Reachability Analysis for Hybrid Dynamic Systems*

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Motivation for Modeling by Hybrid Dynamic Systems

Hybrid Dynamic Systems (HDS):

„*Discrete event system equipped with continuous-valued dynamics*"

$\vee$ „*Continuous dynamics enriched by discontinuities (switching, jumps)*"

<u>Examples:</u>

(a) Walking humanoid robots

[Ulbrich et al., TUM]

- x_i: joint positions, velocities, …
- z_i: ground contact situation

(b) Autonomously driving cars

[Wünsche et al., UniBW]

- x_i: distances, velocities, …
- z_i: driving modes (gears; accelerating, braking, …)

$$\boxed{x_i \in IR, \ \ z_i \in IN}$$

Motivation for Modeling by Hybrid Dynamic Systems

Examples (ctd.):

(c) Manufacturing plants

hierarchical, distributed heterogenous control

[Zäh et al., TUM]

- x_i: work piece positions, robot arm control , …
- z_i: processing status, resource conditions, …)

(d) Chemical processing systems

[BASF]

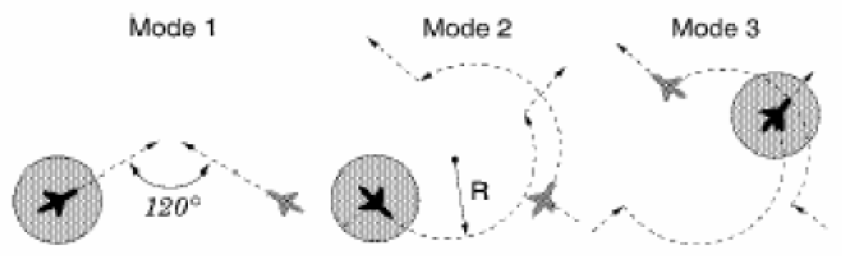
- x_i: temperature, levels, concentrations, …
- z_i: production phase, actuator state, …)

$\boxed{x_i \in IR,\ \ z_i \in IN}$

Motivation for Modeling by Hybrid Dynamic Systems

Examples (ctd.):

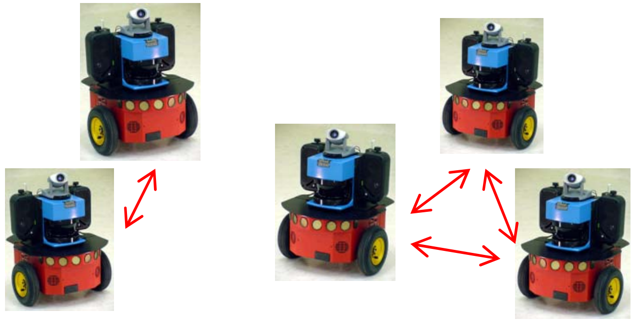
(e) Air conflict resolution



[Tomlin et al., Stanford]

- x_i: speed,
  orientation, …
- z_i: flight mode,
  (cruise, conflict resolution)

(f) Multi robot systems



[LSR, TUM]

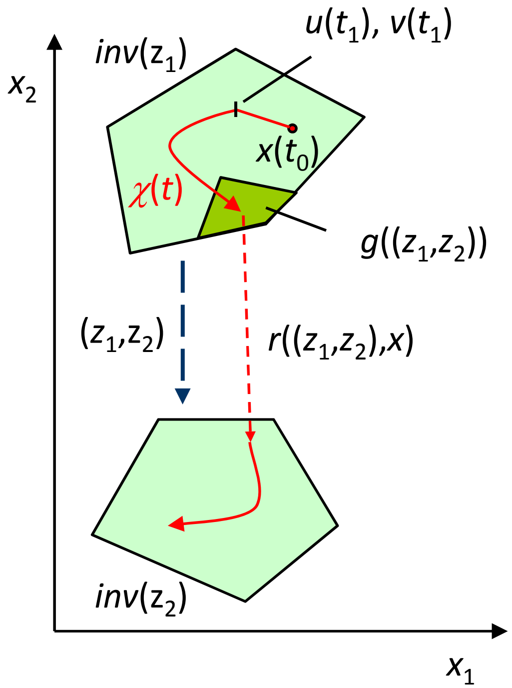
- x_i: position,
  speed, …
- z_i: communcation topology,
  formation, …

$$\boxed{x_i \in IR,\ \ z_i \in IN}$$

Hybrid Dynamic Systems: Syntax

Hybrid Automaton: $HA = (X, U, V, Z, inv, \Theta, g, r, f)$

- Continuous states: $x \in X \subseteq R^{n_x}$
- Continuous inputs: $u \in U = [u_1^-, u_1^+] \times \ldots \times [u_{n_u}^-, u_{n_u}^+]$
- Discrete inputs: $v \in V = \{v_1, \ldots, v_{n_d}\},\ v_j \in R^{n_v}$
- Discrete states (locations): $Z = \{z_1, \ldots, z_{n_z}\}$
- Invariants: $inv : Z \rightarrow 2^X$
- Transitions: $(z_1, z_2) \in \Theta \subseteq Z \times Z$
- Transition guards: $g : \Theta \rightarrow 2^X$
- Reset functions: $r : \Theta \times X \rightarrow X$
- Continuous dynamics: $f : Z \times X \times U \times V \rightarrow R^{n_x}$, such that: $\dot{x} = f(z, x, u, v)$

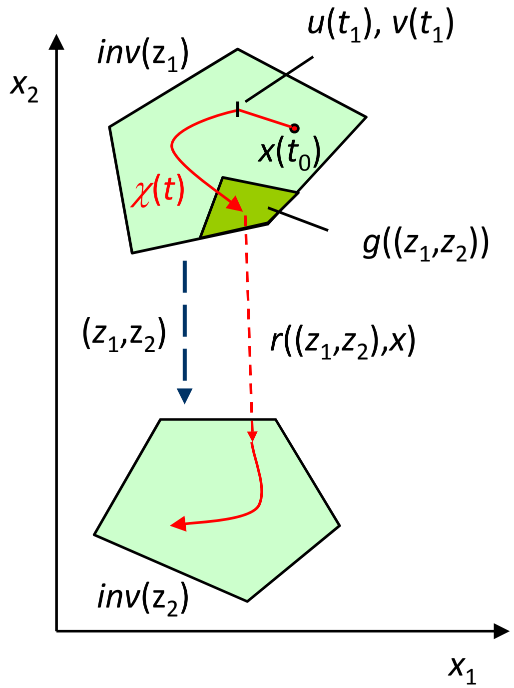


Hybrid Dynamic Systems: Semantics

Set of event times: $T = \{t_0, t_1, t_2, ...\}$

Input trajectories: $\phi_u = (u_0, u_1, ...) \in \Phi_u$,
$\phi_v = (v_0, v_1, ...) \in \Phi_v$
with inputs u_k, v_k on $t \in [t_k, t_{k+1}[$

Hybrid States: $s_k \in (z_k, x_k) \in S$ with:
$x_k = x(t_k)$, $z_k = z(t_k)$

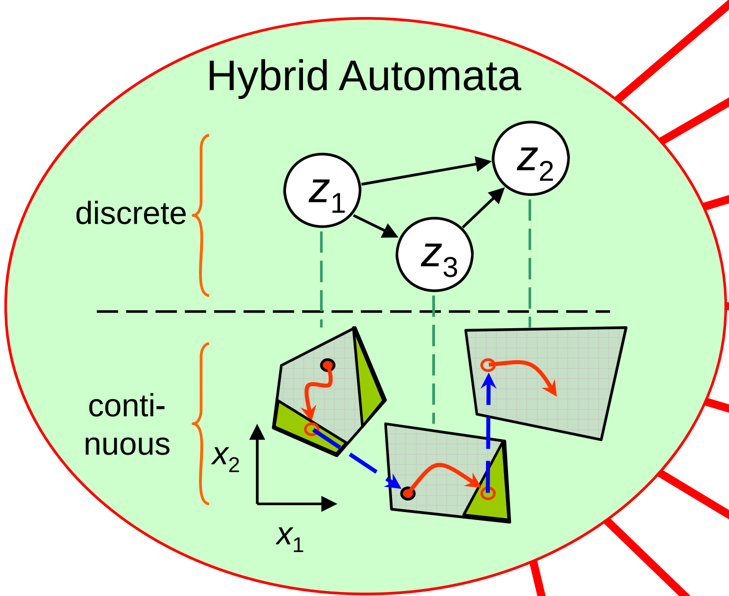


Feasible execution for given s_0, ϕ_u, ϕ_v :
$\phi_s = (s_0, s_1, s_2, ...)$ with s_k from:

(i) contin. evolution: $\chi(0) = x_k$ and $\chi(t)$ is unique solution of ODEs for $t \in [0, \tau]$;
$\chi(t) \in inv(z_k)$, (optional: $\chi(t) \notin g((z_k, \bullet))$ für $t < \tau$)

(ii) transition: $(z_k, z_{k+1}) \in \Theta$, $\chi(\tau) \in g((z_k,z_{k+1}))$ and
$x_{k+1} = r((z_k,z_{k+1}), \chi(\tau)) \in inv(z_{k+1})$

Investigations for Hybrid Dynamic Systems



- Simulation (including sliding modes)
- Abstraction and Refinement
- Reachability Analysis and Verification
- Optimal Control
- Scheduling for TA
- Controller Synthesis
- Design of SFC controllers with subsequent testing / verification
- Networked Control Systems

Design Tasks for HDS using Reachability Computation

(a) Verification:

given:
- plant P of type HA
- controller C of type HA
- specification γ

  (e.g. safety: $AG \neg S_{unsafe}$)

show that: $P \,||\, C \models \gamma$

if false, redesign C

(b) Controller Synthesis:

given:
- plant P of type HA
- specification γ

  (goal attainment: $AF\; S_{target}$)

generate C such that:

$$P \,||\, C \models \gamma$$

(c) Optimal Control:

given:
- plant P of type HA
- goal and safety spec. γ
- performance measure ψ

compute C such that:

$$\min_{\phi_u, \phi_v} \psi$$

$$\text{s.t.}: P \,||\, C \models \gamma$$

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Reachable Set Computation for HDS

Def.: Reachable Set of *HA*

given:
- initialization $S_0 \subset S$,
- sets of input trajectories Φ_u, Φ_v

$$R := \left\{ \begin{array}{l} s \in S \mid \exists s_0 \in S_0, \phi_u \in \Phi_u, \\ \phi_v \in \Phi_v : s \text{ is reached along} \\ \text{any } \phi_s \in \Phi_s \end{array} \right\}$$

Assumption: $M := P \,||\, C$, i.e. M is autonomous (Φ_u, Φ_v restricted by C)

Standard algorithm for computing R:

$S_0 := \{z_0\} \times X_0$, $k := 0$
$D := S_0$, $R := \emptyset$
WHILE ($D \neq \emptyset$) *Termination?*
  $k := k + 1$
  $R := R \cup D$
  $S_k := Reach(D)$
  $D := S_k \setminus R$
END

step k:
- transition
- time step

The Challenge

Problem: scales badly in most respects!

- infinitely many executions of *M* must be analyzed
- reachable sets have to be represented efficiently
- set intersection, subtraction, and union must be computable

In contrast:

For finite state automata $A = (Z, z_0, \Theta)$, the reachable set:

$R := \{z_k \in Z \mid \exists z_0 \in Z_0 : (z_0, z_1) \in \Theta, \ldots, (z_{k-1}, z_k) \in \Theta\}$ is efficiently computable.

[Verification of $A \models \gamma$ successfully reported for systems with $|Z| \approx 10^{20}$.]

<u>Approach:</u>

- Use of abstractions *A* of *HA*
- Consider the specification of analysis, synthesis, or optimization for computing *R* $\rightarrow$ reduce use of *Reach*, $\cup$, */*

Abstraction-Based Reachability Analysis: Principle

Objective: identify evolutions of *M* that potentially violate γ based on abstractions *A* and evaluate *Reach*(*D*) only for these evolutions!

Principle:

– Generate a discrete abstraction *A* of the hybrid model *M* (*A*: state transition system)

– Determine counterexamples (CEs) for *A* as evolutions of *M* that potentially violate the specification (CE: run of *A* that connects the initial and critical state)

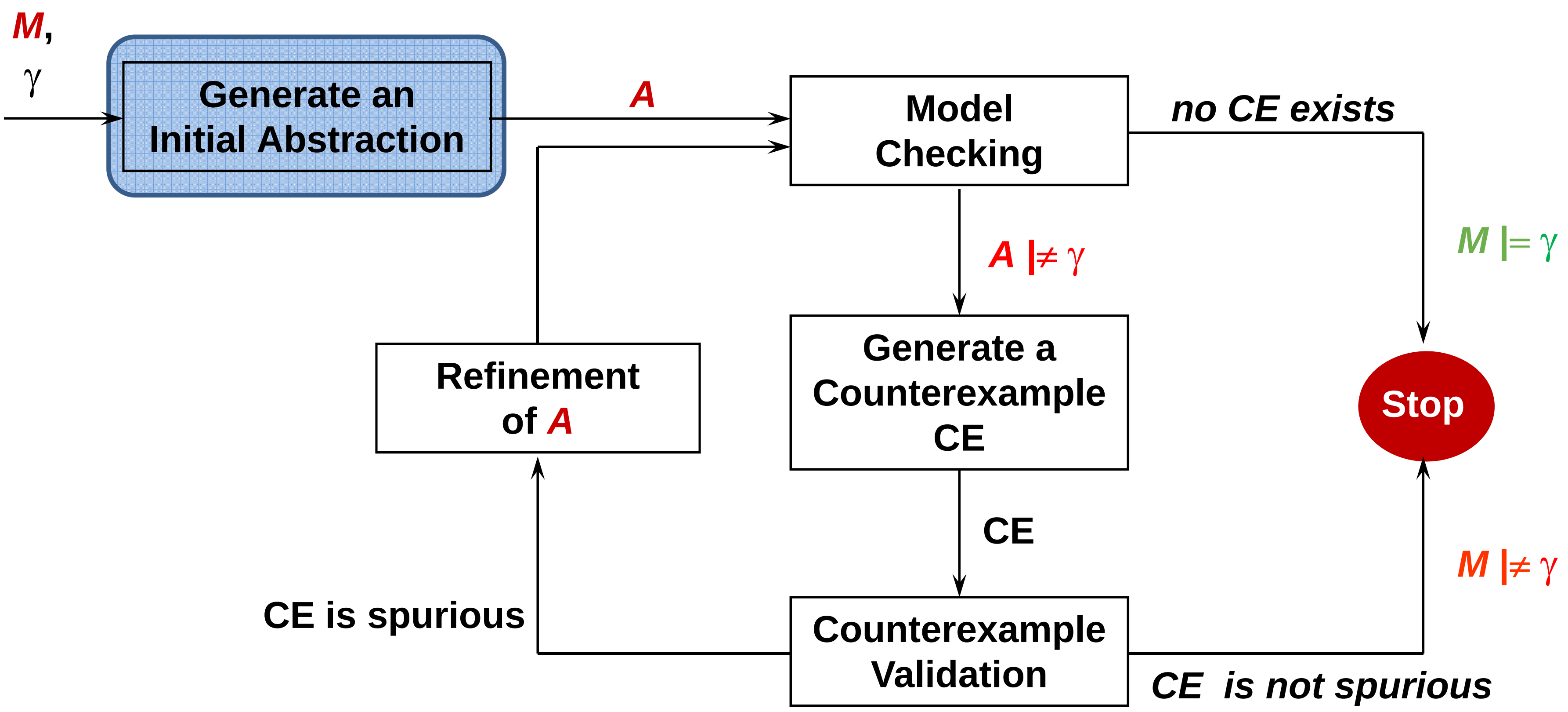
– (In-)Validate CE for *M*

– if CE is invalid, add details to *A* (refinement)

Assumption: let γ denote a safety specification:

given $S_{unsafe} \subset S: \neg\exists(s_0 \in S_0, \phi_s \in \Phi_s): (z, \chi(\tau)) \in S_{unsafe}$

Abstraction-Based Reachability Analysis: Principle



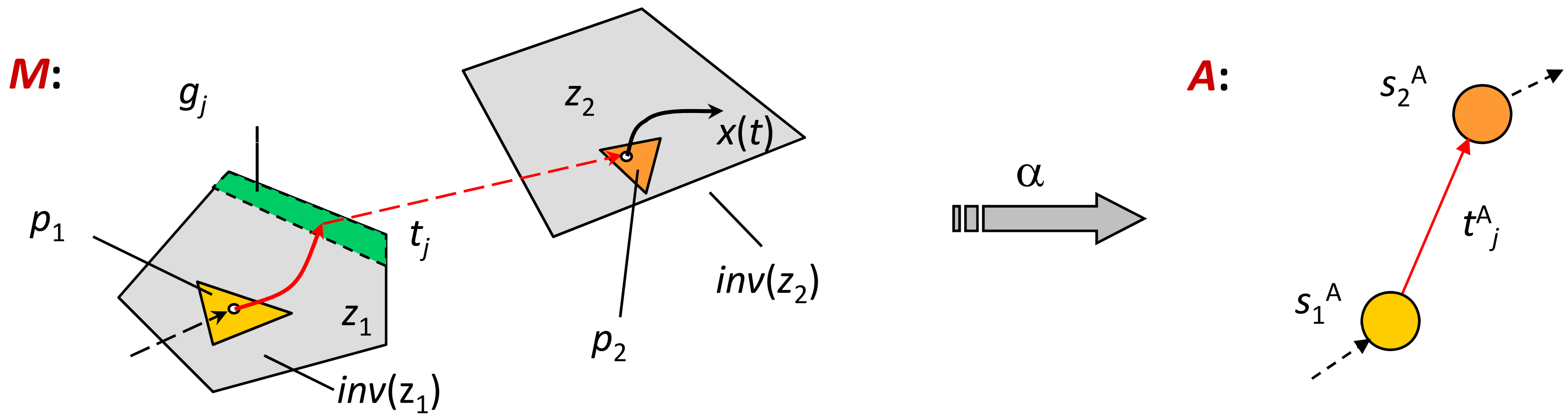
[for discrete automata: Clarke et al., 2000]

Initial Abstraction

Abstract away the continuous part of *M*, retain the discrete dynamics:

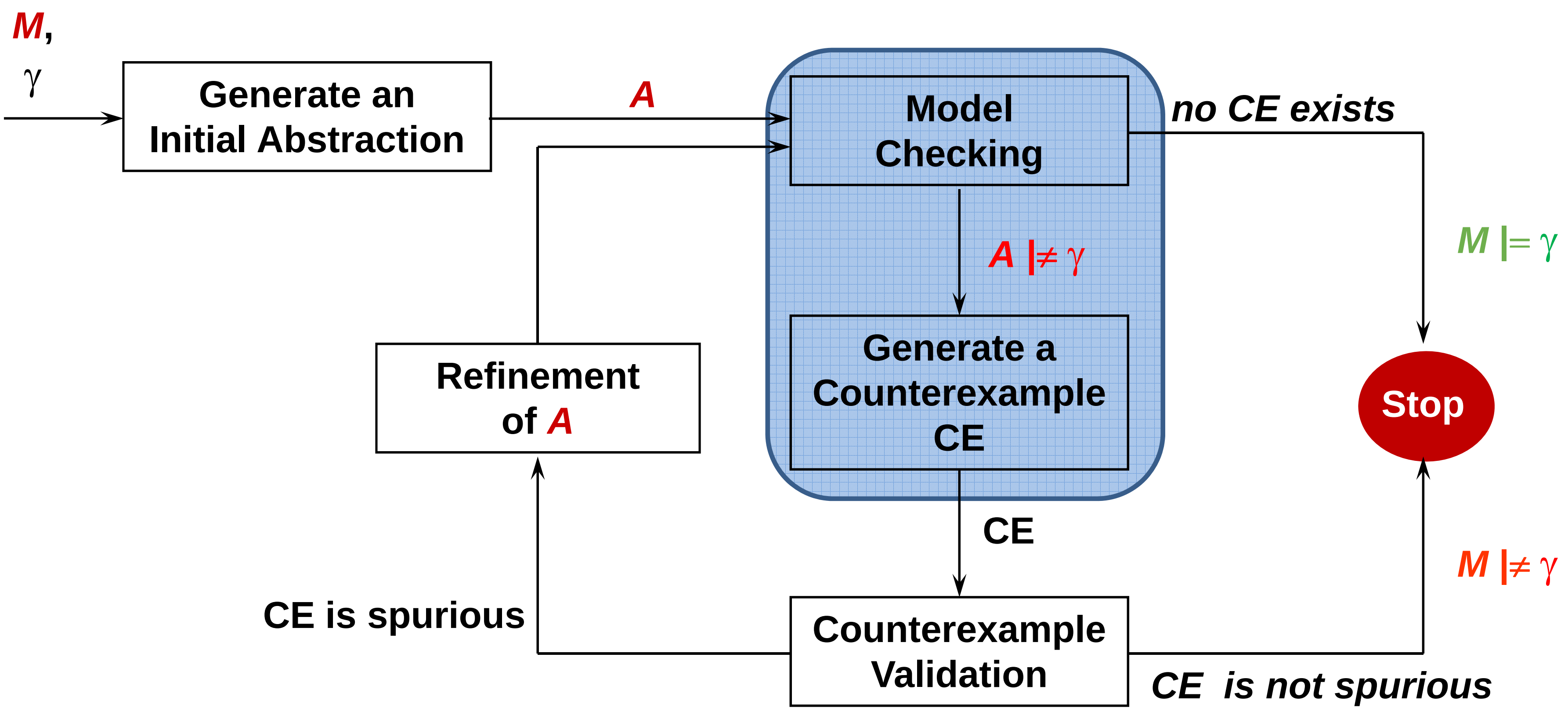
– a state in *A* represents a location in *M* (exception: initial location)

– one transition in *A* for each transition in *M*



- *A* is a simple state transition system: $A = (S^A, s^A_0, \Theta^A)$
- *A* is an *abstraction*: – contains all evolutions of *M*
  – can contain additional behaviors

Abstraction-Based Reachability Analysis: Principle

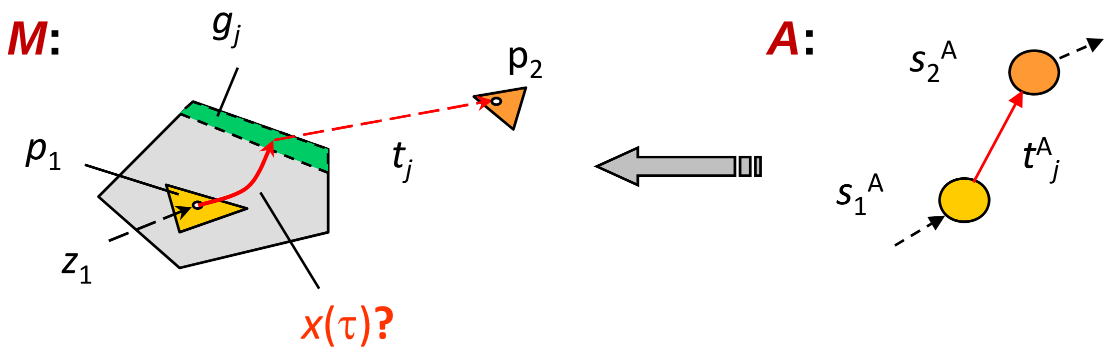


Model Checking and Counterexamples

- Standard model checking for FSA can be applied (breadth-first search for S^A starting from s^A_0)
- if γ is violated, i.e., a critical state s^A_f is reachable:

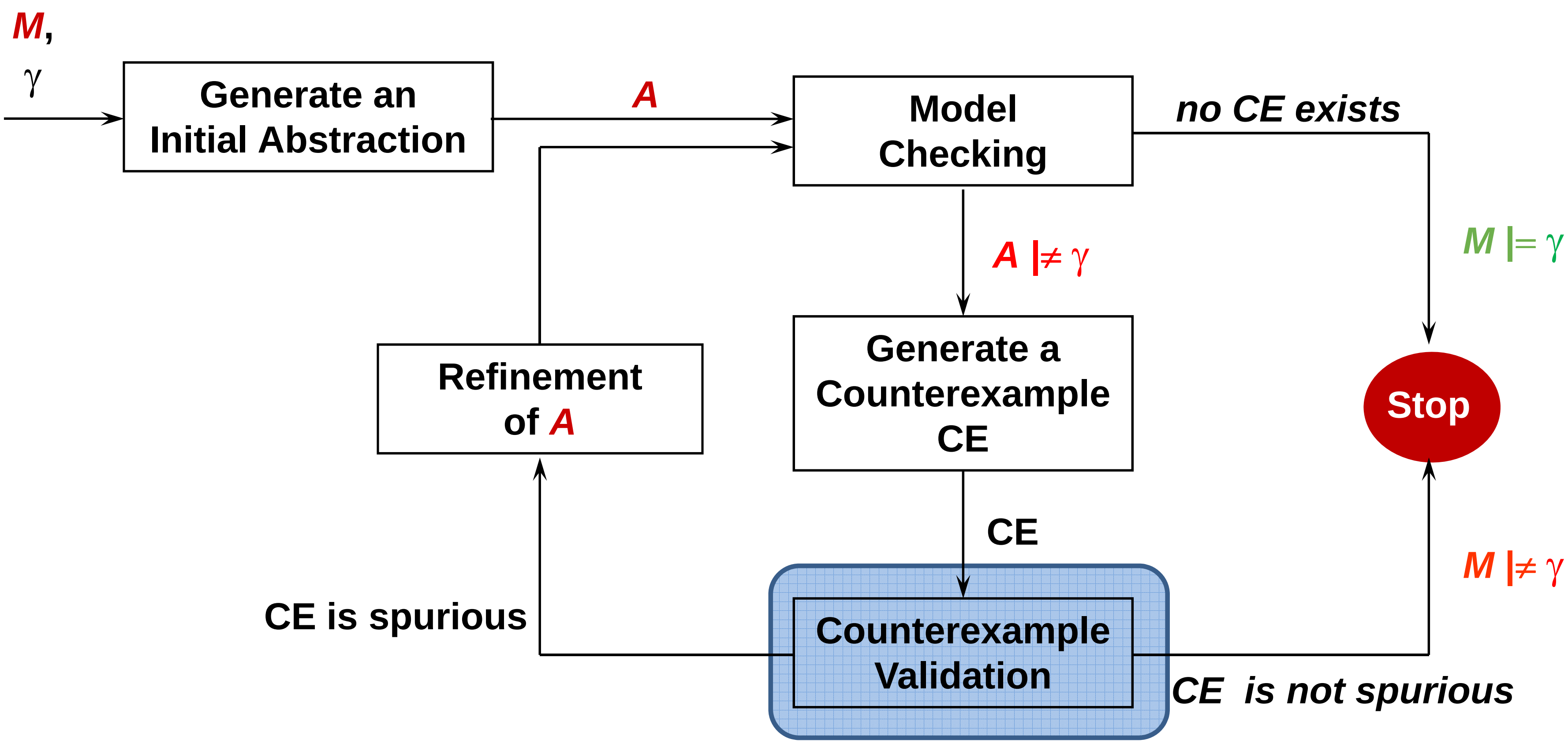
  counterexample (CE): $(s^A_0, s^A_1, ..., s^A_f)$

Question: Does a corresponding evolution exist for *M*?



$\Rightarrow$ validation along the counterexample

Abstraction-Based Reachability Analysis: Principle

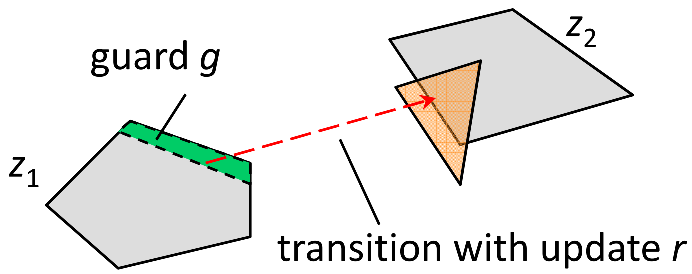


Validation of Counterexamples (1)

VM1: Intersection Check

transition of *A* is invalid if:

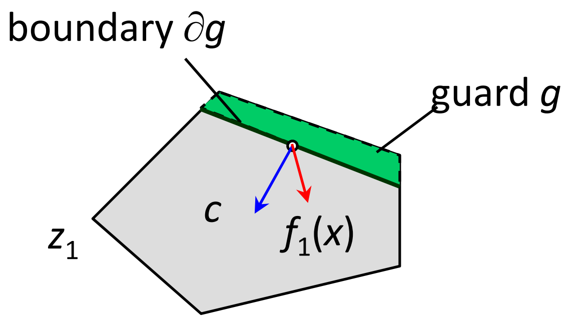
$r_j(x) \notin inv(z_2) \ \forall \ x \in g_j$



VM2: Gradient Check

determine gradient on the guard boundaries

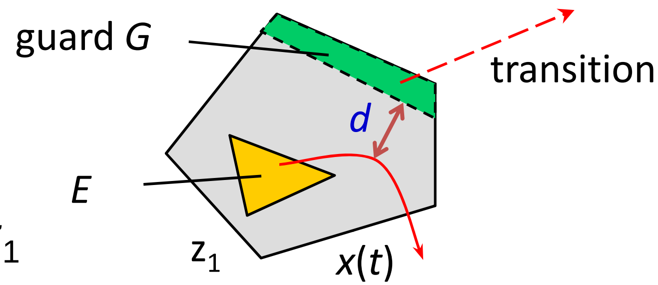
transition of *A* is invalid if: $\min_{x \in \partial g} (c^T \cdot f(x)) > 0$



VM3: Connectivity Check

transition of *A* is invalid if: $\min_{x_0 \in E}\{d\} > 0$

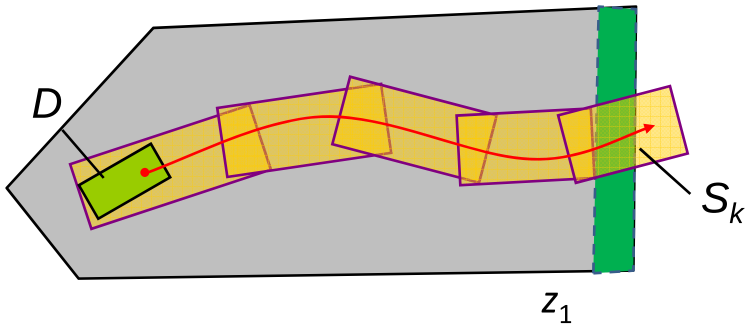
s.t. dynamics of *HA* in z_1



Validation of Counterexamples (2)

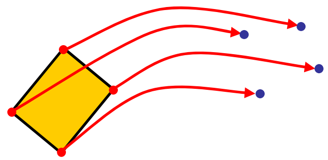
VM4: Flowpipe Approximation

computation of S_k = Reach(D):

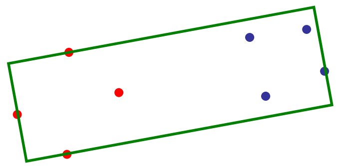


for each segment:

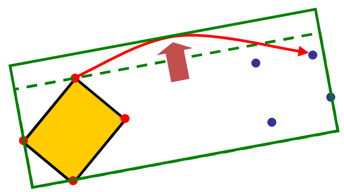
(1) simulate vertices for a timestep



(2) determine an oriented hyper-rectangle
(orientation from sample covariance matrix)



(3) enlarge hull (nonlinear optimization with embedded simulation)



Validation of Counterexamples (3)

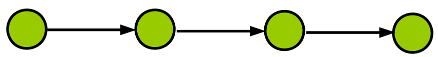
Four methods to refute the existence of counterexamples:

VM1 VM2 VM3 VM4

$$\xrightarrow{\qquad\qquad} \quad \frac{\text{accuracy (over-approximation)}}{\text{computational cost}}$$

$\Rightarrow$ *refute with the least effort possible*

Application mode:

CE: 

***sequential*:**

VM1 VM1 VM1
VM2 VM2 VM2
VM3 ...

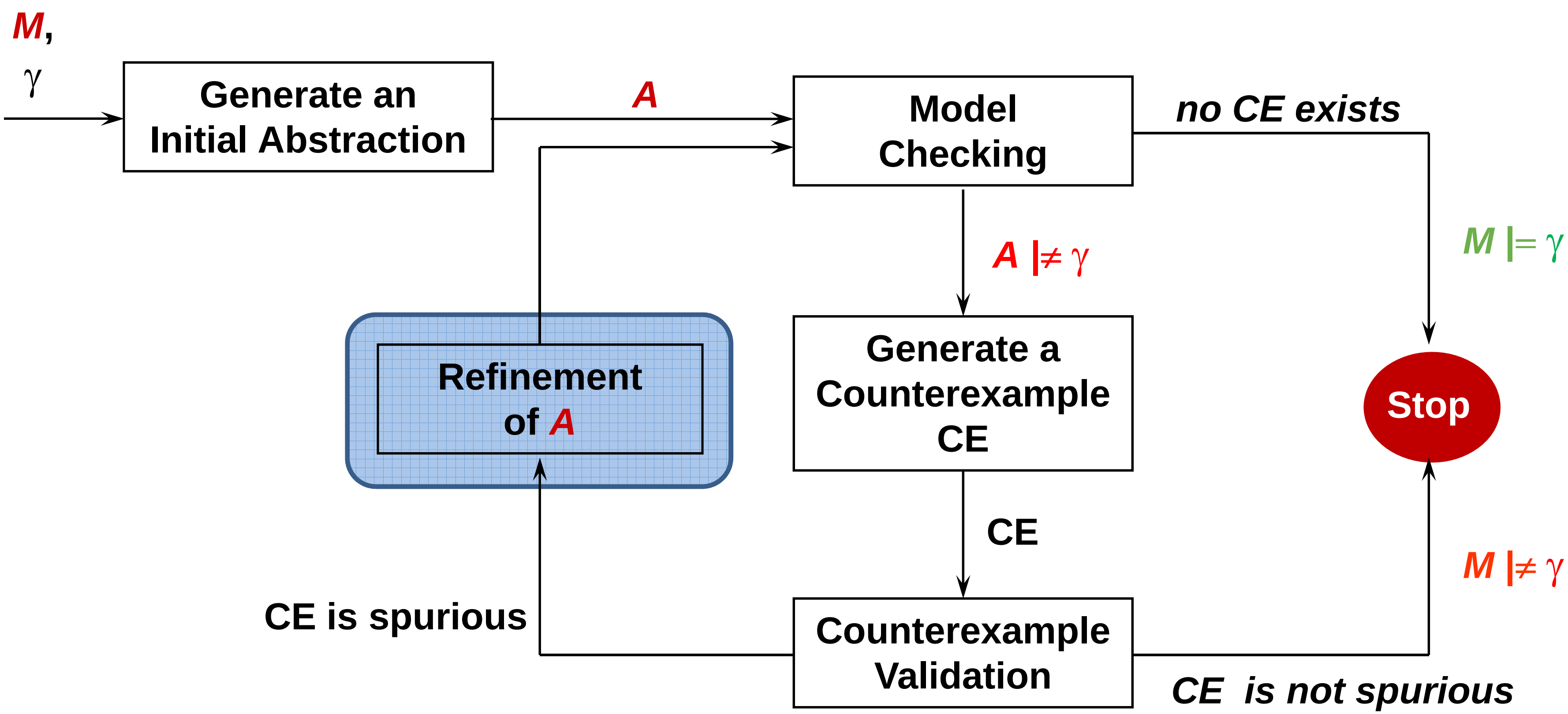
***alternating*:**

VM1
VM2
VM3
VM4 VM1
VM2
...

progress

terminate process as soon as one transition is refuted!

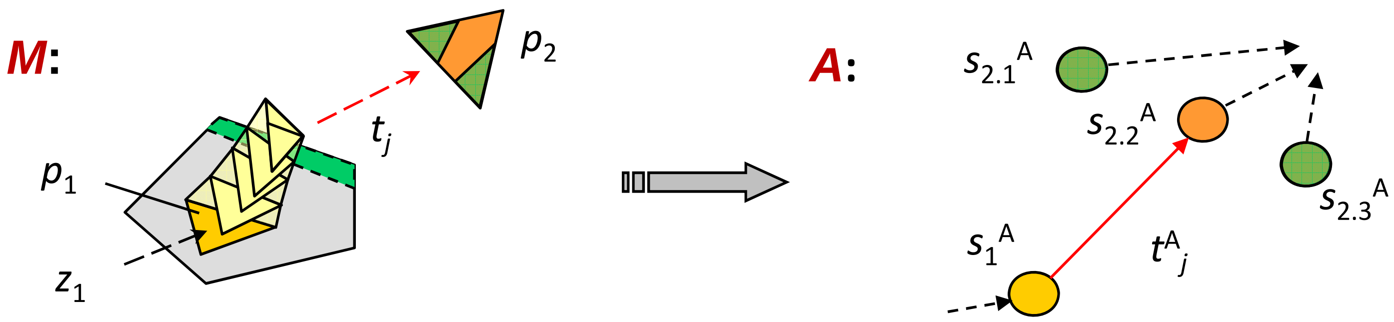
Abstraction-Based Reachability Analysis: Principle



Refinement of *A*

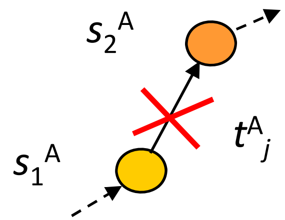
Refinement based on the flowpipe approximation:

If $x(\tau)$ exists, i.e. the transition $s_1^A \rightarrow s_2^A$ is validated, the automaton *A* is refined by splitting s_2^A:



Purging of *A*:

If $x(\tau)$ does not exist, the corresponding transition is removed from *A*, and the method proceeds with a new CE.

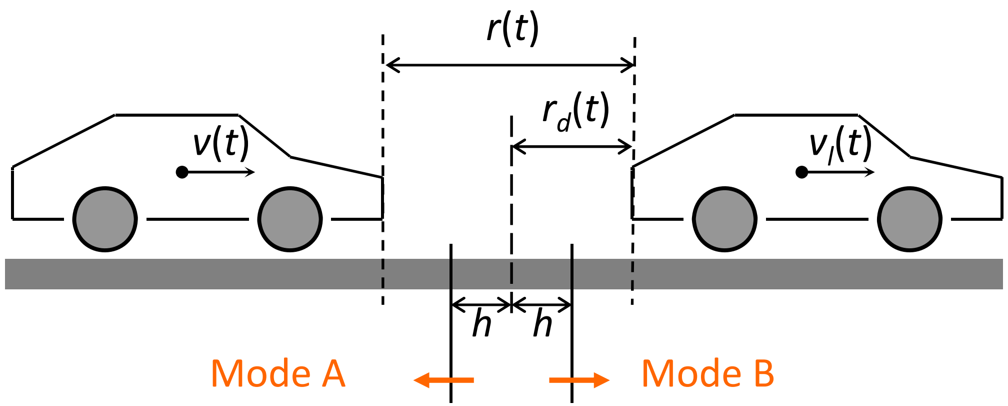


Example: Verification of a Cruise Controller (1)

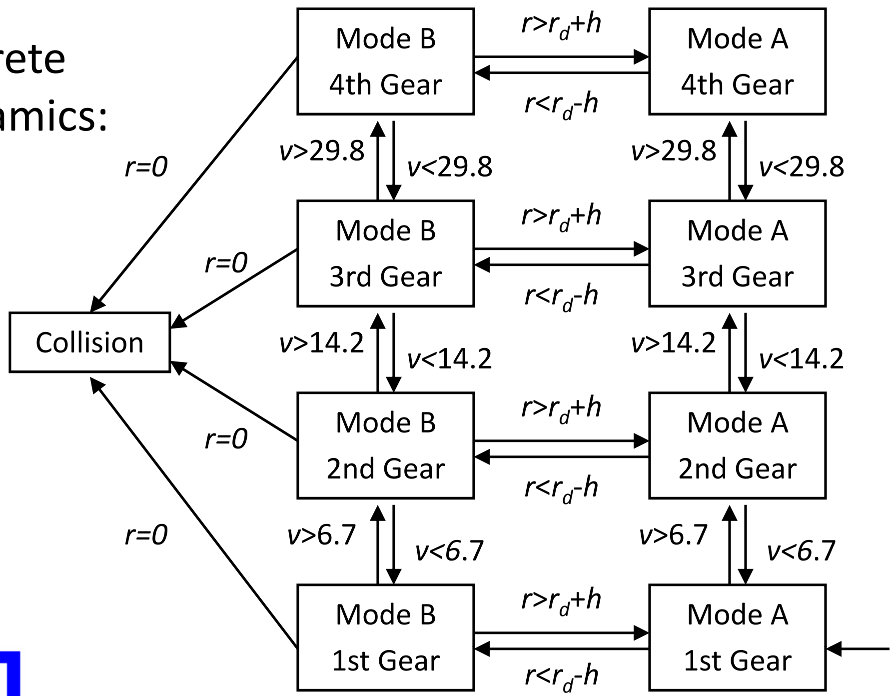
Control objectives:

- Mode A: constant speed
- Mode B: distance control

Safety specification γ: $\neg(r < r_{critical})$



Discrete Dynamics:



Continuous Dynamics:

- in „Collision“: $\dot{r} = 0, \ \dot{v} = 0$
- else:

$$\dot{r} = v_l - v$$

$$\dot{v} = min\left(max\left(0, \frac{a_d + 3.5}{a_{par} + 3.5}, 1\right) \cdot (a_{par} + 3.5) - 3.5 \right)$$

$$a_d = f(Mode, v, v_d, v_l, r), \ a_{par} = f(gear)$$

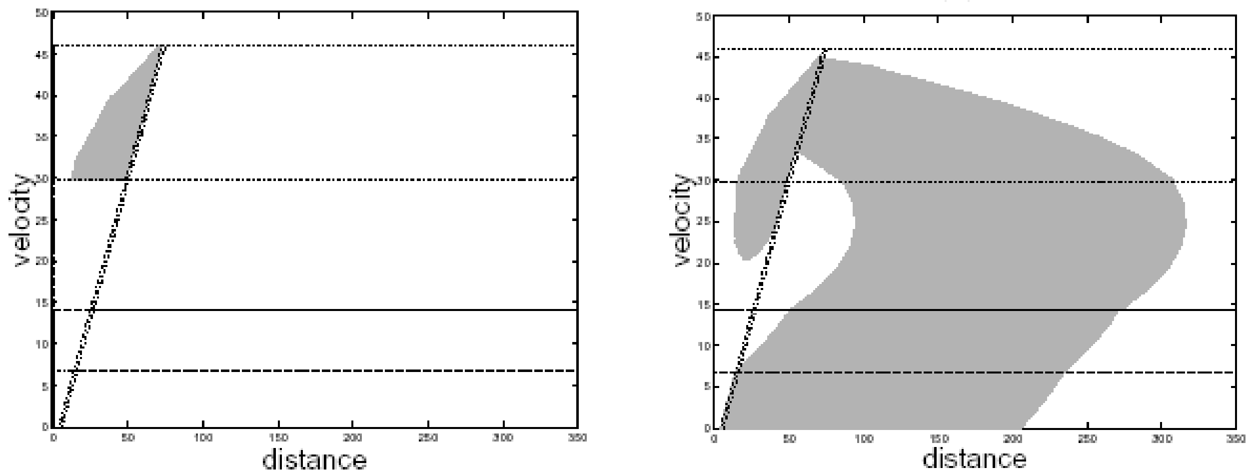
Example: Verification of a Cruise Controller (2)

Verification for given parametrization:

- 10 counterexamples
- VM4 (reachable set computation) only applied once
- final abstract model *A*: 11 states
- computation time: < 10 seconds on a standard PC

Result: $\neg(r < r_{critical})$ does hold

Reachable sets:



abstraction-based complete *R*-computation

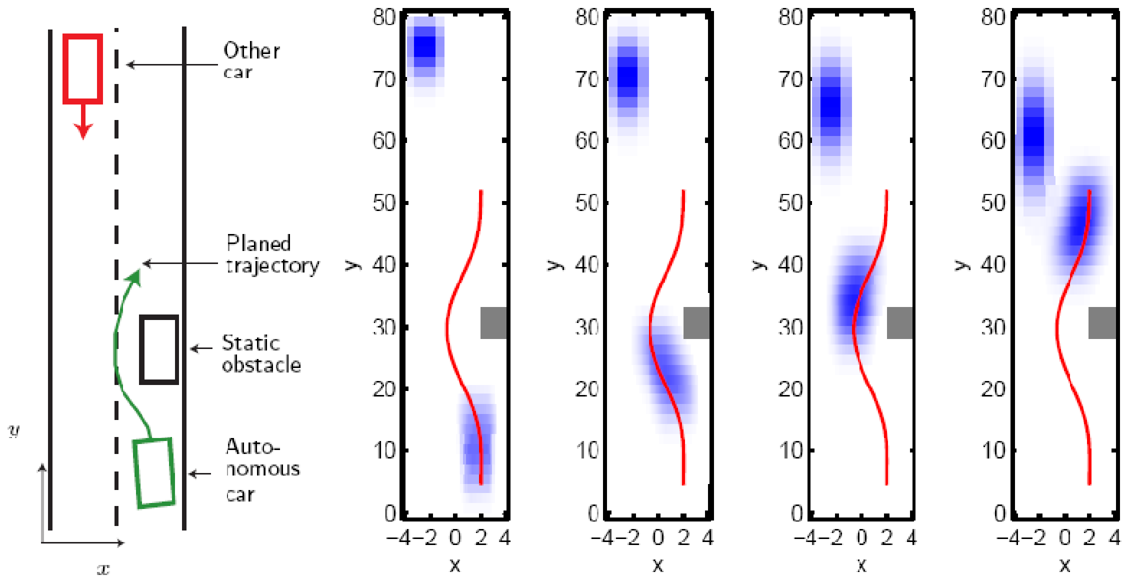
Extension to Stochastic Verification

Context:

iterative online verification of driving strategies for autonomous cars

<u>Method:</u>

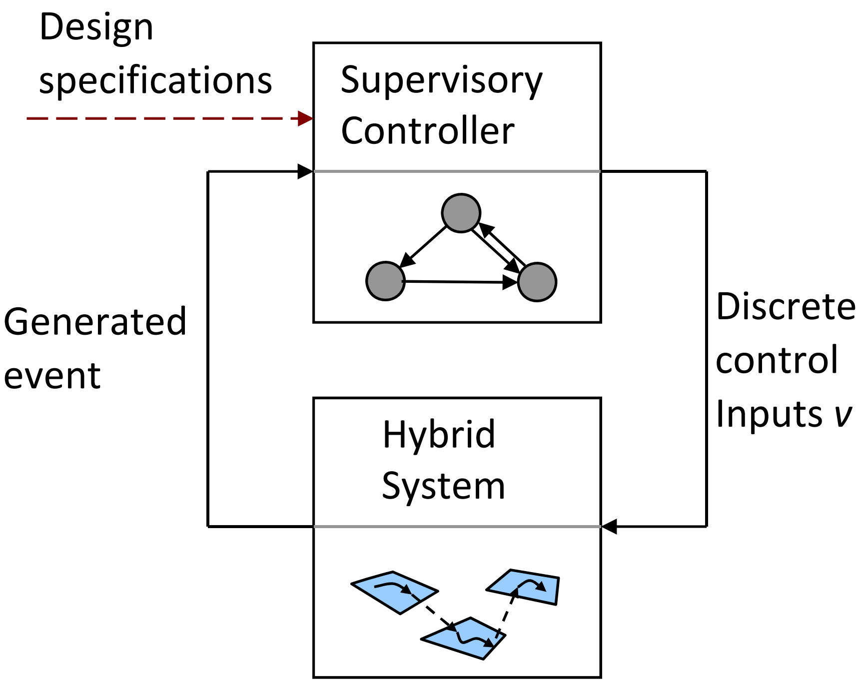
- hybrid model with uncertain dynamics: $\dot{x} = [H]_{z_i} \cdot x + [v]_{z_i}$
- reachable set computation based on zonotopes
- abstraction into Markov chains
- computation of collision probabilities



(e) Scenario (f) t=0-0.8 sec (g) t=0.8-1.6 sec (h) t=1.6-2.4 sec (i) t=2.4-3.2 sec

Computation time: 0.88 seconds for 3.2 seconds in real time using Matlab on a notebook processor (1.66 GHz).

Synthesis of Supervisory Controllers



Modification of *HA*:

- no continuous inputs u
- discrete input v changes only when a transition is taken
- if $x(t)$ enters g, the transition must be taken before g is left

Given sets (one z_i, compact in X)

- initial set: S_0
- forbidden sets: $S_{F,i}$
- goal set: S_G

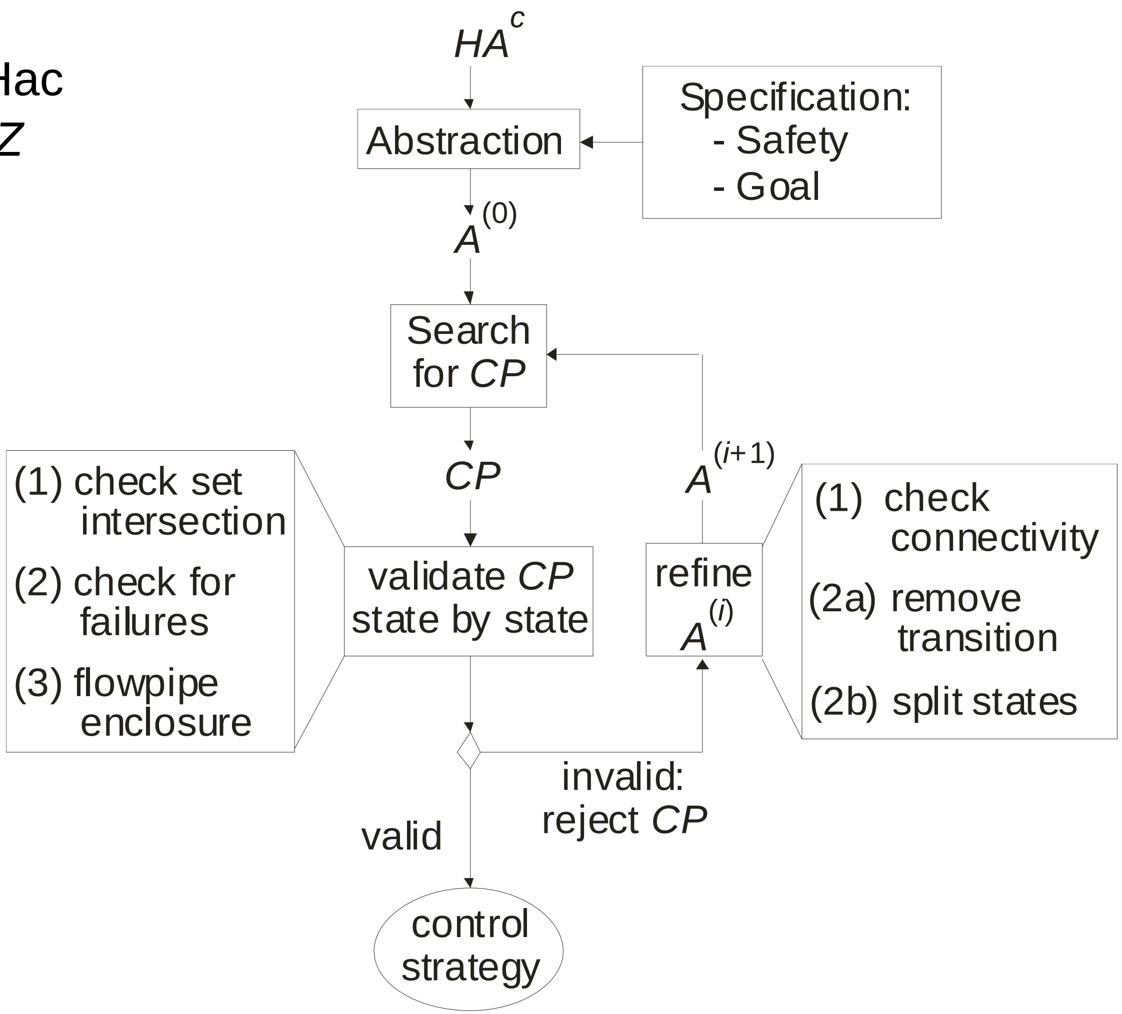
Synthesis Problem:
compute $\phi_v = (v_0, v_1, v_2,...)$ such that any $(z_0, x_0) \in S_0$ is driven into S_G by a feasible run of *HA* that never encounters any $s \in S_F = \bigcup_i S_{F,i}$

Abstraction-Based Synthesis: Principle

Principle:

- rewrite HA into closed system Hac by considering any $v \in V_z$, $z \in Z$
- use an abstract model to identify promising evolutions: *candidate paths CP*
- validate *CP* for the original model with lowest possible computational effort
- if necessary: refine the abstract model for the next iteration

→ a validated *CP* represents a proper control strategy



Abstraction and Candidate Paths

Abstract Model: $A^{(0)} = (S^A, s^A{}_0, \Theta^A)$

represents the discrete dynamics of HA^c (as in verification)

Candidate Path: $CP = \left(s_0^A, s_1^A, \ldots, s_p^A\right)$ with $s_0^A \in S_0^A$, $s_p^A \in S_G^A$ and $s_k^A \notin S_F^A$ for all $k \in \{0,\ldots,p\}$

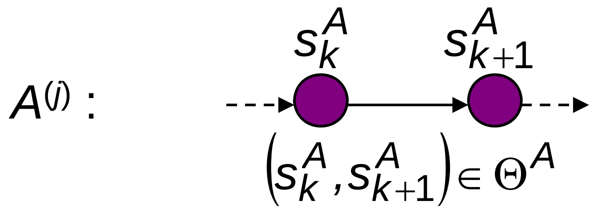
search for CP: standard forward breadth-first algorithm

$\rightarrow$ returns one of the shortest candidate paths existing for $A^{(j)}$

Validation (1)

Check for any pair $\left(s_k^A, s_{k+1}^A\right)$ of CP whether it realizes a feasible control action for HA^c:

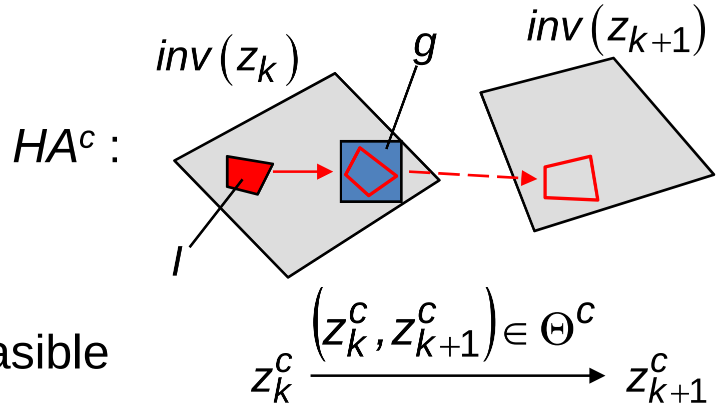
$I \subset inv(z_k)$ - set of continuous states represented by s_k^A



$\Rightarrow$ any state $x \in I$ must be transferred into $inv(z_{k+1}^c)$ by:

(i.) continuous evolution

(ii.) transition and reset



Validation procedure: determine with an as small effort as possible that the control action is not feasible

(1) intersection check
(2) search for invalidating trajectories
(3) flowpipe enclosure

stricter condition, higher computational effort

Validation (2)

(1.) Intersection Check: control action is invalid, if no $x \in g\left(\left(z_k^c, z_{k+1}^c\right)\right)$ is mapped into $inv\left(z_{k+1}^c\right)$ by $r\left(\left(z_k^c, z_{k+1}^c\right), x\right)$

(2.) Search for invalidate trajectories:

target set T : subset of $g\left(\left(z_k^c, z_{k+1}^c\right)\right)$ that is mapped into $inv\left(z_{k+1}^c\right)$ by the reset.

control action is invalid, if any $x(t_f) \notin T$ is found during solving:

$$\max_{x_0 \in I} \left\| x(t_f) - x_{cent,g} \right\|_2$$

with the terminal state $x(t_f)$ determined by numeric simulation as:

(a) $x(t_f) \in T$

(b) $x(t_f) \in g\left(\left(z_k^c, z_q^c\right)\right)$ with $z_q^c \neq z_{k+1}^c$

(c) $x(t_f) \in F$

(d) $x(t_f) \notin inv\left(z_k^c\right)$ and $x\left(t_f^-\right) \in inv\left(z_k^c\right)$

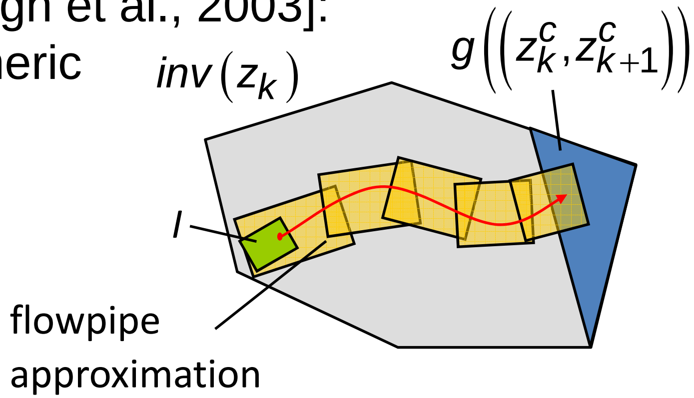
(e) $x(t) \in inv\left(z_k^c\right)$ and $t_{max} < t \left(< t_f^*\right)$

invalidating cases

Validation (3)

(3.) Flowpipe enclosure:
over-approximation of the continuous set reachable inside of $inv\left(z_k^c\right)$ starting from I

$\rightarrow$ series of oriented hyper-rectangles [Krogh et al., 2003]: each hyper-rectangle computed by numeric simulation embedded into optimization



control action is invalid, if the flowpipe does not completely lead into the set $g\left(\left(z_k^c, z_{k+1}^c\right)\right)$

$\rightarrow$ if a control action is invalid, refute *CP*!

$\rightarrow$ if a control action is valid: continue with the next step of *CP*.

Refinement of A

$A^{(j)}$ is refined to $A^{(j+1)}$ in the following cases:

(1) if the intersection check shows that $\left(z_k^c, z_{k+1}^c\right) \in \Theta^c$ can never be taken, the corresponding transition $\left(s_k^A, s_{k+1}^A\right)$ is removed from $\hat{E}$.

(2) if the other two validation methods show that $\left(s_k^A, s_{k+1}^A\right)$ is invalid, it can not be removed from Θ^A immediately.

[Krogh et al., 2003: optimization-based method to show that $\left(z_k^c, z_{k+1}^c\right) \in \Theta^c$ cannot occur]

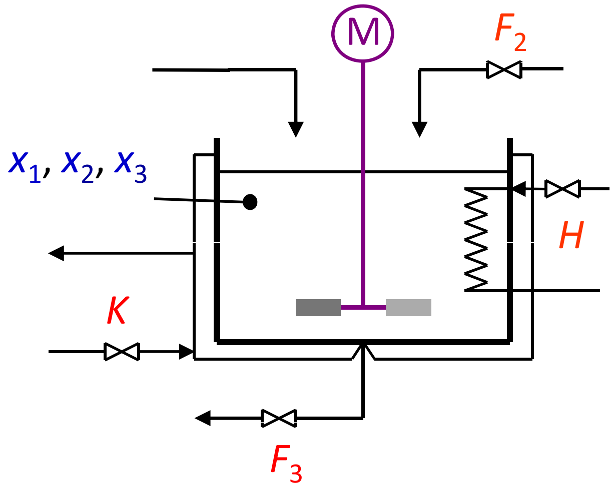
(3) if flowpipe approximation validates a control action, state splitting can be used optionally:

- new abstract state for the reachable subset of $inv\left(z_{k+1}^c\right)$
- transition set Θ^A modified according to the reachability result
- can be advantageous to (in-)validate a *CP* computed later

Application: Chemical Reactor

Continuous chemical liquid-phase reactor:

- exothermic reaction of two components
- state variables: x_1 (level), x_2 (temperature), x_3 (concentration)
- inputs: F_2 (flow), F_3 (flow), K (cooling), H (heating) 16 possible combinations of values
- hybrid automaton:
  - 12 locations (32 for HA^c), 22 transitions
  - dynamics for $x_1 < 0.8$:



for $x_1 \geq 0.8$:

$$\dot{x}_1 = k_1 + F_2 + F_3,\ \dot{x}_2 = \frac{k_2\left(k_3 - x_2\right) + F_2\left(k_4 - x_2\right)}{x_1} + k_5 K\left(k_6 - x_2\right)\left(\frac{k_7}{x_1} + k_8\right),\ \dot{x}_3 = \left(k_9 - \left(k_{10} + F_2\right)x_3\right)/x_1$$

$$\dot{x}'_2 = \dot{x}_2 + k_{11}\left(k_{12} - x_2\right)\left(k_{13} - k_{14}/x_1\right)H$$

for $\left(0, k_{15}, k_{16}\right) \cdot x \geq k_{17}$:

$$\dot{x}''_2 = \dot{x}'_2 + x_3\left(k_{18} + k_{19}x_2^2\right),\ \dot{x}'_3 = \dot{x}_3 + k_{20} \cdot \exp\left(k_{21}/x_2\right)$$

Application: Chemical Reactor

Task: find ϕ_v for start-up from S_0 into nominal operation S_G
(S_0: reactor empty and cold; S_G: high level, temperature, and yield)

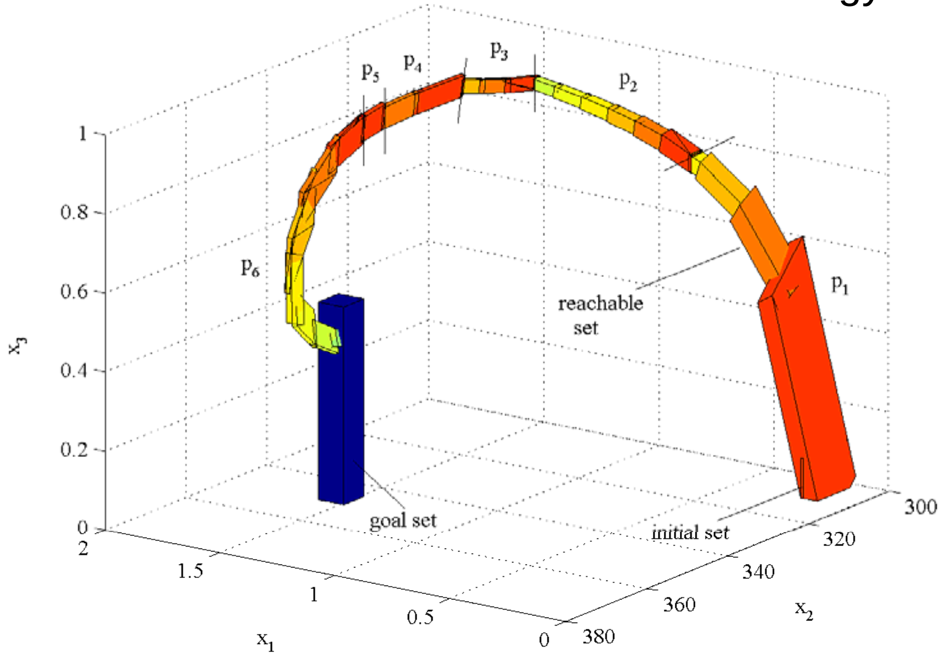
Control synthesis:

- 17^{th} *CP*: feasible strategy
- six phases p_1 to p_6:

$$\begin{array}{c} \begin{matrix} v_{p1} & & & & & v_{p6} \\ \downarrow & & & & & \downarrow \end{matrix} \\ \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

- 16 *CP* invalidated by the 2^{nd} validation method
- obtained in approx. 4 minutes on a standard PC (P4-1.5GHz)

reachable set for the control strategy:



Abstraction-Based Optimal Control of HDS

Model: *HA* as defined initially, but transitions are taken deterministically

Sets:
- initial states: $s_0 \in S_0$ with $z_0 \in Z, x_0 \in inv(z_0)$
- goal states: $s_G \in S_G$ with $z_G \in Z, x_G \in inv(z_G)$
- unsafe states: $S_F = \{S_{F,1}, \ldots, S_{F,p}\}$

Performance criterion ψ with the number of event times $n_e = |\phi_s|$:

$$J(z,x,u,v) = \sum_{k=1}^{n_e-1} \int_{t_{k-1}}^{t_k} L(x,u,v)dt + \sum_{k=1}^{n_e-2} c_{disc}(z_{k-1}, z_k)$$

Hybrid optimal control problem:

$$\min_{\phi_u, \phi_v} J(z,x,u,v)$$

subject to : ϕ_u, ϕ_v lead to a feasible run of *HA*

$$s_0 \in S_0, s(t_f) \in S_G, s(t) \notin S_F \ \forall t \in [t_0, t_f]$$

Scheme of Abstraction-Based Optimal Control

Idea:

- simplify hybrid optimal control problem by abstraction
- use reachability analysis for updating the abstracted model and cost function

Steps:

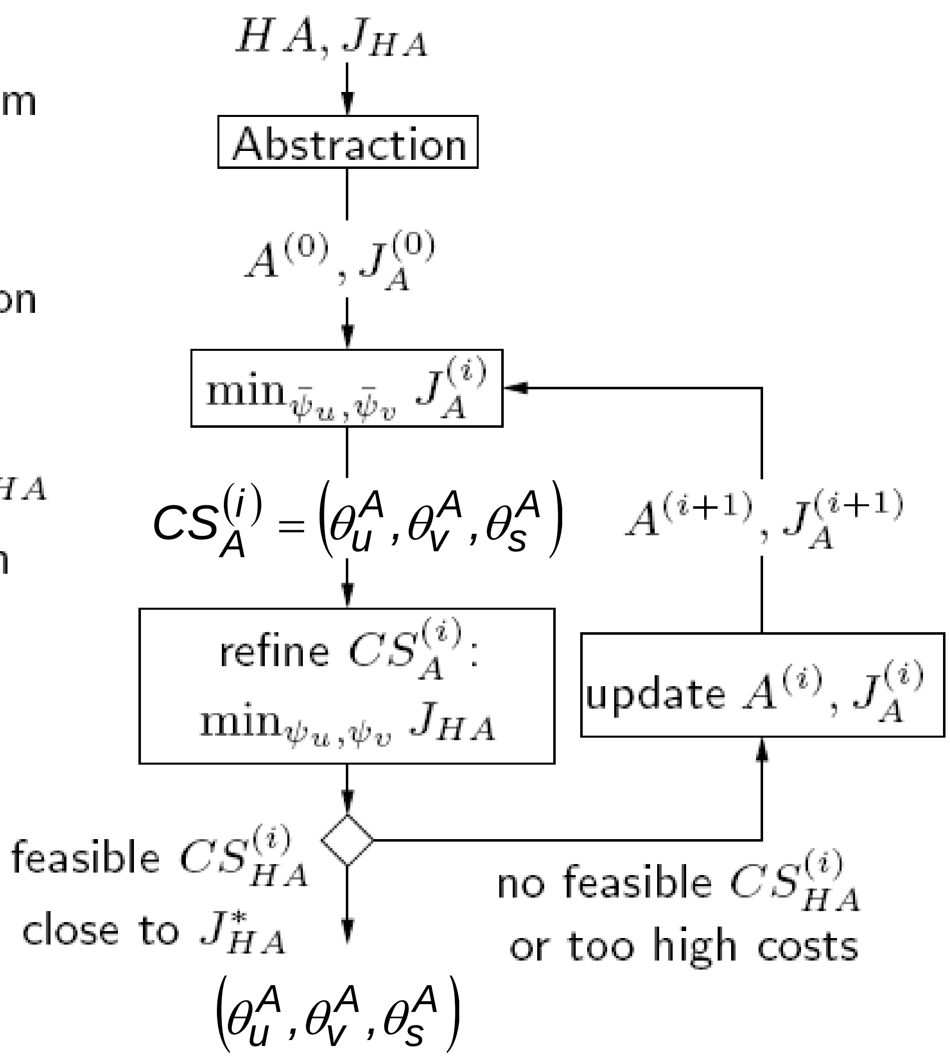
(1) define abstraction maps for HA & J_{HA}

(2) solve abstracted optimization problem

(3) refine to trajectories of HA by reduced optimization problem

(4) evaluate trajectory in terms of original costs

(5) update abstract model and cost criterion iteratively



Conclusions and Future Work

- Modeling with *HA* useful for a wide range of applications
- Computation of *R* is the core of many design techniques for *HA*
- Reachability analysis is computationally costly: use of abstractions can reduce the load
- Choice of abstraction (preserved property, degree of detail) is crucial

  [use of *A* such that it encodes discrete dynamics of *HA* is not always a suitable choice]

Future work:

- Extend verification to other specifications γ than safety
- Improve efficiency of computation with respect to *n*
- Use hierarchies of abstractions
- Include model uncertainties and robustness