LAGRANGIAN AND HAMILTONIAN STRUCTURE OF COMPLEX FLUIDS

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PLAN OF THE PRESENTATION

• Affine Lagrangian and Hamiltonian reduction

• Example 1: Ericksen-Leslie equations

• Example 2: Eringen equations

PUNCHLINE: All these equations are obtained by Euler-Poincaré and Lie-Poisson reduction from material representation. These reduction procedures need to be extended to include affine terms and the groups have a relatively complicated internal structure adapted to complex fluids.

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AFFINE LAGRANGIAN AND HAMILTONIAN REDUCTION

\( \rho : G \rightarrow \text{Aut}(V) \) \textit{right} representation, \( S = G \circledast V \); multiplication is

\[(g_1, v_1)(g_2, v_2) = (g_1g_2, v_2 + \rho_{g_2}(v_1)).\]

The Lie algebra \( \mathfrak{s} = \mathfrak{g} \circledast V \) of \( S \) has bracket

\[\text{ad}_{(\xi_1, v_1)}(\xi_2, v_2) = [(\xi_1, v_1), (\xi_2, v_2)] = (\lbrack \xi_1, \xi_2 \rbrack, v_1\xi_2 - v_2\xi_1),\]

where \( v\xi \) denotes the induced action of \( \mathfrak{g} \) on \( V \), that is,

\[v\xi := \frac{d}{dt} \bigg|_{t=0} \rho_{\exp(t\xi)}(v) \in V.\]

If \( (\xi, v) \in \mathfrak{s} \) and \( (\mu, a) \in \mathfrak{s}^* \) we have

\[\text{ad}_{(\xi, v)}^*(\mu, a) = (\text{ad}_{\xi}^* \mu + v \circ a, a\xi),\]

where \( a\xi \in V^* \) and \( v \circ a \in \mathfrak{g}^* \) are given by

\[a\xi := \frac{d}{dt} \bigg|_{t=0} \rho_{\exp(-t\xi)}^*(a) \quad \text{and} \quad \langle v \circ a, \xi \rangle_{\mathfrak{g}} := -\langle a\xi, v \rangle_V,\]

\( \langle \cdot, \cdot \rangle_{\mathfrak{g}} : \mathfrak{g}^* \times \mathfrak{g} \rightarrow \mathbb{R} \) and \( \langle \cdot, \cdot \rangle_V : V^* \times V \rightarrow \mathbb{R} \) are the duality parings.
Lagrangian semidirect product theory

• $L : TG \times V^* \to \mathbb{R}$ which is right $G$-invariant.

• So, if $a_0 \in V^*$, define the Lagrangian $L_{a_0} : TG \to \mathbb{R}$ by $L_{a_0}(v_g) := L(v_g, a_0)$. Then $L_{a_0}$ is right invariant under the lift to $TG$ of the right action of $G_{a_0}$ on $G$, where $G_{a_0} := \{ g \in G \mid \rho^*_g a_0 = a_0 \}$.

• Right $G$-invariance of $L$ permits us to define $l : \mathfrak{g} \times V^* \to \mathbb{R}$ by

$$l(T_gR_{g^{-1}}(v_g), \rho^*_g(a_0)) = L(v_g, a_0).$$

• For a curve $g(t) \in G$, let $\xi(t) := TR_{g(t)}^{-1}(\dot{g}(t))$ and define the curve $a(t)$ as the unique solution of the following linear differential equation with time dependent coefficients

$$\dot{a}(t) = -a(t)\xi(t),$$

with initial condition $a(0) = a_0$. Solution is $a(t) = \rho^*_{g(t)}(a_0)$.
i With $a_0$ held fixed, Hamilton’s variational principle

$$\delta \int_{t_1}^{t_2} L_{a_0}(g(t), \dot{g}(t)) dt = 0,$$

holds, for variations $\delta g(t)$ of $g(t)$ vanishing at the endpoints.

ii $g(t)$ satisfies the Euler-Lagrange equations for $L_{a_0}$ on $G$.

iii The constrained variational principle

$$\delta \int_{t_1}^{t_2} l(\xi(t), a(t)) dt = 0,$$

holds on $g \times V^*$, upon using variations $(\delta \xi, \delta a)$ of the form

$$\delta \xi = \frac{\partial \eta}{\partial t} - [\xi, \eta], \quad \delta a = - a \eta,$$

where $\eta(t) \in g$ vanishes at the endpoints.

iv The Euler-Poincaré equations hold on $g \times V^*$:

$$\frac{\partial}{\partial t} \frac{\delta l}{\delta \xi} = - \text{ad}^* \frac{\delta l}{\delta \xi} + \frac{\delta l}{\delta a} \diamond a.$$
Hamiltonian semidirect product theory

• \( H : T^*G \times V^* \to \mathbb{R} \) which is right \( G \)-invariant.

• So, if \( a_0 \in V^* \), define the Hamiltonian \( H_{a_0} : TG \to \mathbb{R} \) by \( H_{a_0}(\alpha_g) := H(\alpha_g, a_0) \). Then \( H_{a_0} \) is right invariant under the lift to \( TG \) of the right action of \( G_{a_0} \) on \( G \).

• Right \( G \)-invariance of \( H \) permits us to define \( h : g^* \times V^* \to \mathbb{R} \) by

\[
h(T^*_e R_g(\alpha_g), \rho_g^*(a_0)) = H(\alpha_g, a_0).
\]

For \( \alpha(t) \in T^*_{g(t)}G \) and \( \mu(t) := T^* R_{g(t)}(\alpha(t)) \in g^* \), the following are equivalent:

\[ \alpha(t) \text{ satisfies Hamilton's equations for } H_{a_0} \text{ on } T^*G. \]
The Lie-Poisson equation holds on $\mathfrak{s}^*$:

$$\frac{\partial}{\partial t}(\mu, a) = -\text{ad}^*_{\delta h \delta h}(\mu, a) = -\left(\text{ad}^*_{\delta h \delta \mu} \mu + \frac{\delta h}{\delta a} \diamond a, a \frac{\delta h}{\delta \mu}\right), \quad a(0) = a_0$$

where $\mathfrak{s}$ is the semidirect product Lie algebra $\mathfrak{s} = \mathfrak{g} \circledast \mathfrak{v}$. The associated Poisson bracket is the Lie-Poisson bracket on the semidirect product Lie algebra $\mathfrak{s}^*$, that is,

$$\{f, g\}(\mu, a) = \left\langle \mu, \left[\frac{\delta f}{\delta \mu}, \frac{\delta g}{\delta \mu}\right]\right\rangle + \left\langle a, \frac{\delta f}{\delta a} \frac{\delta g}{\delta \mu} - \frac{\delta g}{\delta a} \frac{\delta f}{\delta \mu}\right\rangle.$$

As on the Lagrangian side, the evolution of the advected quantities is given by $a(t) = \rho^*_{g(t)}(a_0)$.

**Legendre transformation:** $h(\mu, a) := \langle \mu, \xi \rangle - l(\xi, a)$, where $\mu = \frac{\delta l}{\delta \xi}$. If it is invertible, since

$$\frac{\delta h}{\delta \mu} = \xi \quad \text{and} \quad \frac{\delta h}{\delta a} = -\frac{\delta l}{\delta a},$$

the Lie-Poisson equations for $h$ are equivalent to the Euler-Poincaré equations for $l$ together with the advection equation $\dot{a} + a\xi = 0$.

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Let $c \in \mathcal{F}(G, V^*)$ be a right one-cocycle, that is, it verifies the property $c(fg) = \rho_{g^{-1}}^*(c(f)) + c(g)$ for all $f, g \in V^*$. This implies that $c(e) = 0$ and $c(g^{-1}) = -\rho_g^*(c(g))$. Instead of the contragredient representation $\rho_{g^{-1}}^*$ of $G$ on $V^*$ form the affine right representation

$$\theta_g(a) = \rho_{g^{-1}}^*(a) + c(g).$$

Note that

$$\left. \frac{d}{dt} \right|_{t=0} \theta_{\exp(t\xi)}(a) = a\xi + dc(\xi).$$

and

$$\langle a\xi + dc(\xi), v \rangle_V = \langle dc^T(v) - v \diamond a, \xi \rangle_g,$$

where $dc : g \to V^*$ is defined by $dc(\xi) := T_{e\xi}(\xi)$, and $dc^T : V \to g^*$ is defined by

$$\langle dc^T(v), \xi \rangle_g := \langle dc(\xi), v \rangle_V.$$
\begin{itemize}
    
    \item $L : TG \times V^* \to \mathbb{R}$ right $G$-invariant under the affine action $(v_h, a) \mapsto (T_h R_g(v_h), \theta_g(a)) = (T_h R_g(v_h), \rho_{g^{-1}}^*(a) + c(g))$.

    \item So, if $a_0 \in V^*$, define $L_{a_0} : TG \to \mathbb{R}$ by $L_{a_0}(v_g) := L(v_g, a_0)$. Then $L_{a_0}$ is right invariant under the lift to $TG$ of the right action of $G_{a_0}^c$ on $G$, where $G_{a_0}^c := \{g \in G \mid \theta_g(a_0) = a_0\}$.

    \item Right $G$-invariance of $L$ permits us to define $l : g \times V^* \to \mathbb{R}$ by $l(T_g R_{g^{-1}}(v_g), \theta_{g^{-1}}(a_0)) = L(v_g, a_0)$.

    \item For a curve $g(t) \in G$, let $\xi(t) := TR_{g(t)^{-1}}(\dot{g}(t))$ and define the curve $a(t)$ as the unique solution of the following affine differential equation with time dependent coefficients

    $$\dot{a}(t) = -a(t)\xi(t) - dc(\xi(t)),$$

    with initial condition $a(0) = a_0$. The solution can be written as $a(t) = \theta_{g(t)^{-1}}(a_0)$.

\end{itemize}
With \( a_0 \) held fixed, Hamilton’s variational principle

$$\delta \int_{t_1}^{t_2} L_{a_0}(g(t), \dot{g}(t))dt = 0,$$

holds, for variations \( \delta g(t) \) of \( g(t) \) vanishing at the endpoints.

\( g(t) \) satisfies the Euler-Lagrange equations for \( L_{a_0} \) on \( G \).

The constrained variational principle

$$\delta \int_{t_1}^{t_2} l(\xi(t), a(t))dt = 0,$$

holds on \( g \times V^* \), upon using variations of the form

$$\delta \xi = \frac{\partial \eta}{\partial t} - [\xi, \eta], \quad \delta a = -a\eta - dc(\eta),$$

where \( \eta(t) \in g \) vanishes at the endpoints.

The affine Euler-Poincaré equations hold on \( g \times V^* \):

$$\frac{\partial}{\partial t} \frac{\delta l}{\delta \xi} = -ad^*_\xi \frac{\delta l}{\delta \xi} + \frac{\delta l}{\delta a} \diamond a - dc^T \left( \frac{\delta l}{\delta a} \right).$$
Lagrangian Approach to Continuum Theories of Perfect Complex Fluids

Two key observations:

1. Enlarge the configuration manifold $\text{Diff}(\mathcal{D})$ to a bigger group $G$ that contains variables in the Lie group $\mathcal{O}$ of order parameters.

2. The usual advection equations (for the mass density, the entropy, the magnetic field, etc) need to be augmented by a new advected quantity on which the group $G$ acts by an affine representation.

$\mathcal{O}$ the order parameter Lie group, $\mathcal{F}(\mathcal{D}, \mathcal{O}) := \{\chi : \mathcal{D} \to \mathcal{O} \text{ smooth}\}$

Basic idea for complex fluids: enlarge the “particle relabeling group” $\text{Diff}(\mathcal{D})$ to the semidirect product $G = \text{Diff}(\mathcal{D}) \semidirect \mathcal{F}(\mathcal{D}, \mathcal{O})$. 

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\( \text{Diff}(\mathcal{D}) \) acts on \( \mathcal{F}(\mathcal{D}, \mathcal{O}) \) via the \textit{right} action

\[ (\eta, \chi) \in \text{Diff}(\mathcal{D}) \times \mathcal{F}(\mathcal{D}, \mathcal{O}) \mapsto \chi \circ \eta \in \mathcal{F}(\mathcal{D}, \mathcal{O}). \]

Therefore, the group multiplication is given by

\[ (\eta, \chi)(\varphi, \psi) = (\eta \circ \varphi, (\chi \circ \varphi)\psi). \]

Fix a volume form \( \mu \) on \( \mathcal{D} \), so identify densities with functions, one-form densities with one-forms, etc. But the dual actions will be of course different once these identifications are used.

The Lie algebra \( \mathfrak{g} \) of the semidirect product group is

\[ \mathfrak{g} = \mathfrak{X}(\mathcal{D}) \bar{\bigodot} \mathcal{F}(\mathcal{D}, o), \]

and the Lie bracket is computed to be

\[ \text{ad}_{(u, \nu)}(v, \zeta) = (\text{ad}_u v, \text{ad}_\nu \zeta + d\nu \cdot v - d\zeta \cdot u), \]

where \( \text{ad}_u v = -[u, v] \), \( \text{ad}_\nu \zeta \in \mathcal{F}(\mathcal{D}, o) \) is given by \( \text{ad}_\nu \zeta(x) := \text{ad}_{\nu(x)} \zeta(x) \), and \( d\nu \cdot v \in \mathcal{F}(\mathcal{D}, o) \) is given by \( d\nu \cdot v(x) := d\nu(x)(v(x)) \).
\[ g^* = \Omega^1(\mathcal{D}) \otimes \mathcal{F}(\mathcal{D}, \mathfrak{o}^*) \]

through the pairing
\[ \langle (m, \kappa), (u, \nu) \rangle = \int_{\mathcal{D}} (m \cdot u + \kappa \cdot \nu) \mu. \]

The dual map to \( \text{ad}_{(u, \nu)} \) is
\[ \text{ad}_{(u, \nu)}^*(m, \kappa) = \left( \mathcal{L}_u m + (\text{div } u) m + \kappa \cdot d\nu, \text{ad}_{\nu}^* \kappa + \text{div}(u\kappa) \right). \]

**Explanation of the symbols:**

- \( \kappa \cdot d\nu \in \Omega^1(\mathcal{D}) \) denotes the one-form defined by
  \[ \kappa \cdot d\nu(v_x) := \kappa(x)(d\nu(v_x)) \]
- \( \text{ad}_{\nu}^* \kappa \in \mathcal{F}(\mathcal{D}, \mathfrak{o}^*) \) denotes the \( \mathfrak{o}^* \)-valued mapping defined by
  \[ \text{ad}_{\nu}^* \kappa(x) := \text{ad}_{\nu(x)}^* \kappa(x). \]
- \( u\kappa \) is the 1-contravariant tensor field with values in \( \mathfrak{o}^* \) defined by
  \[ u\kappa(\alpha_x) := \alpha_x(u(x)) \kappa(x) \in \mathfrak{o}^*. \]
So \( u_\kappa \) is a generalization of the notion of a vector field. \( \mathcal{X}(\mathcal{D}, \sigma^*) \) denotes the space of all \( \sigma^* \)-valued 1-contravariant tensor fields.

- \( \text{div}(u) \) denotes the divergence of the vector field \( u \) with respect to the fixed volume form \( \mu \). Recall that it is defined by the condition

\[
(\text{div} u)_\mu = \mathcal{L}_u \mu.
\]

This operator can be naturally extended to the space \( \mathcal{X}(\mathcal{D}, \sigma^*) \) as follows. For \( w \in \mathcal{X}(\mathcal{D}, \sigma^*) \) we write \( w = w_a \varepsilon^a \) where \( (\varepsilon^a) \) is a basis of \( \sigma^* \) and \( w_a \in \mathcal{X}(\mathcal{D}) \). We define \( \text{div} : \mathcal{X}(\mathcal{D}, \sigma^*) \to \mathcal{F}(\mathcal{D}, \sigma^*) \) by

\[
\text{div} w : = (\text{div} w_a) \varepsilon^a.
\]

Note that if \( w = u_\kappa \) we have

\[
\text{div}(u_\kappa) = d_\kappa \cdot u + (\text{div} u)_\kappa.
\]

Split the space of advected quantities in two: usual ones and new ones that involve affine actions and cocycles.
Affine representation space: $V_1^* \oplus V_2^*$, $V_i^*$ are subspaces of the space of all tensor fields on $\mathcal{D}$, possibly with values in a vector space.

- $V_1^*$ is only acted upon by the component $\text{Diff}(\mathcal{D})$ of $G$.

- The action of $G$ on $V_2^*$ is affine, with the restriction that the affine term only depends on the second component $\mathcal{F}(\mathcal{D}, \mathcal{O})$ of $G$.

- Right affine representation of $G = \text{Diff}(\mathcal{D}) \circledS \mathcal{F}(\mathcal{D}, \mathcal{O})$ on $V_1^* \oplus V_2^*$:

  $$(a, \gamma) \in V_1^* \oplus V_2^* \mapsto (a\eta, \gamma(\eta, \chi) + C(\chi)) \in V_1^* \oplus V_2^*,$$

  where $\gamma(\eta, \chi)$ denotes the representation of $(\eta, \chi) \in G$ on $\gamma \in V_2^*$, and $C \in \mathcal{F}(\mathcal{F}(\mathcal{D}, \mathcal{O}), V_2^*)$ satisfies the cocycle identity

  $$C((\chi \circ \varphi)\psi) = C(\chi)(\varphi, \psi) + C(\psi).$$

  The representation $\rho$ and the affine term $c$ in the general theory are

  $$\rho_{(\eta, \chi)}^{-1}(a, \gamma) = (a\eta, \gamma(\eta, \chi)) \quad \text{and} \quad c(\eta, \chi) = (0, C(\chi)).$$
• The infinitesimal action of \((u, \nu) \in g\) on \(\gamma \in V_2^*\) is:
\[
\gamma(u, \nu) = \gamma u + \gamma \nu.
\]

• The diamond operation: for \((v, w) \in V_1 \oplus V_2\) we have
\[
(v, w) \diamond (a, \gamma) = (v \diamond a + w \diamond_1 \gamma, w \diamond_2 \gamma),
\]
where \(\diamond_1\) and \(\diamond_2\) are associated to the induced representations of the first and second component of \(G\) on \(V_2^*\). On the right hand side, \(\diamond\) is associated to the representation of \(\text{Diff}(\mathcal{D})\) on \(V_1^*\). Usually, \(V_1^*\) is naturally the dual of some space \(V_1\) of tensor fields on \(\mathcal{D}\). For example the \((p, q)\) tensor fields are naturally in duality with the \((q, p)\) tensor fields. For \(a \in V_1^*\) and \(v \in V_1\), the duality pairing is given by
\[
\langle a, v \rangle = \int_{\mathcal{D}} (a \cdot v) \mu,
\]
where \(\cdot\) denotes the contraction of tensor fields.

• The affine cocycle is \(c(\eta, \chi) = (0, C(\chi))\). Hence
\[
dcT(v, w) = (0, dC^T(w)).
\]

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• For a Lagrangian \( l = l(u, \nu, a, \gamma) : [\mathcal{X}(\mathcal{D}) \otimes \mathcal{F}(\mathcal{D}, \sigma)] \otimes [V_1^* \oplus V_2^*] \to \mathbb{R} \), the affine Euler-Poincaré equations become

\[
\begin{align*}
\frac{\partial}{\partial t} \frac{\delta l}{\delta u} & = -\mathcal{L}_u \frac{\delta l}{\delta u} - (\text{div} \, u) \frac{\delta l}{\delta u} - \frac{\delta l}{\delta \nu} \cdot \text{d}\nu - \frac{\delta l}{\delta a} \triangledown a + \frac{\delta l}{\delta \gamma} \triangledown_1 \gamma \\
\frac{\partial}{\partial t} \frac{\delta l}{\delta \nu} & = -\text{ad}^*_{\nu} \frac{\delta l}{\delta \nu} - \text{div} \left( u \frac{\delta l}{\delta \nu} \right) + \frac{\delta l}{\delta \gamma} \triangledown_2 \gamma - \text{d}C^T \left( \frac{\delta l}{\delta \gamma} \right),
\end{align*}
\]

and the advection equations are

\[
\begin{align*}
\dot{a} + au & = 0 \\
\dot{\gamma} + \gamma u + \gamma \nu + \text{d}C(\nu) & = 0.
\end{align*}
\]
Complex Fluids Example

\[ V_1 = \mathfrak{X}(\mathcal{D}, \sigma^*), \ V_2^* := \Omega^1(\mathcal{D}, \sigma) \]

**Affine representation:**

\[(a, \gamma) \mapsto (a\eta, \text{Ad}_{\chi^{-1}} \eta^* \gamma + \chi^{-1} T\chi), \]

where \( \text{Ad}_{\chi^{-1}} \eta^* \gamma + \chi^{-1} T\chi \) is the \( \sigma \)-valued one-form given by

\[
(\text{Ad}_{\chi^{-1}} \eta^* \gamma + \chi^{-1} T\chi)(v_x) := \text{Ad}_{\chi(x)^{-1}}(\eta^*(\gamma(v_x))) + \chi(x)^{-1} T_{x\chi}(v_x),
\]

for \( v_x \in T_x\mathcal{D} \). One can check that \( \gamma(\eta, \chi) := \text{Ad}_{\chi^{-1}} \eta^* \gamma \) is a right representation of \( G \) on \( V_2^* \) and that \( C(\chi) = \chi^{-1} T\chi \) verifies the condition cocycle condition.

This formula corresponds to the action of the automorphism group of the trivial principal bundle \( \mathcal{O} \times \mathcal{D} \) on the space connections.
For this example we have

\[ \gamma u = \mathcal{L}_u \gamma, \quad \gamma \nu = -\text{ad}_\nu \gamma \quad \text{and} \quad \text{d}C(\nu) = \text{d}\nu, \]

where \( \text{ad}_\nu \gamma \in \Omega^1(M, \mathfrak{o}) \) and \( \text{d}\nu \in \Omega^1(D, \mathfrak{o}) \) are the one-forms

\[ (\text{ad}_\nu \gamma)(v_x) := \text{ad}_{\nu(x)}(\gamma(v_x)) = [\nu(x), \gamma(v_x)], \quad \text{d}\nu(v_x) := T_x \nu(v_x) \in \mathfrak{o}. \]

A direct computation shows that

\[ w \triangledown_1 \gamma = (\text{div} \, w) \cdot \gamma - w \cdot i_\cdot \text{d}\gamma \in \Omega^1(D), \]

\[ w \triangledown_2 \gamma = -\text{Tr}(\text{ad}_\gamma^* \text{w}) \in \mathcal{F}(D, \mathfrak{o}^*), \]

\[ \text{d}C^T(w) = -\text{div} \, w \in \mathcal{F}(D, \mathfrak{o}^*), \]

where \( \text{Tr} \) denotes the trace of the \( \mathfrak{o}^* \)-valued \( (1, 1) \) tensor

\[ \text{ad}_\gamma^* \text{w} : T^*D \times TD \to \mathfrak{o}^*, \quad (\alpha, v_x) \mapsto \text{ad}_\gamma^*(v_x)(w(\alpha_x)). \]

In coordinates we have \( \text{Tr}(\text{ad}_\gamma^* \text{w}) = \text{ad}_\gamma^* w^i. \)
The affine Euler-Poincaré equations become in this case

\[
\begin{align*}
\frac{\partial}{\partial t} \frac{\delta l}{\delta u} &= -\mathcal{L}_u \frac{\delta l}{\delta u} - (\text{div } u) \frac{\delta l}{\delta \nu} - \frac{\delta l}{\delta \nu} \cdot d\nu + \frac{\delta l}{\delta a} \circ a + \left( \text{div } \frac{\delta l}{\delta \gamma} \right) \cdot \gamma - \frac{\delta l}{\delta \gamma} \cdot i_d \gamma
\end{align*}
\]

and the advection equations are

\[
\begin{align*}
\frac{\partial}{\partial t} \frac{\delta l}{\delta \nu} &= -\text{ad}^*_\nu \frac{\delta l}{\delta \nu} + \text{div} \left( \frac{\delta l}{\delta \gamma} - u \frac{\delta l}{\delta \nu} \right) - \text{Tr} \left( \text{ad}^*_\gamma \frac{\delta l}{\delta \gamma} \right),
\end{align*}
\]

and the advection equations are

\[
\begin{align*}
\dot{a} + a u &= 0
\end{align*}
\]

\[
\begin{align*}
\dot{\gamma} + \mathcal{L}_u \gamma - \text{ad}_\nu \gamma + d\nu &= 0.
\end{align*}
\]

These are, up to sign conventions, the equations for complex fluids given by Holm[2002].

Write these equations more geometrically; \( \gamma \) defines a connection:

\[(v_x, \xi_h) \in T_x \mathcal{D} \times T_h \mathcal{O} \mapsto \text{Ad}_{h^{-1}}(\gamma(x)(v_x) + TR_{h^{-1}}(\xi_h)) \in \mathfrak{o}.
\]

Covariant differential is denoted by \( d^\gamma \). For a function \( \nu \in \mathcal{F}(\mathcal{D}, \mathfrak{o}) \)

\[
d^\gamma \nu(v) := d\nu(v) + [\gamma(v), \nu].
\]

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The covariant divergence of \( w \in \mathfrak{X}(\mathcal{D}, \mathfrak{o}^*) \) is the function
\[
div^\gamma w := \text{div} w - \text{Tr}(\text{ad}_\gamma^* w) \in \mathcal{F}(\mathcal{D}, \mathfrak{o}^*),
\]
defined as minus the adjoint of the covariant differential, that is,
\[
\int_{\mathcal{D}} (\text{d}^\gamma \nu \cdot w) \mu = -\int_{\mathcal{D}} (\nu \cdot \text{div}^\gamma w) \mu
\]
for all \( \nu \in \mathcal{F}(\mathcal{D}, \mathfrak{o}) \).

Note that the Lie derivative of \( \gamma \in \Omega^1(\mathcal{D}, \mathfrak{o}) \) can be written as
\[
\mathcal{L}_u \gamma(v) = d(\gamma(u))(v) + i_u d\gamma(v)
\]
\[
= d^\gamma(\gamma(u))(v) - [\gamma(v), \gamma(u)] + d\gamma^*(u,v) - [\gamma(u), \gamma(v)]
\]
\[
= d^\gamma(\gamma(u))(v) + i_u B(v),
\]
where
\[
B := d^\gamma \gamma = d\gamma + [\gamma, \gamma],
\]
is the \textit{curvature} of the connection induced by \( \gamma \).

Note also that, using covariant differentiation, we have
\[
w \diamond_1 \gamma = (\text{div} w) \cdot \gamma - w \cdot i_\_ d\gamma = (\text{div}^\gamma w) \cdot \gamma - w \cdot i_\_ B.
\]
Therefore, in terms of $d^\gamma$, $\text{div}^\gamma$, and $B = d^\gamma \gamma$, the equations read

$$\frac{\partial}{\partial t} \frac{\delta l}{\delta u} = -\mathcal{L}_u \frac{\delta l}{\delta u} - (\text{div} \, u) \frac{\delta l}{\delta \nu} \cdot d\nu + \frac{\delta l}{\delta a} \diamond a + \left( \text{div}^\gamma \frac{\delta l}{\delta \gamma} \right) \cdot \gamma - \frac{\delta l}{\delta \gamma} \cdot i_B$$

$$\frac{\partial}{\partial t} \frac{\delta l}{\delta \nu} = -\text{ad}^*_\nu \frac{\delta l}{\delta \nu} - \text{div} \left( u \frac{\delta l}{\delta \nu} \right) + \text{div}^\gamma \frac{\delta l}{\delta \gamma},$$

and

$$\begin{cases} \dot{a} + a u = 0 \\ \dot{\gamma} + d^\gamma(\gamma(u)) + i_u B + d^\gamma \nu = 0. \end{cases}$$

The Curvature Representation

Want to reformulate the reduction process and the equations of motion in terms of $(u, \nu, a, B)$, instead of $(u, \nu, a, \gamma)$, where $B = d^\gamma \gamma = d\gamma + [\gamma, \gamma] \in \Omega^2(D, \mathfrak{g})$ is the curvature of $\gamma$. With this choice of variables the action of $G$ becomes linear instead of affine. We shall also assume that the Lagrangian $L$, and hence also $l$, depend on $\gamma$ only through $B$. We shall use therefore standard Euler-Poincaré reduction for semidirect products.

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If \( \gamma' = \text{Ad}_{\chi^{-1}} \eta^* \gamma + \chi^{-1} T\chi \) then \( d\gamma' = \text{Ad}_{\chi^{-1}} \eta^* d\gamma \). Thus the representation of \( \text{Diff}(\mathcal{D}) \otimes \mathcal{F}(\mathcal{D}, \mathcal{O}) \) on \( V^1_1 + \Omega^2(\mathcal{D}, \mathcal{o}) \) is given by

\[
(a, B) \mapsto (a\eta, \text{Ad}_{\chi^{-1}} \eta^* B).
\]

The associated infinitesimal action of \((u, \nu) \in \mathcal{X}(\mathcal{D}) \otimes \mathcal{F}(\mathcal{D}, \mathcal{o})\) is

\[
(a, B)(u, \nu) = (au, B(u, \nu)) = (au, \mathcal{L}_u B - \text{ad}_\nu B).
\]

**Duality:** contraction and integration with respect to the fixed volume form \( \mu \), so the space \( \Omega^k(\mathcal{D}, \mathcal{o}^*) \) of \( k \)-contravariant skew symmetric tensor fields with values in \( \mathcal{o}^* \) is dual to \( \Omega^{k-1}(\mathcal{D}, \mathcal{o}) \). Define the divergence operators, \( \text{div}, \text{div}^\gamma : \Omega^k(\mathcal{D}, \mathcal{o}^*) \to \Omega^{k-1}(\mathcal{D}, \mathcal{o}) \), to be minus the adjoint of the exterior derivatives \( d \) and \( d^\gamma \), respectively. For example, \( \text{div}^\gamma \) is defined on \( \Omega^k(\mathcal{D}, \mathcal{o}^*) \) by

\[
\int_\mathcal{D} (d^\gamma \alpha \cdot \omega) \mu = -\int_\mathcal{D} (\alpha \cdot \text{div}^\gamma \omega) \mu,
\]

where \( \alpha \in \Omega^{k-1}(\mathcal{D}, \mathcal{o}) \) and \( \omega \in \Omega^k(\mathcal{D}, \mathcal{o}^*) \). Note that we have used the notations \( \Omega_1(\mathcal{D}, \mathcal{o}^*) = \mathcal{X}(\mathcal{D}, \mathcal{o}^*) \) and \( \Omega_0(\mathcal{D}, \mathcal{o}^*) = \mathcal{F}(\mathcal{D}, \mathcal{o}^*) \).
If \((v, b) \in V_1 \oplus \Omega_2(\mathcal{D}, \mathfrak{o}^*)\) and \((a, B) \in V_1^* \oplus \Omega^2(\mathcal{D}, \mathfrak{o})\), then
\[
(v, b) \diamond (a, B) = (v \diamond a + b \diamond_1 B, b \diamond_2 B),
\]
where
\[
b \diamond_1 B = (\text{div} \, b) \cdot \mathbf{i}_B - b \cdot \mathbf{i}_d B \in \Omega^1(\mathcal{D})
\]
and
\[
b \diamond_2 B = - \text{Tr}(\text{ad}^*_B b) = - \text{ad}^*_{B_{ij}} b^{ij} \in \mathcal{F}(\mathcal{D}, \mathfrak{o}^*).
\]
The (usual semidirect product) Euler-Poincaré equation are
\[
\begin{align*}
\frac{\partial}{\partial t} \frac{\delta l}{\delta u} &= - \mathbf{L}_u \frac{\delta l}{\delta u} - (\text{div} \, u) \frac{\delta l}{\delta \nu} - \frac{\delta l}{\delta \nu} \cdot d \nu + \frac{\delta l}{\delta a} \diamond a \\
&\quad + \left( \text{div} \frac{\delta l}{\delta B} \right) \cdot \mathbf{i}_B - \frac{\delta l}{\delta B} \cdot \mathbf{i}_d B \\
\frac{\partial}{\partial t} \frac{\delta l}{\delta \nu} &= - \text{ad}^*_\nu \frac{\delta l}{\delta \nu} + \frac{\delta l}{\delta B} \diamond_2 B - \text{Tr} \left( \text{ad}^*_B \frac{\delta l}{\delta B} \right),
\end{align*}
\]
and the advection equations are
\[
\begin{align*}
\dot{a} + a u &= 0 \\
\dot{B} + \mathbf{L}_u B - \text{ad}_\nu B &= 0.
\end{align*}
\]

The affine EP equations imply these standard EP equations.
Affine Hamiltonian semidirect product theory

$R_g^{T^*}$ is the lift of right translation on $G$: $R_g^{T^*}(\alpha_f) = T^*R_{g^{-1}}(\alpha_f)$.

Let $C : G \times G \to T^*G$ be a smooth map such that $C_g(f) := C(g, f) \in T_{fg}^*G$, for all $f, g \in G$ and define $\Psi_g : T^*G \to T^*G$ by

$$\Psi_g(\alpha_f) := R_g^{T^*}(\alpha_f) + C_g(f),$$

where $C : G \times G \to T^*G$ satisfies $C_g(f) \in T_{fg}^*G$, for all $f, g \in G$.

The following are equivalent.

i $\Psi_g$ is a right action.

ii For all $f, g, h \in G$, the affine term $C$ verifies the property

$$C_{gh}(f) = C_h(fg) + R_h^{T^*}(C_g(f)).$$

iii There exists $\alpha \in \Omega^1(G)$ such that $C_g(f) = \alpha(fg) - R_g^{T^*}(\alpha(f))$.  

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The one-form $\alpha$ is unique if we assume that $\alpha(e) = 0$, which is what we will do from now on. Let $\Omega^1_0(G) := \{\alpha \in \Omega^1(G) \mid \alpha(e) = 0\}$ and $\mathcal{C}(G) = \{C : G \times G \to T^*G \mid C_{gh}(f) = C_h(fg) + R^T_{gh}(C_g(f)), \forall f,g,h \in G\}$.

Let $G$ act on $(T^*G, \Omega_{\text{can}})$ by the right affine action

$$\Psi_g(\beta_f) := R^T_g(\beta_f) + C_g(f),$$

where $C \in \mathcal{C}(G)$. Let $\alpha \in \Omega^1_0(G)$ be the one-form associated to $C$.

**i** $t_\alpha : \beta_q \in (T^*G, \Omega_{\text{can}}) \mapsto \beta_q - \alpha(q) \in (T^*G, \Omega_{\text{can}} - \pi^*_G d\alpha)$ symplectic. The action induced by $\Psi_g$ on $(T^*G, \Omega_{\text{can}} - \pi^*_G d\alpha)$ through $t_\alpha$ is $R^T_g$.

**ii** Suppose that $d\alpha$ is $G$-invariant. Then the action $\Psi_g$ is symplectic relative to the canonical symplectic form $\Omega_{\text{can}}$. 

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Suppose that there is a smooth map $\phi : G \to \mathfrak{g}^*$ that satisfies

$$i_{\xi L}d\alpha = d\langle \phi, \xi \rangle$$

for all $\xi \in \mathfrak{g}$. Here $\xi^L(g) := T_eL_g\xi$. Then $J_\alpha = J_R \circ t_\alpha - \phi \circ \pi_G$ is a momentum map for the action $\Psi_g$ relative to $\Omega_{\text{can}}$. We can always choose $\phi$ such that $\phi(e) = 0$. Then, the nonequivariance one-cocycle of $J_\alpha$ is $-\phi$.

$G^\phi_\mu$ is the isotropy group of $\mu$ relative to the affine action $\mu \mapsto \text{Ad}^*_g(\mu) - \phi(g)$. The symplectic reduced space $(J^{-1}_\alpha(\mu)/G^\phi_\mu, \Omega_\mu)$ is symplectically diffeomorphic to the affine coadjoint orbit $(O^\phi_\mu, \omega^+_d)$, the symplectic diffeomorphism being induced by the $G^\phi_\mu$-invariant smooth map $\alpha_g \in J^{-1}_\alpha(\mu) \mapsto \Psi_{g^{-1}}(\alpha_g) \in O^\sigma_\mu$. Here

$$\omega^+_d(\lambda) \left( \text{ad}^*_\xi \lambda - \Sigma(\xi, \cdot), \text{ad}^*_\eta \lambda - \Sigma(\eta, \cdot) \right)$$

$$= \langle \lambda, [\xi, \eta] \rangle - \Sigma(\xi, \eta),$$

where $\Sigma(\xi, \cdot) := T_e\phi(\xi)$.

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Now work on the semidirect product $G \circledast V$. Modify the cotangent lift of right translation by adding the term

$$C_{(g,v)}(f,u) := (0_{fg}, v + \rho_g(u), c(g)),$$

where $c \in \mathcal{F}(G,V^*)$ is a group one-cocycle, that is, it verifies $c(fg) = \rho^*_{g^{-1}}(c(f)) + c(g)$. Thus, the affine right action on $T^*S$ is:

$$\Psi_{(g,v)}(\alpha_f, (u, a)) = (R^*_{T_g} (\alpha_f)_f, v + \rho_g(u), \rho^*_{g^{-1}}(a) + c(g))$$

All properties of the preceding theorem hold (long calculations). For example, $\alpha \in \Omega^1_0(S)$ associated to the affine term $C$ is given by

$$\alpha(g, v)(\xi_g, (v, u)) = \langle c(g), u \rangle,$$

for $(\xi_g, (v, u)) \in T_{(g,v)}S$,

$$J_{\alpha}(\beta_f, (u, a)) = (T^*L_f(\beta_f) + u \diamond a - dc^T(u), a)$$

and

$$-\phi(f, u) = (u \diamond c(f) - dc^T(u), c(f)) \in s^*.$$

So, by the theorem.
$J^{-1}_\alpha(\mu, a)/S^{\phi}_{(\mu, a)}$ is symplectomorphic to

$O^{\phi}_{(\mu, a)} = \left\{ (\text{Ad}_g^* \mu + u \diamond (\rho^*_{g^{-1}}(a) + c(g)) - dc^T(u), \rho^*_{g^{-1}}(a) + c(g)) \mid (g, u) \in S \right\}$

relative to the symplectic form

$\omega^+_B(\lambda, b) \left( (\text{ad}_\xi^* \lambda + u \diamond b - dc^T(u), b\xi + dc(\xi)), (\text{ad}_\eta^* \lambda + w \diamond b - dc^T(w), b\eta + dc(\eta)) \right) = \langle \lambda, [\xi, \eta] \rangle + \langle b, u\eta - w\xi \rangle + \langle dc(\eta), u \rangle - \langle dc(\xi), w \rangle$.  

Recall that the affine coadjoint orbits $O^{\phi}_{(\mu, a)}$ are the symplectic leaves of the Poisson manifold $s^*$ with Poisson bracket

$\{ f, g \}_{d{\alpha}}(\mu, a) = \langle \mu, \left[ \frac{\delta f}{\delta \mu}, \frac{\delta g}{\delta \mu} \right] \rangle + \langle a, \frac{\delta f}{\delta a} \frac{\delta g}{\delta \mu} - \frac{\delta g}{\delta a} \frac{\delta f}{\delta \mu} \rangle$

$+ \langle dc \left( \frac{\delta f}{\delta \mu} \right), \frac{\delta g}{\delta a} \rangle - \langle dc \left( \frac{\delta g}{\delta \mu} \right), \frac{\delta f}{\delta a} \rangle$.

With this geometric background we can state the Hamiltonian analogue of the affine Lagrangian semidirect product theorem.
$H : T^*G \times V^* \to \mathbb{R}$ right-invariant under the $G$-action

$$(\alpha_h, a) \mapsto (R_g^{T^*}(\alpha_h), \theta_g(a)) := (R_g^{T^*}(\alpha_h), \rho_g^{-1}(a) + c(g)).$$

In particular, the function $H_{a_0} := H|_{T^*G \times \{a_0\}} : T^*G \to \mathbb{R}$ is invariant under the induced action of the isotropy subgroup $G_{a_0}^c$ of $a_0$ relative to the affine action $\theta$, for any $a_0 \in V^*$. Recall that

$$\theta_g(a) := ag + c(g)$$

for any $g \in G$ and $a \in V^*$.

For $\alpha(t) \in T^*_{g(t)}G$ and $\mu(t) := T_e^*R_{g(t)}(\alpha(t)) \in g^*$, the following are equivalent:

i $\alpha(t)$ satisfies Hamilton's equations for $H_{a_0}$ on $T^*G$.

ii The following affine Lie-Poisson equation holds on $s^*$:

$$\frac{\partial}{\partial t} (\mu, a) = \left( - \text{ad}_{\frac{\delta h}{\delta \mu}}^* \mu - \frac{\delta h}{\delta a} \diamond a + dc^T \left( \frac{\delta h}{\delta a} \right), -a \frac{\delta h}{\delta \mu} - dc \left( \frac{\delta h}{\delta \mu} \right) \right), \ a(0) = a_0.$$

The evolution of the advected quantities is given by $a(t) = \theta_{g(t)}^{-1}(a_0)$.

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Hamiltonian Approach to Continuum Theories of Perfect Complex Fluids

This is the counterpart of the Lagrangian approach, so the Lie-Poisson space is

\[
\left( [\mathcal{X}(\mathcal{D}) \otimes \mathcal{F}(\mathcal{D}, o)] \otimes [V_1 \oplus V_2] \right)^* \cong \Omega^1(\mathcal{D}) \times \mathcal{F}(\mathcal{D}, o^*) \times V_1^* \times V_2^*.
\]

with affine Lie-Poisson bracket given by

\[
\{f, g\}(m, \kappa, a, \gamma) = \int_{\mathcal{D}} m \cdot \left[ \frac{\delta f}{\delta m}, \frac{\delta g}{\delta m} \right] \mu
\]

\[
+ \int_{\mathcal{D}} \kappa \cdot \left( \text{ad}_{\frac{\delta f}{\delta \kappa}} \frac{\delta g}{\delta \kappa} + \text{d} \frac{\delta f}{\delta \kappa} \cdot \frac{\delta g}{\delta \kappa} \right) \mu
\]

\[
+ \int_{\mathcal{D}} a \cdot \left( \frac{\delta f}{\delta a} \frac{\delta g}{\delta m} - \frac{\delta g}{\delta a} \frac{\delta f}{\delta m} \right)
\]

\[
+ \int_{\mathcal{D}} \gamma \cdot \left( \frac{\delta f}{\delta \gamma} \frac{\delta g}{\delta m} + \frac{\delta f}{\delta \gamma} \frac{\delta g}{\delta \kappa} - \frac{\delta g}{\delta \gamma} \frac{\delta f}{\delta m} - \frac{\delta g}{\delta \gamma} \frac{\delta f}{\delta \kappa} \right) \mu
\]

\[
+ \int_{\mathcal{D}} \left( \text{d}C \left( \frac{\delta f}{\delta \kappa} \right) \cdot \frac{\delta g}{\delta \gamma} - \text{d}C \left( \frac{\delta g}{\delta \kappa} \right) \cdot \frac{\delta f}{\delta \gamma} \right) \mu.
\]
For a Hamiltonian $h = h(m, \kappa, a, \gamma) : \Omega^1(D) \times \mathcal{F}(D, \sigma^*) \times V_1^* \times V_2^* \rightarrow \mathbb{R}$, the affine Lie-Poisson equations become

\[
\begin{align*}
\frac{\partial}{\partial t} m &= -\mathcal{L}_{\frac{\delta h}{\delta m}} m - \text{div} \left( \frac{\delta h}{\delta m} \right) m - \kappa \cdot d \frac{\delta h}{\delta \kappa} - \frac{\delta h}{\delta a} \diamond a - \frac{\delta h}{\delta \gamma} \diamond_1 \gamma \\
\frac{\partial}{\partial t} \kappa &= -\text{ad}^*_{\frac{\delta h}{\delta \kappa}} \kappa - \text{div} \left( \frac{\delta h}{\delta m} \right) \kappa - \frac{\delta h}{\delta \gamma} \diamond_2 \gamma + dC^T \left( \frac{\delta h}{\delta \gamma} \right) \\
\frac{\partial}{\partial t} a &= -a \frac{\delta h}{\delta \gamma} \\
\frac{\partial}{\partial t} \gamma &= -\gamma \frac{\delta h}{\delta m} - \gamma \frac{\delta h}{\delta \kappa} - dC \left( \frac{\delta h}{\delta \kappa} \right).
\end{align*}
\]

**Complex Fluids Example**

$V_1 = \mathfrak{X}(D, \sigma^*)$, $V_2^* := \Omega^1(D, \sigma)$ and all formulas were already presented. The affine Lie-Poisson equations become in this case:
\[
\begin{align*}
\frac{\partial}{\partial t} m &= -\mathcal{L}_{\delta h} m - \text{div} \left( \frac{\delta h}{\delta m} \right) m - \kappa \cdot d\frac{\delta h}{\delta \kappa} - d\frac{\delta h}{\delta a} \diamond a \\
&\quad - \left( \text{div}^\gamma \frac{\delta h}{\delta \gamma} \right) \gamma + \frac{\delta h}{\delta \gamma} \cdot i \text{ d}^\gamma \gamma
\end{align*}
\]

or, in matrix notation (like in Holm[2002] up to sign conventions)

\[
\begin{bmatrix}
m_i \\
\kappa_a \\
a \\
\gamma^a_i
\end{bmatrix}
= -
\begin{bmatrix}
m_k \partial_i + \partial_k m_i & \kappa_b \partial_i & (\Box \diamond a)_i & \partial_j \gamma_i^b - \gamma_{j,i}^b \\
\partial_k \kappa_a & \kappa_c C_{ba}^c & 0 & \delta_a \partial_j - C_{ca}^b \gamma_j^c \\
a \Box \partial_k & 0 & 0 & \delta h/\delta a \\
\gamma_k^a \partial_i + \gamma_{i,k}^a & \partial_b \partial_i + C_{cb}^a \gamma_i^c & 0 & \text{(} \delta h/\delta \gamma \text{)}_b^j
\end{bmatrix}
\begin{bmatrix}
(\delta h/\delta m)_k \\
(\delta h/\delta \kappa)_b \\
\delta h/\delta a \\
(\delta h/\delta \gamma)_b^j
\end{bmatrix}
\]

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The associated affine Lie-Poisson bracket is

\[ \{f, g\}(m, \kappa, a, \gamma) = \int_D m \cdot \left[ \frac{\delta f}{\delta m}, \frac{\delta g}{\delta m} \right] \mu + \int_D \kappa \cdot \left( \text{ad}_{\frac{\delta f}{\delta \kappa}} \frac{\delta g}{\delta \kappa} + d\frac{\delta f}{\delta \kappa} \cdot \frac{\delta g}{\delta m} - d\frac{\delta g}{\delta \kappa} \cdot \frac{\delta f}{\delta m} \right) \mu + \int_D a \cdot \left( \frac{\delta f}{\delta a} \frac{\delta g}{\delta m} - \frac{\delta g}{\delta a} \frac{\delta f}{\delta m} \right) \mu + \int_D \left[ (d\gamma \frac{\delta f}{\delta \kappa} + \mathcal{L}_{\frac{\delta f}{\delta m}} \gamma) \cdot \frac{\delta g}{\delta \gamma} - (d\gamma \frac{\delta g}{\delta \kappa} + \mathcal{L}_{\frac{\delta g}{\delta m}} \gamma) \cdot \frac{\delta f}{\delta \gamma} \right] \mu. \]

**Curvature Representation**

The affine action on connections becomes a linear action on the curvature and one can therefore reduce. The relevant group is

\[ [\text{Diff}(D) \circledcirc \mathcal{F}(D, \mathcal{O})] \circledcirc [V^*_1 \oplus \Omega^2(D, \mathcal{O})], \]

where \( \text{Diff}(D) \circledcirc \mathcal{F}(D, \mathcal{O}) \) acts on \( \Omega^2(D, \mathcal{O}) \) by the representation

\[ B \mapsto \text{Ad}_{\chi^{-1}} \eta^* B, \]

and where the space \( V^*_1 \) is only acted upon by the subgroup \( \text{Diff}(D) \).

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The Lie-Poisson equations for semidirect products are

\[
\begin{align*}
\frac{\partial}{\partial t} m &= -\mathcal{L}_{\frac{\delta h}{\delta m}} m - \text{div} \left( \frac{\delta h}{\delta m} \right) m - \kappa \cdot \delta \kappa - \frac{\delta h}{\delta a} \odot a \\
&\quad - \text{div} \frac{\delta h}{\delta B} \cdot \mathbf{i}_B + \frac{\delta h}{\delta B} \cdot \mathbf{i}_d B \\
\frac{\partial}{\partial t} \kappa &= -\text{ad}^*_{\frac{\delta h}{\delta \kappa}} \kappa - \text{div} \left( \frac{\delta h}{\delta m} \kappa \right) + \text{Tr} \left( \text{ad}^*_B \frac{\delta h}{\delta B} \right) \\
\frac{\partial}{\partial t} a &= -a \frac{\delta h}{\delta m} \\
\frac{\partial}{\partial t} B &= -\mathcal{L}_{\frac{\delta h}{\delta m}} B + \text{ad}_{\frac{\delta h}{\delta \kappa}} B
\end{align*}
\]

if \( h \) depends on the connection only through the curvature.

The Lie-Poisson bracket is in this case:
\{f, g\}(m, \kappa, a, B) = \int_{D} m \cdot \left[ \frac{\delta f}{\delta m}, \frac{\delta g}{\delta m} \right] \mu \\
+ \int_{D} \kappa \cdot \left( \text{ad} \frac{\delta g}{\delta \kappa} + d \frac{\delta f}{\delta \kappa} \cdot \frac{\delta g}{\delta m} - d \frac{\delta g}{\delta \kappa} \cdot \frac{\delta f}{\delta m} \right) \mu \\
+ \int_{D} a \cdot \left( \frac{\delta f}{\delta a} \frac{\delta g}{\delta m} - \frac{\delta g}{\delta a} \frac{\delta f}{\delta m} \right) \mu \\
+ \int_{D} \left[ \left( \mathcal{L}_{\frac{\delta f}{\delta m}} B - \text{ad} \frac{\delta f}{\delta \kappa} B \right) \cdot \frac{\delta g}{\delta B} - \left( \mathcal{L}_{\frac{\delta g}{\delta m}} B - \text{ad} \frac{\delta g}{\delta \kappa} B \right) \cdot \frac{\delta f}{\delta B} \right] \mu.

The map

\((m, \nu, a, \gamma) \mapsto (m, \nu, a, d^{\gamma} \gamma)\)

is a Poisson map relative to the affine Lie-Poisson bracket and this Lie-Poisson bracket.
THE CIRCULATION THEOREMS

For compressible adiabatic fluids the Kelvin circulation theorem is

$$\frac{d}{dt} \oint_{\gamma_t} u^b = \oint_{\gamma_t} T ds,$$

where $\gamma_t \subset \mathcal{D}$ is a closed curve which moves with the fluid velocity $u$, $T = \partial e/\partial s$ is the temperature, and $e, s$ denote respectively the specific internal energy and the specific entropy.

Abstract Lagrangian version: Work under the hypotheses of the affine Euler-Poincaré reduction. Let $\mathcal{C}$ be a manifold on which $G$ acts on the left and suppose we have an equivariant map $\mathcal{K} : \mathcal{C} \times V^* \rightarrow g^{**}$, that is, for all $g \in G, a \in V^*, c \in \mathcal{C}$, we have

$$\langle \mathcal{K}(gc, \theta_g(a)), \mu \rangle = \langle \mathcal{K}(c, a), \text{Ad}_g^* \mu \rangle,$$

where $gc$ denotes the action of $G$ on $C$, and $\theta_g$ is the affine action of $G$ on $V^*$.

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Define the Kelvin-Noether quantity $I : \mathcal{C} \times \mathfrak{g} \times V^* \to \mathbb{R}$ by

$$I(c, \xi, a) := \left\langle \mathcal{K}(c, a), \frac{\delta l}{\delta \xi}(\xi, a) \right\rangle.$$

Fixing $c_0 \in \mathcal{C}$, let $\xi(t), a(t)$ satisfy the affine Euler-Poincaré equations and define $g(t)$ to be the solution of $\dot{g}(t) = T R_{g(t)} \xi(t)$ and, say, $g(0) = e$. Let $c(t) = g(t) c_0$ and $I(t) := I(c(t), \xi(t), a(t))$. Then

$$\frac{d}{dt} I(t) = \left\langle \mathcal{K}(c(t), a(t)), \frac{\delta l}{\delta a} \diamond a - dc^T \left( \frac{\delta l}{\delta a} \right) \right\rangle.$$

**Abstract Hamiltonian version:** Some examples do not admit a Lagrangian formulation. Nevertheless, a Kelvin-Noether theorem is still valid for the Hamiltonian formulation. The Kelvin-Noether quantity is now the mapping $J : \mathcal{C} \times \mathfrak{g}^* \times V^* \to \mathbb{R}$ defined by

$$J(c, \mu, a) := \langle \mathcal{K}(c, a), \mu \rangle.$$
Fixing $c_0 \in \mathcal{C}$, let $\mu(t), a(t)$ satisfy the affine Lie-Poisson equations and define $g(t)$ to be the solution of

$$ \dot{g}(t) = TR_{g(t)} \left( \frac{\delta h}{\delta \mu} \right), \quad g(0) = e. $$

Let $c(t) = g(t)c_0$ and $J(t) := J(c(t), \mu(t), a(t))$. Then

$$ \frac{d}{dt} J(t) = \left\langle K(c(t), a(t)), -\frac{\delta h}{\delta a} \diamond a + dc^T \left( \frac{\delta h}{\delta a} \right) \right\rangle. $$

In the case of dynamics on the group $G = \text{Diff}(\mathcal{D})$, the standard choice for the equivariant map $K$ is

$$ \langle K(c, a), m \rangle := \oint_c \frac{1}{\rho} m, $$

where $c \in \mathcal{C} = \text{Emb}(S^1, \mathcal{D})$, the manifold of all embeddings of the circle $S^1$ in $\mathcal{D}$, $m \in \Omega^1(\mathcal{D})$, and $\rho$ is advected as $(J\eta)(\rho \circ \eta)$.
Consider the affine Euler-Poincaré equations for complex fluids. Suppose that one of the linear advected variables, say $\rho$, is advected as $(J\eta)(\rho \circ \eta)$. Then
\[
\frac{d}{dt} \oint_{c_t} \rho \frac{1}{\delta l} \delta u = \oint_{c_t} \rho \left( -\frac{\delta l}{\delta \nu} \cdot d\nu + \frac{\delta l}{\delta a} \diamond a + \frac{\delta l}{\delta \gamma} \diamond \gamma \right),
\]
where $c_t$ is a loop in $\mathcal{D}$ which moves with the fluid velocity $u$.

Similarly, consider the affine Lie-Poisson equations for complex fluids. Suppose that one of the linear advected variables, say $\rho$, is advected as $(J\eta)(\rho \circ \eta)$. Then
\[
\frac{d}{dt} \oint_{c_t} \rho \frac{1}{\delta m} m = \oint_{c_t} \rho \left( \kappa \cdot d\frac{\delta h}{\delta \kappa} - \frac{\delta h}{\delta a} \diamond a - \frac{\delta h}{\delta \gamma} \diamond \gamma \right),
\]
where $c_t$ is a loop in $\mathcal{D}$ which moves with the fluid velocity $u$, defined be the equality
\[
\mathbf{u} := \frac{\delta h}{\delta m}.
\]
There is also a circulation theorem associated to the variable $\gamma$ because of the equation

$$\frac{\partial}{\partial t}\gamma + \mathcal{L}_u \gamma = -d\nu + \text{ad}_\nu \gamma.$$ 

Let $\eta_t$ be the flow of the vector field $u$, let $c_0$ be a loop in $\mathcal{D}$ and let $c_t := \eta_t \circ c_0$. Then, by change of variables, we have

$$\mathcal{d}\frac{d}{dt} \oint_{c_t} \gamma = \mathcal{d}\frac{d}{dt} \oint_{c_0} \eta_t^* \gamma = \oint_{c_0} \eta_t^* \left( \dot{\gamma} + \mathcal{L}_u \gamma \right)$$

$$= \oint_{c_0} \eta_t^* \left( -d\nu + \text{ad}_\nu \gamma \right) = \oint_{c_t} \text{ad}_\nu \gamma \in \mathfrak{o},$$

that is, the $\gamma$-circulation law is

$$\frac{d}{dt} \oint_{c_t} \gamma = \oint_{c_t} \text{ad}_\nu \gamma \in \mathfrak{o}.$$
EXAMPLE 1:
ERICKSEN-LESLIE EQUATIONS

Liquid crystal state: a distinct phase of matter observed between the crystalline (solid) and isotropic (liquid) states. Three main types of liquid crystal states, depending upon the amount of order:

Nematic liquid crystal phase: characterized by rod-like molecules, no positional order, but tend to point in the same direction.

Cholesteric (or chiral nematic) liquid crystal phase: molecules resemble helical springs, which may have opposite chiralities. Molecules exhibit a privileged direction, which is the axis of the helices.

Smectic liquid crystals are essentially different form both nematics and cholesterics: they have one more degree of orientational order. Smectics generally form layers within which there is a loss of positional order, while orientational order is still preserved.
Three main theories:

**Director theory** due to Oseen, Frank, Zöcher, Ericksen and Leslie

**Micropolar** and **microstretch theories**, due to Eringen, which take into account the microinertia of the particles and which is applicable, for example, to *liquid crystal polymers*

**Ordered micropolar** approach, due to Lhuillier and Rey, which combines the director theory with of the micropolar models.

In all that follows $\mathcal{D} \subset \mathbb{R}^3$ and all boundary conditions are ignored: in all integration by parts the boundary terms vanish. We fix a volume form $\mu$ on $\mathcal{D}$.

**EXAMPLE: DIRECTOR THEORY** (nematics, cholesterics)

**Assumption**: only the direction and not the sense of the molecules matter. The preferred orientation of the molecules around a point is described by a unit vector $n : \mathcal{D} \to S^2$, called the *director*, and $n$ and $-n$ are assumed to be equivalent.
**Ericksen-Leslie equations** in a domain $\mathcal{D}$, constraint $\|n\| = 1$, are:

\[
\begin{align*}
&\rho \left( \frac{\partial}{\partial t} u + \nabla u u \right) = \text{grad} \, \frac{\partial F}{\partial \rho^{-1}} - \partial_j \left( \rho \frac{\partial F}{\partial n,j} \cdot \nabla n \right), \\
&\rho J \frac{D^2}{dt^2} n - 2qn + h = 0, \\
&\frac{\partial}{\partial t} \rho + \text{div}(\rho u) = 0
\end{align*}
\]

$u$ **Eulerian velocity**, $\rho$ **mass density**, $n : \mathcal{D} \to \mathbb{R}^3$ **director** ($n$ equivalent to $-n$), $J$ **microinertia constant**, and $F(n, n,i)$ is the **free energy**. The **axiom of objectivity** requires that

\[F(\rho^{-1}, A^{-1}n, A^{-1}\nabla n A) = F(\rho^{-1}, n, \nabla n),\]

for all $A \in O(3)$ for nematics, or for all $A \in SO(3)$ for cholesterics.

\[h := \rho \frac{\partial F}{\partial n} - \partial_i \left( \rho \frac{\partial F}{\partial n,i} \right).\]

is the **h molecular field**. $q$ is unknown and determined by

\[2q := n \cdot h - \rho J \left\| \frac{Dn}{dt} \right\|^2\]
This is seen in the following way.

Take the dot product with $n$ of the second equation to get

$$2q = \rho J n \cdot \frac{D^2}{dt^2} n + n \cdot h = n \cdot h - \rho J \left\| \frac{Dn}{dt} \right\|^2$$

since $\|n\|^2 = 1$ implies $n \cdot \frac{Dn}{dt} = 0$ and hence, taking one more material derivative gives

$$n \cdot \frac{D^2}{dt^2} n = -\left\| \frac{Dn}{dt} \right\|^2.$$

Think of the function $q$ in the Ericksen-Leslie equation the way one regards the pressure in ideal incompressible homogeneous fluid dynamics, namely, the $q$ is an unknown function determined by the imposed constraint $\|n\| = 1$.

What is the structure of these equations?
Let \((u, \rho, n)\) be a solution of the Ericksen-Leslie equations such that \(||n|| = 1\) and define

\[
\nu := n \times \frac{D}{dt} n \in \mathcal{F}(\mathcal{D}, \mathbb{R}^3), \quad \frac{D}{dt} := \frac{\partial}{\partial t} + u \cdot \nabla \quad \text{material derivative}.
\]

Then \((u, \nu, \rho, n)\) is a solution of the equations

\[
\begin{align*}
\rho \left( \frac{\partial}{\partial t} u + \nabla u u \right) &= \text{grad} \frac{\partial F}{\partial \rho^{-1}} - \partial_i \left( \rho \frac{\partial F}{\partial n_i} \cdot \nabla n \right), \\
\rho J \frac{D}{dt} \nu &= h \times n,
\end{align*}
\]

\[
\begin{align*}
\frac{\partial}{\partial t} \rho + \text{div}(\rho u) &= 0, \\
\frac{D}{dt} n &= \nu \times n,
\end{align*}
\]

Evolution of \(\rho, n\) (where \(\eta \in \text{Diff}(\mathcal{D}), \chi \in \mathcal{F}(\mathcal{D}, \text{SO}(3))\) is

\[
\rho = J(\eta^{-1})(\rho_0 \circ \eta^{-1}) \quad \text{and} \quad n = (\chi n_0) \circ \eta^{-1}.
\]
These equations are Euler-Poincaré/Lie-Poisson for the group

\[(\text{Diff}(\mathcal{D}) \circledast \mathcal{F}(\mathcal{D}, \text{SO}(3))) \circledast (\mathcal{F}(\mathcal{D}) \times \mathcal{F}(\mathcal{D}, \mathbb{R}^3))\].

**EXPLANATION:**

- **Diff(\mathcal{D})** acts on \(\mathcal{F}(\mathcal{D}, \text{SO}(3))\) via the right action

\[(\eta, \chi) \in \text{Diff}(\mathcal{D}) \times \mathcal{F}(\mathcal{D}, \text{SO}(3)) \mapsto \chi \circ \eta \in \mathcal{F}(\mathcal{D}, \text{SO}(3)).\]

Therefore, the group multiplication in \(\text{Diff}(\mathcal{D}) \circledast \mathcal{F}(\mathcal{D}, \text{SO}(3))\) is

\[(\eta, \chi)(\varphi, \psi) = (\eta \circ \varphi, (\chi \circ \varphi)\psi).\]

- The bracket of \(\mathcal{X}(\mathcal{D}) \circledast \mathcal{F}(\mathcal{D}, \mathfrak{so}(3))\) is

\[\text{ad}_{(u, \nu)}(v, \zeta) = (\text{ad}_u v, \text{ad}_\nu \zeta + d\nu \cdot v - d\zeta \cdot u),\]

where \(\text{ad}_u v = -[u, v]\), \(\text{ad}_\nu \zeta \in \mathcal{F}(\mathcal{D}, \mathfrak{so}(3))\) is given by \(\text{ad}_\nu \zeta(x) := \text{ad}_{\nu(x)} \zeta(x)\), and \(d\nu \cdot v \in \mathcal{F}(\mathcal{D}, \mathfrak{so}(3))\) is given by \(d\nu \cdot v(x) := d\nu(x)(v(x))\).

- \((\eta, \chi) \in \text{Diff}(\mathcal{D}) \circledast \mathcal{F}(\mathcal{D}, \text{SO}(3))\) acts linearly and on the right on the advected quantities \((\rho, n) \in \mathcal{F}(\mathcal{D}) \times \mathcal{F}(\mathcal{D}, \mathbb{R}^3)\), by

\[(\rho, n) \mapsto (J_\eta(\rho \circ \eta), \chi^{-1}(n \circ \eta)).\]

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• The associated infinitesimal action and diamond operations are
\(\nu u = \nabla n \cdot u, \quad n \nu = n \times \nu, \quad m \diamond_1 n = -\nabla n^T \cdot m \) and \(m \diamond_2 n = n \times m\),
where \(\nu, m, n \in F(D, \mathbb{R}^3)\).

• EP equations for \((\text{Diff}(D) \circledast F(D, \text{SO}(3))) \circledast (F(D) \times F(D, \mathbb{R}^3))\):
\[
\begin{align*}
\frac{\partial}{\partial t} \frac{\delta l}{\delta u} &= -\mathcal{L}_u \frac{\delta l}{\delta u} - \text{div} u \frac{\delta l}{\delta \nu} - \frac{\delta l}{\delta \nu} \cdot d\nu + \rho \frac{\delta l}{\delta \rho} - \left(\nabla n^T \cdot \frac{\delta l}{\delta n}\right)^b, \\
\frac{\partial}{\partial t} \frac{\delta l}{\delta \nu} &= \nu \times \frac{\delta l}{\delta \nu} - \text{div} \left(\frac{\delta l}{\delta \nu} u\right) + n \times \frac{\delta l}{\delta n},
\end{align*}
\]

• The advection equations are:
\[
\begin{align*}
\frac{\partial}{\partial t} \rho + \text{div}(\rho u) &= 0, \\
\frac{\partial}{\partial t} n + \nabla n \cdot u + n \times \nu &= 0.
\end{align*}
\]

• Reduced Lagrangian for nematic and cholesteric liquid crystals:
\[
l(u, \nu, \rho, n) := \frac{1}{2} \int_D \rho \|u\|^2 \mu + \frac{1}{2} \int_D \rho J \|\nu\|^2 \mu - \int_D \rho F(\rho^{-1}, n, \nabla n) \mu.
\]

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The functional derivatives of the Lagrangian $l$ are:

$$m := \frac{\delta l}{\delta u} = \rho u^b, \quad \kappa := \frac{\delta l}{\delta \nu} = \rho J \nu,$$

$$\frac{\delta l}{\delta \rho} = \frac{1}{2} \|u\|^2 + \frac{1}{2} J \|\nu\|^2 - F + \frac{1}{\rho} \frac{\partial F}{\partial \rho}, \quad \frac{\delta l}{\delta \nu} = -\rho \frac{\partial F}{\partial \nu} + \partial_i \left( \rho \frac{\partial F}{\partial \nu, i} \right) = -h.$$

By the Legendre transformation, the Hamiltonian is:

$$h(m, \kappa, \rho, n) := \frac{1}{2} \int_D \frac{1}{\rho} \|m\|^2 \mu + \frac{1}{2} J \int_D \frac{1}{\rho} \|\kappa\|^2 \mu + \int_D \rho F(\rho^{-1}, n, \nabla n) \mu.$$

The Poisson bracket for liquid crystals is given by:

$$\{f, g\}(m, \rho, \kappa, n) = \int_D \frac{\delta f}{\delta m} \cdot \frac{\delta g}{\delta m} \mu$$

$$+ \int_D \kappa \cdot \left( \frac{\delta f}{\delta \kappa} \times \frac{\delta g}{\delta \kappa} + \frac{\delta f}{\delta \kappa} \cdot \frac{\delta g}{\delta \kappa} - \frac{\delta f}{\delta \kappa} \cdot \frac{\delta f}{\delta \kappa} \right) \mu$$

$$+ \int_D \rho \left( \frac{\delta f}{\delta \rho} \cdot \frac{\delta g}{\delta m} - \frac{\delta g}{\delta \rho} \cdot \frac{\delta f}{\delta m} \right) \mu$$

$$+ \int_D \left[ \left( n \times \frac{\delta f}{\delta \kappa} + \nabla n \cdot \frac{\delta f}{\delta m} \right) \frac{\delta g}{\delta n} - \left( n \times \frac{\delta g}{\delta \kappa} + \nabla n \cdot \frac{\delta g}{\delta m} \right) \frac{\delta f}{\delta n} \right] \mu.$$
• The Kelvin circulation theorem for liquid crystals reads:

\[
\frac{d}{dt} \oint_{c_t} u^b = \oint_{c_t} \frac{1}{\rho} \nabla n^T \cdot h \quad \text{where} \quad h = \rho \frac{\partial F}{\partial n} - \partial_i \left( \rho \frac{\partial F}{\partial n_i} \right).
\]

Now do the converse: show that the EP equations imply the Ericksen-Leslie equations. For this one needs to show first that if \( \nu \) and \( n \) are solutions of the EP equations then:

(i) \( \|n_0\| = 1 \) implies \( \|n\| = 1 \) for all time.

(ii) \( \frac{D}{dt}(n \cdot \nu) = 0 \). Therefore, \( n_0 \cdot \nu_0 = 0 \) implies \( n \cdot \nu = 0 \) for all time.

(iii) Suppose that \( n_0 \cdot \nu_0 = 0 \) and \( \|n_0\| = 1 \). Then

\[
\frac{D}{dt} n = \nu \times n \quad \text{becomes} \quad \nu = n \times \frac{D}{dt} n
\]

and

\[
\rho J \frac{D}{dt} \nu = h \times n \quad \text{becomes} \quad \rho J \frac{D^2}{dt^2} n - 2qn + h = 0.
\]
If \((u, \nu, \rho, n)\) is a solution of the Euler-Poincaré equations with initial conditions \(n_0\) and \(\nu_0\) satisfying \(\|n_0\| = 1\) and \(n_0 \cdot \nu_0 = 0\), then \((u, \rho, n)\) is a solution of the Ericksen-Leslie equations.

The \(q\) does not appear in the Euler-Poincaré formulation relative to the variables \((u, \nu, \rho, n)\), since in this case, the constraint \(\|n\| = 1\) is automatically satisfied.

Consequence of this theorem: the Ericksen-Leslie equations are obtained by Lagrangian reduction. Right-invariant Lagrangian

\[ L(\rho_0, n_0) : T[D\mathcal{F}(\mathcal{D}, SO(3))] \rightarrow \mathbb{R} \]

induced by the Lagrangian \(l\) (make it right invariant after freezing the parameters \((\rho_0, n_0)\)). Assume that \(\|n_0\| = 1\) and \(\nu_0 \cdot n_0 = 0\). A curve \((\eta, \chi) \in D\mathcal{F}(\mathcal{D}, SO(3))\) is a solution of the Euler-Lagrange equations for \(L(\rho_0, n_0)\), with initial condition \(u_0, \nu_0\) iff

\[(u, \nu) := (\dot{\eta} \circ \eta^{-1}, \dot{\chi} \chi^{-1} \circ \eta^{-1})\]
is a solution of the Ericksen-Leslie equations, where
\[ \rho = J(\eta^{-1})(\rho_0 \circ \eta^{-1}) \quad \text{and} \quad n = (\chi n_0) \circ \eta^{-1}. \]

The curve \( \eta \in \text{Diff}(\mathcal{D}) \) describes the \textit{Lagrangian motion of the fluid} or \textit{macromotion} and the curve \( \chi \in \mathcal{F}(\mathcal{D}, \text{SO}(3)) \) describes the \textit{local molecular orientation relative to a fixed reference frame} or \textit{micromotion}. Standard choice for the initial value of the director is
\[ n_0(x) := (0, 0, 1), \quad \text{for all} \ x \in \mathcal{D}. \]

In this case we obtain
\[ n = \begin{pmatrix} \chi_1 \ 3 \\ \chi_2 \ 3 \\ \chi_3 \ 3 \end{pmatrix} \circ \eta^{-1}. \]

This relation is usually taken as a definition of the director, when the 3-axis is chosen as the reference axis of symmetry.
Standard choice for $F$ is the **Oseen-Zöcher-Frank free energy**:

$$
\rho F(\rho^{-1}, n, \nabla n) = K_2 (n \cdot \text{curl } n) + \frac{1}{2} K_{11} (\text{div } n)^2 + \frac{1}{2} K_{22} (n \cdot \text{curl } n)^2 + \frac{1}{2} K_{33} ||n \times \text{curl } n||^2,
$$

where $K_2 \neq 0$ for cholesterics and $K_2 = 0$ for nematics. The free energy can also contain additional terms due to external electromagnetic fields. The constants $K_{11}, K_{22}, K_{33}$ are respectively associated to the three principal distinct director axis deformations in nematic liquid crystals, namely, splay, twist, and bend.

**One-constant approximation** : $K_{11} = K_{22} = K_{33} = K$. Free energy is, up to the addition of a divergence,

$$
\rho F(\rho^{-1}, n, \nabla n) = \frac{1}{2} K ||\nabla n||^2.
$$
Recall that the molecular field was given by
\[
\frac{\delta l}{\delta n} = -\rho \frac{\partial F}{\partial n} + \partial_i \left( \rho \frac{\partial F}{\partial n,i} \right) = -h.
\]
In the case of the Oseen-Zöcher-Frank free energy for nematics (that is, \( K_2 = 0 \)), the vector \( h \) is given by
\[
h = K_{11} \text{grad} \text{div} \ n - K_{22}(A \text{curl} \ n + \text{curl}(An)) + K_{33}(B \times \text{curl} \ n + \text{curl}(n \times B)),
\]
where \( A := n \cdot \text{curl} \ n \) and \( B := n \times \text{curl} \ n \).

In the case of the one-constant approximation, \( h = -K \Delta n \).
EXAMPLE 2: ERINGEN EQUATIONS

This is the micropolar theory of liquid crystals. There is a more general approach to microfluids, in general.

Microfluids are fluids whose material points are small deformable particles. Examples of microfluids include liquid crystals, blood, polymer melts, bubbly fluids, suspensions with deformable particles, biological fluids.

SKETCH OF ERINGEN’S THEORY

A material particle $P$ in the fluid is characterized by its position $X$ and by a vector $\Xi$ attached to $P$ that denotes the orientation and intrinsic deformation of $P$. Both $X$ and $\Xi$ have their own motions, $X \mapsto x = \eta(X, t)$ and $\Xi \mapsto \xi = \chi(X, \Xi, t)$, called respectively the macromotion and micromotion.
The material particles are thought of as very small, so a linear approximation in $\Xi$ is permissible for the micromotion:

$$\xi = \chi(X,t)\Xi,$$

where $\chi(X,t) \in \text{GL}(3)^+ := \{ A \in \text{GL}(3) \mid \text{det}(A) > 0 \}$.

The classical Eringen theory considers only three possible groups in the description of the micromotion of the particles:

$$\text{GL}(3)^+ \supset K(3) \supset \text{SO}(3),$$

where

$$K(3) = \left\{ A \in \text{GL}(3)^+ \mid \text{there exists } \lambda \in \mathbb{R} \text{ such that } AA^T = \lambda I_3 \right\}.$$

These cases correspond to micromorphic, microstretch, and micropolar fluids. The Lie group $K(3)$ is a closed subgroup of $\text{GL}(3)^+$ that is associated to rotations and stretch.

The general theory admits other groups describing the micromotion.

Oberwolfach, July 2008
Eringen’s equations for non-dissipative micropolar liquid crystals

\[
\begin{align*}
\rho \frac{D}{dt} \mathbf{u}_l &= \partial_l \frac{\partial \Psi}{\partial \rho} - \partial_k \left( \rho \frac{\partial \Psi}{\partial \gamma_k^a} \gamma_l^a \right), \\
\rho \sigma_l &= \partial_k \left( \rho \frac{\partial \Psi}{\partial \gamma_k^l} \right) - \varepsilon_{lmn} \rho \frac{\partial \Psi}{\partial \gamma_m^a} \gamma_n^a, \\
\frac{D}{dt} \rho + \rho \text{div} \mathbf{u} &= 0, \\
\frac{D}{dt} j_{kl} + (\varepsilon_{kpr} j_{lp} + \varepsilon_{lpr} j_{kp}) \nu_r &= 0, \\
\frac{D}{dt} \gamma_l^a &= \partial_l \nu_a + \nu_{ab} \gamma_l^b - \gamma_r^a \partial_l \nu_r.
\end{align*}
\]

\( \mathbf{u} \in X(\mathcal{D}) \) Eulerian velocity, \( \rho \in \mathcal{F}(\mathcal{D}) \) mass density, \( \nu \in \mathcal{F}(\mathcal{D}, \mathbb{R}^3) \), microrotation rate, where we use the standard isomorphism between \( so(3) \) and \( \mathbb{R}^3 \), \( j_{kl} \in \mathcal{F}(\mathcal{D}, \text{Sym}(3)) \) microinertia tensor (symmetric), \( \sigma_k \), spin inertia is defined by

\[
\sigma_k := j_{kl} \frac{D}{dt} \nu_l + \varepsilon_{klm} j_{mn} \nu_l \nu_n = \frac{D}{dt} (j_{kl} \nu_l),
\]

and \( \gamma = (\gamma_i^{ab}) \in \Omega^1(\mathcal{D}, so(3)) \) wryness tensor. This variable is denoted by \( \gamma = (\gamma_i^a) \) when it is seen as a form with values in \( \mathbb{R}^3 \).

\[
\Psi = \Psi(\rho^{-1}, j, \gamma) : \mathbb{R} \times \text{Sym}(3) \times gl(3) \rightarrow \mathbb{R} \text{ is the free energy.}
\]
The axiom of objectivity requires that
\[ \Psi(\rho^{-1}, A^{-1}jA, A^{-1}\gamma A) = \Psi(\rho^{-1}, j, \gamma), \]
for all \( A \in O(3) \) (for nematics and nonchiral smectics), or for all \( A \in SO(3) \) (for cholesterics and chiral smectics).

These equations are Euler-Poincaré/Lie-Poisson for the group
\[
(Diff(D) \circledast F(D, SO(3))) \circledast (F(D) \times F(D, Sym(3)) \times F(D, so(3))).
\]

**EXPLANATION:**

- \( Diff(D) \) acts on \( F(D, SO(3)) \) via the *right* action

\[
(\eta, \chi) \in Diff(D) \times F(D, SO(3)) \mapsto \chi \circ \eta \in F(D, SO(3)).
\]

Therefore, the group multiplication in \( Diff(D) \circledast F(D, SO(3)) \) is
\[
(\eta, \chi)(\varphi, \psi) = (\eta \circ \varphi, (\chi \circ \varphi)\psi).
\]
• The bracket of $\mathcal{X}(\mathcal{D}) \otimes \mathcal{F}(\mathcal{D}, \mathfrak{so}(3))$ is

$$\text{ad}_{(u, \nu)}(v, \zeta) = (\text{ad}_u v, \text{ad}_\nu \zeta + d\nu \cdot v - d\zeta \cdot u),$$

where $\text{ad}_u v = -[u, v]$, $\text{ad}_\nu \zeta \in \mathcal{F}(\mathcal{D}, \mathfrak{so}(3))$ is given by $\text{ad}_\nu \zeta(x) \coloneqq \text{ad}_{\nu(x)} \zeta(x)$, and $d\nu \cdot v \in \mathcal{F}(\mathcal{D}, \mathfrak{so}(3))$ is given by $d\nu \cdot v(x) \coloneqq d\nu(x)(v(x))$.

• $(\eta, \chi) \in \text{Diff}(\mathcal{D}) \otimes \mathcal{F}(\mathcal{D}, \text{SO}(3))$ acts \textit{linearly and on the right} on the advected quantities $(\rho, j) \in \mathcal{F}(\mathcal{D}) \times \mathcal{F}(\mathcal{D}, \text{Sym}(3))$, by

$$(\rho, j) \mapsto \left(J\eta(\rho \circ \eta), \chi^T(j \circ \eta)\chi\right).$$

• $(\eta, \chi) \in \text{Diff}(\mathcal{D}) \otimes \mathcal{F}(\mathcal{D}, \text{SO}(3))$ acts on $\gamma \in \Omega^1(\mathcal{D}, \mathfrak{so}(3))$ by

$$\gamma \mapsto \chi^{-1}(\eta^*\gamma)\chi + \chi^{-1}T\chi.$$  

This is a \textit{right affine} action. Note that $\gamma$ transforms as a connection.
The reduced Lagrangian

\[ l: \mathcal{X}(D) \otimes \mathcal{F}(D, \mathbb{R}^3) \otimes \mathcal{F}(D) \oplus \mathcal{F}(D, \text{Sym}(3)) \oplus \Omega^1(D, \mathfrak{so}(3)) \rightarrow \mathbb{R} \]

is given by

\[ l(u, \nu, \rho, j, \gamma) = \frac{1}{2} \int_D \rho \|u\|^2 \mu + \frac{1}{2} \int_D \rho (j \nu \cdot \nu) \mu - \int_D \rho \Psi(\rho^{-1}, j, \gamma) \mu. \]

The affine Euler-Poincaré equations for \( l \) are:

\[
\begin{align*}
\rho \left( \frac{\partial}{\partial t} u + \nabla u u \right) &= \text{grad} \frac{\partial \Psi}{\partial \rho} - \partial_k \left( \rho \frac{\partial \Psi}{\partial \gamma^a_k} \gamma^a \right), \\
\frac{D}{dt} \nu - (j \nu) \times \nu &= -\frac{1}{\rho} \text{div} \left( \rho \frac{\partial \Psi}{\partial \gamma} \right) + \gamma^a \times \frac{\partial \Psi}{\partial \gamma^a}, \\
\frac{\partial}{\partial t} \rho + \text{div}(\rho u) &= 0, \\
\frac{\partial}{\partial t} \gamma + \mathcal{L}_u \gamma + d^\gamma \nu &= 0,
\end{align*}
\]

which are the Eringen equations after the change of variables \( \gamma \mapsto -\gamma \). Here \( d^\gamma \nu(v) := d\nu(v) + [\gamma(v), \nu] \).

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$L_{(\rho_0,j_0,\gamma_0)} : T[\text{Diff}(\mathcal{D}) \otimes \mathcal{F}(\mathcal{D}, \text{SO}(3))] \to \mathbb{R}$ induced by the Lagrangian $l$ by right translation and freezing the parameters. A curve $(\eta, \chi) \in \text{Diff}(\mathcal{D}) \otimes \mathcal{F}(\mathcal{D}, \text{SO}(3))$ is a solution of the Euler-Lagrange equations associated to $L_{(\rho_0,j_0,\gamma_0)}$ if and only if the curve

$$(u, v) := (\dot{\eta} \circ \eta^{-1}, \dot{\chi} \chi^{-1} \circ \eta^{-1}) \in \mathfrak{x}(\mathcal{D}) \otimes \mathcal{F}(\mathcal{D}, \text{SO}(3))$$

is a solution of the previous equations with initial conditions $(\rho_0, j_0, \gamma_0)$. The evolution of the mass density $\rho$, the microinertia $j$, and the wryness tensor $\gamma$ is given by

$$\rho = J(\eta^{-1})(\rho_0 \circ \eta^{-1}), \quad j = (\chi j_0 \chi^{-1}) \circ \eta^{-1}, \quad \gamma = \eta^* (\chi \gamma_0 \chi^{-1} + \chi T \chi^{-1}).$$

If the initial value $\gamma_0$ is zero, then the evolution of $\gamma$ is given by

$$\gamma = \eta^* (\chi T \chi^{-1}).$$

This relation is usually taken as a definition of $\gamma$ when using the Eringen equations without the last one. This is often the case in the literature.

Oberwolfach, July 2008
**PROBLEM:** Eringen defines a smectic liquid crystal in the micropolar theory by the condition $\text{Tr}(\gamma) = \gamma_1^1 + \gamma_2^2 + \gamma_3^3 = 0$. But this is *not* preserved by the evolution $\gamma = \eta^*(\chi\gamma_0\chi^{-1} + \chi T\chi^{-1})$, in general. This is consistent with: the equation

$$\frac{\partial \gamma}{\partial t} + \mathcal{L}_u \gamma + d\nu + \gamma \times \nu = 0.$$ 

does not show that if the initial condition for $\gamma$ has trace zero then $\text{Tr} \gamma = 0$ for all time.

So we believe that Eringen’s definition of smectic is incorrect. Here is a proposal. Find a function $F$ that is invariant under the action

$$\gamma \mapsto \chi^{-1}(\eta^*\gamma)\chi + \chi^{-1}T\chi.$$ 

In fact, the $\eta$ plays no role so we need an $\mathcal{F}(\mathcal{D}, \text{SO}(3))$-invariant function under the action

$$\nu \mapsto \chi^{-1}\nu + \chi^{-1}T\chi,$$

where $\nu : \mathcal{D} \rightarrow \mathbb{R}^3$, $\chi : \mathcal{D} \rightarrow \text{SO}(3)$. 

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The affine Lie-Poisson bracket is in this case equal to:

\[
\{f, g\}(m, \kappa, \rho, j) = \int_D m \cdot \left[ \frac{\delta f}{\delta m}, \frac{\delta g}{\delta m} \right] \mu
\]

\[
+ \int_D \kappa \cdot \left( \text{ad}_{\frac{\delta f}{\delta \kappa}} \frac{\delta g}{\delta \kappa} + \frac{\delta f}{\delta \kappa} \cdot \frac{\delta g}{\delta m} - \frac{\delta g}{\delta \kappa} \cdot \frac{\delta f}{\delta m} \right) \mu
\]

\[
+ \int_D \rho \left( \text{d} \left( \frac{\delta f}{\delta \rho} \right) \frac{\delta g}{\delta m} - \text{d} \left( \frac{\delta g}{\delta \rho} \right) \frac{\delta f}{\delta m} \right) \mu
\]

\[
+ \int_D j \cdot \left( \text{div} \left( \frac{\delta f}{\delta j} \frac{\delta g}{\delta m} \right) + \left[ \frac{\delta f}{\delta j}, \frac{\delta g}{\delta \kappa} \right] - \text{div} \left( \frac{\delta g}{\delta j} \frac{\delta f}{\delta m} \right) - \left[ \frac{\delta g}{\delta j}, \frac{\delta f}{\delta \kappa} \right] \right) \mu
\]

\[
+ \int_D \left[ \left( \text{d}^\gamma \frac{\delta f}{\delta \kappa} + \mathcal{L}_{\frac{\delta f}{\delta m}} \gamma \right) \frac{\delta g}{\delta \gamma} - \left( \text{d}^\gamma \frac{\delta g}{\delta \kappa} + \mathcal{L}_{\frac{\delta g}{\delta m}} \gamma \right) \frac{\delta f}{\delta \gamma} \right] \mu
\]

where the brackets in the second to last term denote the usual commutator bracket of matrices. Circulation theorems are:

\[
\frac{d}{dt} \oint_{c_t} u^b = \oint_{c_t} \frac{\partial \psi}{\partial i} \cdot di + \frac{\partial \psi}{\partial \gamma} \cdot i_- d\gamma - \frac{1}{\rho} \text{div} \left( \rho \frac{\partial \psi}{\partial \gamma} \right) \cdot \gamma.
\]

and

\[
\frac{d}{dt} \oint_{c_t} \gamma = \oint_{c_t} \mathbf{v} \times \gamma
\]
One can show that the ordered micropolar theory of Lhuillier-Rey is a direct generalization of the Ericksen-Leslie director theory. So one needs to compare the Lhuillier-Rey theory to the Eringen theory.

**PROBLEM:** How does one pass from ordered micropolar (or Ericksen-Leslie) theory to Eringen theory? Eringen says that it is given by \( \gamma = \nabla n \times n \) and \( j := J(I_3 - n \otimes n) \). If so, then transformation laws should be preserved.

a.) If \( n \mapsto \chi^{-1}(n \circ \eta) \) is the transformation law for \( n \), which is imposed by Lhuillier-Rey (and also Ericksen-Leslie) theory, then \( j \) transforms as \( j \mapsto \chi^T(j \circ \eta) \chi \), which is correct. However, \( \gamma \) does not transform as \( \gamma \mapsto \chi^{-1}(\eta^*\gamma) \chi + \chi^{-1}T \chi \).

b.) One can find, by a brutal computation, what the Eringen equations should be under this transformation, if \((u, v, \rho, j, n)\) are solutions of the Lhuillier-Rey equations. The resulting system is almost the Eringen system: there are two bad factors of \( j/J \).