Bilinear discretization
of quadratic vector fields,
or
The strange case of Prof. Kahan and Hirota-sensei

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Consider a completely integrable flow

\[ \dot{x} = f(x) = \{H, x\} \quad (1) \]

with a Hamilton function $H$ on a Poisson manifold $\mathcal{P}$ with a Poisson bracket $\{\cdot, \cdot\}$. Thus, the flow (1) possesses many functionally independent integrals $I_k(x)$ in involution.

The *problem of integrable discretization*: find a family of diffeomorphisms $\mathcal{P} \rightarrow \mathcal{P}$,

\[ \tilde{x} = \Phi(x; \epsilon), \quad (2) \]

depending smoothly on a small parameter $\epsilon > 0$, with the following properties:
1. The maps (2) approximate the flow (1):

\[ \Phi(x; \epsilon) = x + \epsilon f(x) + O(\epsilon^2). \]

2. The maps (2) are Poisson w. r. t. the bracket \( \{ \cdot, \cdot \} \) or some its deformation \( \{ \cdot, \cdot \}_\epsilon = \{ \cdot, \cdot \} + O(\epsilon) \).

3. The maps (2) are integrable, i.e. possess the necessary number of independent integrals in involution, \( I_k(x; \epsilon) = I_k(x) + O(\epsilon) \).


Reasons for this omission: discretization of the Euler top seemed to be an isolated curiosity; discretization of the Lagrange top seemed to be completely incomprehensible, if not even wrong.
Hirota-Kimura’s discrete time Euler top

\[
\begin{aligned}
\dot{x}_1 &= \alpha_1 x_2 x_3, \\
\dot{x}_2 &= \alpha_2 x_3 x_1, \\
\dot{x}_3 &= \alpha_3 x_1 x_2,
\end{aligned}
\quad \implies \quad
\begin{aligned}
\tilde{x}_1 - x_1 &= \epsilon \alpha_1 (\tilde{x}_2 x_3 + x_2 \tilde{x}_3), \\
\tilde{x}_2 - x_2 &= \epsilon \alpha_2 (\tilde{x}_3 x_1 + x_3 \tilde{x}_1), \\
\tilde{x}_3 - x_3 &= \epsilon \alpha_3 (\tilde{x}_1 x_2 + x_1 \tilde{x}_2).
\end{aligned}
\]

Features:

▶ Equations are linear w.r.t. \( \tilde{x} = (\tilde{x}_1, \tilde{x}_2, \tilde{x}_3)^T \), result in an explicit (rational) map: \( \tilde{x} = f(x, \epsilon) = A^{-1}(x, \epsilon)x \),

\[
A(x, \epsilon) = \begin{pmatrix}
1 & -\epsilon \alpha_1 x_3 & -\epsilon \alpha_1 x_2 \\
-\epsilon \alpha_2 x_3 & 1 & -\epsilon \alpha_2 x_1 \\
-\epsilon \alpha_3 x_2 & -\epsilon \alpha_3 x_1 & 1
\end{pmatrix}.
\]

▶ The map is reversible (therefore birational):

\[
f^{-1}(x, \epsilon) = f(x, -\epsilon).
\]
Explicit formulas rather messy:

\[
\begin{align*}
\tilde{x}_1 &= x_1 + 2\epsilon \alpha_1 x_2 x_3 + \epsilon^2 x_1 (-\alpha_2 \alpha_3 x_1^2 + \alpha_3 \alpha_1 x_2^2 + \alpha_1 \alpha_2 x_3^2) / \Delta(x, \epsilon), \\
\tilde{x}_2 &= x_2 + 2\epsilon \alpha_2 x_3 x_1 + \epsilon^2 x_2 (\alpha_2 \alpha_3 x_1^2 - \alpha_3 \alpha_1 x_2^2 + \alpha_1 \alpha_2 x_3^2) / \Delta(x, \epsilon), \\
\tilde{x}_3 &= x_3 + 2\epsilon \alpha_3 x_1 x_2 + \epsilon^2 x_3 (\alpha_2 \alpha_3 x_1^2 + \alpha_3 \alpha_1 x_2^2 - \alpha_1 \alpha_2 x_3^2) / \Delta(x, \epsilon),
\end{align*}
\]

where

\[
\Delta(x, \epsilon) = \det A(x, \epsilon) = 1 - \epsilon^2 (\alpha_2 \alpha_3 x_1^2 + \alpha_3 \alpha_1 x_2^2 + \alpha_1 \alpha_2 x_3^2) - 2\epsilon^3 \alpha_1 \alpha_2 \alpha_3 x_1 x_2 x_3.
\]

(Try to see reversibility directly from these formulas!)
Two independent integrals:

\[ I_1(x, \epsilon) = \frac{1 - \epsilon^2 \alpha_2 \alpha_3 x_1^2}{1 - \epsilon^2 \alpha_3 \alpha_1 x_2^2}, \quad I_2(x, \epsilon) = \frac{1 - \epsilon^2 \alpha_3 \alpha_1 x_2^2}{1 - \epsilon^2 \alpha_1 \alpha_2 x_3^2}. \]


\[
\dot{x} = Q(x) + Bx \quad \leadsto \quad (\tilde{x} - x)/\epsilon = Q(x, \tilde{x}) + B(x + \tilde{x}),
\]

where \( B \in \mathbb{R}^{n \times n}, \ Q : \mathbb{R}^n \to \mathbb{R}^n \) is a *quadratic* function, and

\[
Q(x, \tilde{x}) = Q(x + \tilde{x}) - Q(x) - Q(\tilde{x})
\]

is the corresponding symmetric *bilinear* function.

Note: equations for \( \tilde{x} \) always linear, \( \tilde{x} = f(x, \epsilon) = A^{-1}(x, \epsilon)x \), the map is always reversible and birational, \( f^{-1}(x, \epsilon) = f(x, -\epsilon) \).
Kahan’s integrator for the Lotka-Volterra system:

\[
\begin{align*}
\dot{x} &= x(1 - y), \\
\dot{y} &= y(x - 1),
\end{align*}
\]

\[
\begin{align*}
\tilde{x} - x &= \epsilon(\tilde{x} + x) - \epsilon(\tilde{x}y + x\tilde{y}), \\
\tilde{y} - y &= \epsilon(\tilde{x}y + x\tilde{y}) - \epsilon(\tilde{y} + y).
\end{align*}
\]

Explicitly:

\[
\begin{align*}
\tilde{x} &= x \frac{(1 + \epsilon)^2 - \epsilon(1 + \epsilon)x - \epsilon(1 - \epsilon)y}{1 - \epsilon^2 - \epsilon(1 - \epsilon)x + \epsilon(1 + \epsilon)y}, \\
\tilde{y} &= y \frac{(1 - \epsilon)^2 + \epsilon(1 + \epsilon)x + \epsilon(1 - \epsilon)y}{1 - \epsilon^2 - \epsilon(1 - \epsilon)x + \epsilon(1 + \epsilon)y}.
\end{align*}
\]
Left: three orbits of Kahan’s discretization with $\epsilon = 0.1$, right: one orbit of the explicit Euler with $\epsilon = 0.01$.


A sort of an explanation of a non-spiralling behavior: Kahan’s integrator for the Lotka-Volterra system in Poisson.
Hirota-Kimura’s discrete time Lagrange top

\[
\begin{cases}
\dot{\omega}_1 = (1 - \alpha)\omega_2\omega_3 + Z_0\gamma_2, \\
\dot{\omega}_2 = -(1 - \alpha)\omega_3\omega_1 - Z_0\gamma_1, \\
\dot{\omega}_3 = 0, \\
\dot{\gamma}_1 = \omega_3\gamma_2 - \omega_2\gamma_3, \\
\dot{\gamma}_2 = \omega_1\gamma_3 - \omega_3\gamma_1, \\
\dot{\gamma}_3 = \omega_2\gamma_1 - \omega_1\gamma_2,
\end{cases}
\]

which gives a birational map \((\tilde{\omega}, \tilde{\gamma}) = f(\omega, \gamma, \epsilon)\).
Consider

\[ A = \omega_1^2 + \omega_2^2 - B\gamma_3 - C\gamma_3^2, \]

\[ D = \omega_1\gamma_1 + \omega_2\gamma_2 - E\gamma_3 - F\gamma_3^2, \quad K = \gamma_1^2 + \gamma_2^2 - L\gamma_3 - M\gamma_3^2. \]

Determine \(A, \ldots, M\) by requiring that they are conserved quantities. For instance, for \(A, B, C\), solve the system of three equations for these three unknowns:

\[
\begin{cases}
A + B\tilde{\gamma}_3 + C\tilde{\gamma}_3^2 = \tilde{\omega}_1^2 + \tilde{\omega}_2^2, \\
A + B\gamma_3 + C\gamma_3^2 = \omega_1^2 + \omega_2^2, \\
A + B\tilde{\gamma}_3 + C\tilde{\gamma}_3^2 = \tilde{\omega}_1^2 + \tilde{\omega}_2^2
\end{cases}
\]

with \((\tilde{\omega}, \tilde{\gamma}) = f(\omega, \gamma, \epsilon)\) and \((\omega, \gamma) = f^{-1}(\omega, \gamma, \epsilon)\). Then check that \(A, B, C = A, B, C(\omega, \gamma, \epsilon)\) are conserved quantities, indeed.

Does this make any sense for you???

Nevertheless, this turns out to be not only true but also remarkably deep (as everything by R. Hirota...).
Clebsch system

Clebsch system describes the motion of a rigid body in an ideal fluid:

\[
\begin{align*}
\dot{m}_1 &= (\omega_3 - \omega_2)p_2p_3, \\
\dot{m}_2 &= (\omega_1 - \omega_3)p_3p_1, \\
\dot{m}_3 &= (\omega_2 - \omega_1)p_1p_2, \\
\dot{p}_1 &= m_3p_2 - m_2p_3, \\
\dot{p}_2 &= m_1p_3 - m_3p_1, \\
\dot{p}_3 &= m_2p_1 - m_1p_2.
\end{align*}
\]

It is Hamiltonian w.r.t. Lie-Poisson bracket of \(e(3)\), has four functionally independent integrals in involution:

\[
I_i = p_i^2 + \frac{m_j^2}{\omega_k - \omega_i} + \frac{m_k^2}{\omega_j - \omega_i}, \quad (i, j, k) = \text{c.p.}(1, 2, 3),
\]

and \(H_4 = m_1p_1 + m_2p_2 + m_3p_3\).
Hirota-Kimura discretization of the Clebsch system

Hirota-Kimura-type discretization (proposed by T. Ratiu on Oberwolfach Meeting “Geometric Integration”, March 2006):

\[ \tilde{m}_1 - m_1 = \epsilon(\omega_3 - \omega_2)(\tilde{p}_2p_3 + p_2\tilde{p}_3), \]
\[ \tilde{m}_2 - m_2 = \epsilon(\omega_1 - \omega_3)(\tilde{p}_3p_1 + p_3\tilde{p}_1), \]
\[ \tilde{m}_3 - m_3 = \epsilon(\omega_2 - \omega_1)(\tilde{p}_1p_2 + p_1\tilde{p}_2), \]
\[ \tilde{p}_1 - p_1 = \epsilon(\tilde{m}_3p_2 + m_3\tilde{p}_2) - \epsilon(\tilde{m}_2p_3 + m_2\tilde{p}_3), \]
\[ \tilde{p}_2 - p_2 = \epsilon(\tilde{m}_1p_3 + m_1\tilde{p}_3) - \epsilon(\tilde{m}_3p_1 + m_3\tilde{p}_1), \]
\[ \tilde{p}_3 - p_3 = \epsilon(\tilde{m}_2p_1 + m_2\tilde{p}_1) - \epsilon(\tilde{m}_1p_2 + m_1\tilde{p}_2). \]

What follows is based on a joint work with Matteo Petrera (Rome) and Andreas Pfadler (München).
A birational map

\[
\begin{pmatrix}
\tilde{m} \\
\tilde{p}
\end{pmatrix} = f(m, p, \epsilon) = M^{-1}(m, p, \epsilon) \begin{pmatrix} m \\ p \end{pmatrix},
\]

\[
M(m, p, \epsilon) = \begin{pmatrix}
1 & 0 & 0 & 0 & \epsilon \omega_{31} p_3 & \epsilon \omega_{31} p_2 \\
0 & 1 & 0 & \epsilon \omega_{31} p_3 & 0 & \epsilon \omega_{31} p_1 \\
0 & 0 & 1 & \epsilon \omega_{12} p_3 & \epsilon \omega_{12} p_1 & 0 \\
0 & \epsilon p_3 & -\epsilon p_2 & 1 & -\epsilon m_3 & \epsilon m_2 \\
-\epsilon p_3 & 0 & \epsilon p_1 & \epsilon m_3 & 1 & -\epsilon m_1 \\
\epsilon p_2 & -\epsilon p_1 & 0 & -\epsilon m_2 & \epsilon m_1 & 1
\end{pmatrix},
\]

with \( \omega_{ij} = \omega_i - \omega_j \). The usual reversibility:

\[
f^{-1}(m, p, \epsilon) = f(m, p, -\epsilon).
\]

Numerators and denominators of components of \( \tilde{m}, \tilde{p} \) are polynomials of degree 6, the numerators of \( \tilde{p}_i \) consist of 31 monomials, the numerators of \( \tilde{m}_i \) consist of 41 monomials, the common denominator consists of 28 monomials.
An orbit of the discrete Clebsch system with $\omega_1 = 0.1$, $\omega_2 = 0.2$, $\omega_3 = 0.3$ and $\epsilon = 1$; projections to $(m_1, m_2, m_3)$ and to $(p_1, p_2, p_3)$; initial point $(m_0, p_0) = (1, 1, 1, 1, 1, 1)$.
An orbit of the discrete Clebsch system with $\omega_1 = 1$, $\omega_2 = 0.2$, $\omega_3 = 30$ and $\epsilon = 1$; projections to $(m_1, m_2, m_3)$ and to $(p_1, p_2, p_3)$; initial point $(m_0, p_0) = (1, 1, 1, 1, 1, 1)$. 
**Definition.** For a given birational map \( f : \mathbb{R}^n \to \mathbb{R}^n \), a set of functions \( \Phi = (\varphi_1, \ldots, \varphi_l) \), linearly independent over \( \mathbb{R} \), is called a **HK-basis**, if for every \( x_0 \in \mathbb{R}^n \) there exists a vector \( c = (c_1, \ldots, c_l) \neq 0 \) such that

\[
c_1 \varphi_1(f^i(x_0)) + \ldots + c_l \varphi_l(f^i(x_0)) = 0 \quad \forall i \in \mathbb{Z}.
\]

For a given \( x_0 \in \mathbb{R}^n \), the set of all vectors \( c \in \mathbb{R}^l \) with this property will be denoted by \( K_\Phi(x_0) \) and called the null-space of the basis \( \Phi \) (at the point \( x_0 \)). This set clearly is a vector space.

Note: we cannot claim that \( h = c_1 \varphi_1 + \ldots + c_l \varphi_l \) is an integral of motion, since vectors \( c \in K_\Phi(x_0) \) vary from one initial point \( x_0 \) to another.

However: existence of a HK-basis \( \Phi \) with \( \dim K_\Phi(x_0) = d \) confines the orbits of \( f \) to \((n - d)\)-dimensional invariant sets.
Proposition. If $\Phi$ is a HK-basis for a map $f$, then $K_\Phi(f(x_0)) = K_\Phi(x_0)$.

Thus, the $d$-dimensional null-space $K_\Phi(x_0)$ is a $Gr(d, l)$-valued integral. Its Plücker coordinates are real-valued integrals:

Corollary. Let $\Phi$ be a HK-basis for $f$ with $\dim K_\Phi(x_0) = d$ for all $x_0 \in \mathbb{R}^n$. Take a basis of $K_\Phi(x_0)$ consisting of $d$ vectors $c^{(i)} \in \mathbb{R}^l$ and put them into the columns of a $l \times d$ matrix $C(x_0)$. For any $d$-index $\alpha = (\alpha_1, \ldots, \alpha_d) \subset \{1, 2, \ldots, n\}$ let $C_\alpha = C_{\alpha_1 \ldots \alpha_d}$ denote the $d \times d$ minor of the matrix $C$ built from the rows $\alpha_1, \ldots, \alpha_d$. Then for any two $d$-indices $\alpha, \beta$ the function $C_\alpha / C_\beta$ is an integral of $f$. 
Especially simple is the situation when the null-space of a HK-basis has dimension $d = 1$.

**Corollary.** Let $\Phi$ be a HK-basis for $f$ with $\dim K_\Phi(x_0) = 1$ for all $x_0 \in \mathbb{R}^n$. Let $K_\Phi(x_0) = [c_1(x_0) : \ldots : c_l(x_0)] \in \mathbb{RP}^{l-1}$. Then the functions $c_j/c_k$ are integrals of motion for $f$.

In other words, normalizing $c_l(x_0) = 1$ (say), we find that all other $c_j$ ($j = 1, \ldots, l - 1$) are integrals of motion. It is not clear whether one can say something general about the number of functionally independent integrals among them. It varies in examples (sometimes just $= 1$ and sometimes $> 1$).
Finding HK-bases

**Theorem.** Let, for all \( x_0 \in \mathbb{R}^n \), the dimension of the solution space of the homogeneous system for \( c_1, \ldots, c_l \),

\[
c_1 \varphi_1(f^i(x_0)) + \ldots + c_l \varphi_l(f^i(x_0)) = 0, \quad i = 0, \ldots, s - 1,
\]

be equal to \( l - s \) for \( 1 \leq s \leq l - d \) and to \( d \) for \( s = l - d + 1 \). Then \( K_\Phi(x_0) \) coincides with the solution space for \( s = l - d \), and, in particular, \( \dim K_\Phi(x_0) = d \).

**Numerical algorithm:**

\( (N) \) For several randomly chosen initial points \( x_0 \in \mathbb{R}^n \), compute the dimension of the solution space of the above system for \( 1 \leq s \leq l \). If for every \( x_0 \) the dimension fails to drop after \( s = l - d \) with one and the same \( d \geq 1 \), then \( \Phi \) is likely to be a HK-basis for \( f \), with \( \dim K_\Phi(x_0) = d \).

Especially important case: \( d = 1 \).
To prove that some $\Phi$ is a HK-basis, have to check the conditions of the theorem symbolically. Some possible tricks (for $d = 1$):

(A) Consider the non-homogeneous system of $l - 1$ equations

$$c_1 \varphi_1(f^i(x_0)) + \ldots + c_{l-1} \varphi_{l-1}(f^i(x_0)) = \varphi_l(f^i(x_0))$$

for two different but overlapping ranges $i \in [i_0, i_0 + l - 2]$ and $i \in [i_1, i_1 + l - 2]$. If the solutions coincide, then $\Phi$ is a HK-basis with $d = 1$.

(B) Consider the above system for the index range $i \in [i_0, i_0 + l - 2]$ which contains 0 but is non-symmetric. If the solution functions $c_1(x_0, \epsilon), \ldots, c_{l-1}(x_0, \epsilon)$ are even w.r.t. $\epsilon$, then $\Phi$ is a HK-basis with $d = 1$. 
Theorem. a) The set of functions

\[ \Phi = (p_1^2, p_2^2, p_3^2, m_1^2, m_2^2, m_3^2, m_1 p_1, m_2 p_2, m_3 p_3, 1) \]

is a HK-basis for \( f \), with \( \dim K_\Phi(m, p) = 4 \). Thus, any orbit of \( f \) lies on an intersection of four quadrics in \( \mathbb{R}^6 \).

b) The following four sets of functions are HK-bases for \( f \) with one-dimensional null-spaces:

\[
\begin{align*}
\Phi_0 &= (p_1^2, p_2^2, p_3^2, 1), \\
\Phi_1 &= (p_1^2, p_2^2, p_3^2, m_1^2, m_2^2, m_3^2, m_1 p_1), \\
\Phi_2 &= (p_1^2, p_2^2, p_3^2, m_1^2, m_2^2, m_3^2, m_2 p_2), \\
\Phi_3 &= (p_1^2, p_2^2, p_3^2, m_1^2, m_2^2, m_3^2, m_3 p_3).
\end{align*}
\]

There holds: \( K_\Phi = K_{\Phi_0} \oplus K_{\Phi_1} \oplus K_{\Phi_2} \oplus K_{\Phi_3} \).
The claims in part b) of the above theorem refer to the solutions of the following systems:

\[(c_1p_1^2 + c_2p_2^2 + c_3p_3^2) \circ f^i = 1,\]

\[(\alpha_1p_1^2 + \alpha_2p_2^2 + \alpha_3p_3^2 + \alpha_4m_1^2 + \alpha_5m_2^2 + \alpha_6m_3^2) \circ f^i = m_1p_1 \circ f^i,\]

\[(\beta_1p_1^2 + \beta_2p_2^2 + \beta_3p_3^2 + \beta_4m_1^2 + \beta_5m_2^2 + \beta_6m_3^2) \circ f^i = m_2p_2 \circ f^i,\]

\[(\gamma_1p_1^2 + \gamma_2p_2^2 + \gamma_3p_3^2 + \gamma_4m_1^2 + \gamma_5m_2^2 + \gamma_6m_3^2) \circ f^i = m_3p_3 \circ f^i.\]

The first one has to be solved for one non-symmetric range of \(l - 1 = 3\) values of \(i\), or for two different such ranges. The last three systems have to be solved for a non-symmetric range of \(l - 1 = 6\) values of \(i\). This can be done numerically (in rational arithmetic) without any difficulties, but becomes (nearly) impossible for a symbolic computation, due to complexity of \(f^2\).
### Complexity of $f^2$

#### Degrees of numerators and denominators of $f^2$:

<table>
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<tr>
<th></th>
<th>deg</th>
<th>$\deg_{p_1}$</th>
<th>$\deg_{p_2}$</th>
<th>$\deg_{p_3}$</th>
<th>$\deg_{m_1}$</th>
<th>$\deg_{m_2}$</th>
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<tr>
<td>Denom. of $f^2$</td>
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<td>12</td>
<td>12</td>
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<tr>
<td>Num. of $p_1 \circ f^2$</td>
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<td>25</td>
<td>24</td>
<td>24</td>
<td>12</td>
<td>12</td>
<td>12</td>
</tr>
<tr>
<td>Num. of $p_2 \circ f^2$</td>
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<td>24</td>
<td>25</td>
<td>24</td>
<td>12</td>
<td>12</td>
<td>12</td>
</tr>
<tr>
<td>Num. of $p_3 \circ f^2$</td>
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<td>24</td>
<td>24</td>
<td>25</td>
<td>12</td>
<td>12</td>
<td>12</td>
</tr>
<tr>
<td>Num. of $m_1 \circ f^2$</td>
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<td>28</td>
<td>28</td>
<td>28</td>
<td>15</td>
<td>14</td>
<td>14</td>
</tr>
<tr>
<td>Num. of $m_2 \circ f^2$</td>
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<td>28</td>
<td>28</td>
<td>28</td>
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<tr>
<td>Num. of $m_3 \circ f^2$</td>
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<td>28</td>
<td>28</td>
<td>14</td>
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<td>15</td>
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</tbody>
</table>

The numerator of the $p_1$-component of $f^2(m,p)$, as a polynomial of $m_k, p_k$, contains 64,056 monomials; as a polynomial of $m_k, p_k$, and $\omega_k$, it contains 1,647,595 terms.

Need new ideas! The main one: find (observe numerically) linear relations between the components of $K_\Phi(x_0)$, and then use them to replace the dynamical relations.
Theorem. At each point \((m, p) \in \mathbb{R}^6\) there holds:

\[
K_{\Phi_0}(m, p) = \left[ \frac{1}{J_0} + \epsilon^2 \omega_1 : \frac{1}{J_0} + \epsilon^2 \omega_2 : \frac{1}{J_0} + \epsilon^2 \omega_3 : -1 \right],
\]

where

\[
J_0(m, p, \epsilon) = \frac{p_1^2 + p_2^2 + p_3^2}{1 - \epsilon^2(\omega_1 p_1^2 + \omega_2 p_2^2 + \omega_3 p_3^2)}.
\]

This function is an integral of motion of the map \(f\).
This is the only “simple” integral of \(f\)!

Plot of solutions \((c_1, c_2, c_3)\) of

\[
(c_1 p_1^2 + c_2 p_2^2 + c_3 p_3^2) \circ f^i = 1, \quad i = 0, 1, 2.
\]

Straight line (two linear relations)!
Additional HK-basis $\Psi = (p_1^2, p_2^2, p_3^2, m_1 p_1, m_2 p_2, m_3 p_3)$

Important (numerical) observation: the homogeneous system

$$(d_1 p_1^2 + d_2 p_2^2 + d_3 p_3^2 + d_7 m_1 p_1 + d_8 m_2 p_2 + d_9 m_3 p_3) \circ f^i = 0$$

has a 1-dim space of solutions with $d_1 = d_2 = d_3$. Normalizing this to $-1$, consider the non-homogeneous system

$$(d_7 m_1 p_1 + d_8 m_2 p_2 + d_9 m_3 p_3) \circ f^i = (p_1^2 + p_2^2 + p_3^2) \circ f^i, \quad i = 0, 1, 2.$$
HK-basis $\Psi$ (continued)

Solutions $(d_7, d_8, d_9)$ lie (visually) on a plane in $\mathbb{R}^3$. Equation of this plane can be determined (numerically) with the PSLQ algorithm:

$$(\omega_2 - \omega_3)d_7 + (\omega_3 - \omega_1)d_8 + (\omega_1 - \omega_2)d_9 = 0.$$ 

This equation replaces the one with $i = 2$ in the above system. The resulting system can be solved symbolically, with solutions $(d_7, d_8, d_9)$ being even functions in $\epsilon$ (each of them takes 3 pages of MAPLE output). This proves the next theorem.
Theorem. At each point \((m, p) \in \mathbb{R}^6\) there holds:

\[
K_{\Psi}(m, p) = [-1 : -1 : -1 : d_7 : d_8 : d_9],
\]

with

\[
d_k = \frac{(p_1^2 + p_2^2 + p_3^2)(1 + \epsilon^2 d_k^{(2)} + \epsilon^4 d_k^{(4)} + \epsilon^6 d_k^{(6)})}{\Delta}, \quad k = 7, 8, 9,
\]

\[
\Delta = m_1 p_1 + m_2 p_2 + m_3 p_3 + \epsilon^2 \Delta^{(4)} + \epsilon^4 \Delta^{(6)} + \epsilon^6 \Delta^{(8)},
\]

where \(d_k^{(2q)}\) and \(\Delta^{(2q)}\) are homogeneous polynomials of degree \(2q\) in phase variables. The functions \(d_7, d_8, d_9\) are integrals of the map \(f\). Any two of them together with \(J_0\) are functionally independent.
Theorem. At each point \((m, p) \in \mathbb{R}^6\) there holds:

\[
K_{\Phi_1}(m, p) = [\alpha_1 : \alpha_2 : \alpha_3 : \alpha_4 : \alpha_5 : \alpha_6 : -1],
\]

\[
K_{\Phi_2}(m, p) = [\beta_1 : \beta_2 : \beta_3 : \beta_4 : \beta_5 : \beta_6 : -1],
\]

\[
K_{\Phi_3}(m, p) = [\gamma_1 : \gamma_2 : \gamma_3 : \gamma_4 : \gamma_5 : \gamma_6 : -1],
\]

where \(\alpha_j, \beta_j, \text{ and } \gamma_j\) are rational functions of \((m, p)\), even with respect to \(\epsilon\). They are integrals of motion of the map \(f\). For \(j = 1, 2, 3\) they are of the form

\[
h = \frac{h^{(2)} + \epsilon^2 h^{(4)} + \epsilon^4 h^{(6)} + \epsilon^6 h^{(8)} + \epsilon^8 h^{(10)} + \epsilon^{10} h^{(12)}}{2\epsilon^2(p_1^2 + p_2^2 + p_3^2)\Delta},
\]

where \(h\) stands for any of the functions \(\alpha_j, \beta_j, \gamma_j, j = 1, 2, 3\), and the corresponding \(h^{(2q)}\) are homogeneous polynomials in phase variables of degree \(2q\). For instance,
where $H_3 = p_1^2 + p_2^2 + p_3^2$. The four integrals $J_0, \alpha_1, \beta_1$ and $\gamma_1$ are functionally independent.
Concluding remarks

We established the integrability of the Hirota-Kimura discretization of the Clebsch system, in the sense of

- existence, for every initial point \((m, p) \in \mathbb{R}^6\), of a four-dimensional pencil of quadrics containing the orbit of this point;
- existence of four functionally independent integrals of motion (conserved quantities).

This remains true also for an arbitrary flow of the Clebsch system (with one “simple” and three very big integrals).

Our proofs are computer assisted. We did not find a general structure, which would provide us with less computational proofs and with more insight. In particular, nothing like a Lax representation has been found. Nothing is known about the existence of an invariant Poisson structure for these maps.
Conjecture. *For any algebraically completely integrable system with a quadratic vector field, its Hirota-Kimura discretization remains algebraically completely integrable.*

Supported by the previous discussion and preliminary results on:

- Zhukovsky-Volterra gyrostat;
- \(\text{so}(4)\) Euler top and its commuting flows;
- Volterra lattice;
- Toda lattice;
- classical Gaudin magnet;
- Suslov system (see posting by Dragovic and Gajic on arXiv from July 18).

If true, this statement could be related to addition theorems for multi-dimensional theta-functions.