaces the one with $i = 2$ in the above system. The resulting system can be solved symbolically, with solutions (d_7, d_8, d_9) being even functions in ϵ (each of them takes 3 pages of MAPLE output). This *proves* the next theorem.

Theorem. *At each point* $(m,p)\in\mathbb{R}^6$ *there holds:*

$$K_\Psi(m,p)=[-1:-1:-1:d_7:d_8:d_9],$$

with

$$d_k=\frac{(p_1^2+p_2^2+p_3^2)(1+\epsilon^2 d_k^{(2)}+\epsilon^4 d_k^{(4)}+\epsilon^6 d_k^{(6)})}{\Delta},\quad k=7,8,9,$$

$$\Delta=m_1p_1+m_2p_2+m_3p_3+\epsilon^2\Delta^{(4)}+\epsilon^4\Delta^{(6)}+\epsilon^6\Delta^{(8)},$$

where $d_k^{(2q)}$ *and* $\Delta^{(2q)}$ *are homogeneous polynomials of degree* $2q$ *in phase variables. The functions* d_7, d_8, d_9 *are integrals of the map* f. *Any two of them together with* J_0 *are functionally independent.*

HK-bases Φ_1, Φ_2, Φ_3

Theorem. *At each point* $(m,p) \in \mathbb{R}^6$ *there holds:*

$$\begin{aligned} K_{\Phi_1}(m,p) &= [\alpha_1 : \alpha_2 : \alpha_3 : \alpha_4 : \alpha_5 : \alpha_6 : -1], \\ K_{\Phi_2}(m,p) &= [\beta_1 : \beta_2 : \beta_3 : \beta_4 : \beta_5 : \beta_6 : -1], \\ K_{\Phi_3}(m,p) &= [\gamma_1 : \gamma_2 : \gamma_3 : \gamma_4 : \gamma_5 : \gamma_6 : -1], \end{aligned}$$

where α_j,β_j, *and* γ_j *are rational functions of* (m,p), *even with respect to* ϵ. *They are integrals of motion of the map* f. *For* $j = 1, 2, 3$ *they are of the form*

$$h = \frac{h^{(2)} + \epsilon^2 h^{(4)} + \epsilon^4 h^{(6)} + \epsilon^6 h^{(8)} + \epsilon^8 h^{(10)} + \epsilon^{10} h^{(12)}}{2\epsilon^2(p_1^2 + p_2^2 + p_3^2)\Delta},$$

where h *stands for any of the functions* $\alpha_j, \beta_j, \gamma_j$, $j = 1, 2, 3$, *and the corresponding* $h^{(2q)}$ *are homogeneous polynomials in phase variables of degree* $2q$. *For instance,*

HK-bases Φ_1, Φ_2, Φ_3 (continued)

$$\begin{aligned}
\alpha_1^{(2)} &= H_3 - I_1, & \alpha_2^{(2)} &= -I_1, & \alpha_3^{(2)} &= -I_1,\\
\beta_1^{(2)} &= -I_2, & \beta_2^{(2)} &= H_3 - I_2, & \beta_3^{(2)} &= -I_2,\\
\gamma_1^{(2)} &= -I_3, & \gamma_2^{(2)} &= -I_3, & \gamma_3^{(2)} &= H_3 - I_3,
\end{aligned}$$

where $H_3 = p_1^2 + p_2^2 + p_3^2$. *The four integrals* J_0, α_1, β_1 *and* γ_1 *are functionally independent.*

Concluding remarks

We established the integrability of the Hirota-Kimura discretization of the Clebsch system, in the sense of

- existence, for every initial point $(m, p) \in \mathbb{R}^6$, of a four-dimensional pencil of quadrics containing the orbit of this point;
- existence of four functionally independent integrals of motion (conserved quantities).

This remains true also for an arbitrary flow of the Clebsch system (with one “simple” and three very big integrals).

Our proofs are computer assisted. We did not find a general structure, which would provide us with less computational proofs and with more insight. In particular, nothing like a Lax representation has been found. Nothing is known about the existence of an invariant Poisson structure for these maps.

Conjecture

Conjecture. *For any algebraically completely integrable system with a quadratic vector field, its Hirota-Kimura discretization remains algebraically completely integrable.*

Supported by the previous discussion and preliminary results on:

- Zhukovsky-Volterra gyrostat;
- $so(4)$ Euler top and its commuting flows;
- Volterra lattice;
- Toda lattice;
- classical Gaudin magnet;
- Suslov system (see posting by Dragovic and Gajic on arXiv from July 18).

If true, this statement could be related to addition theorems for multi-dimensional theta-functions.