Bifurcations of Relative Equilibria near Zero Momentum

James Montaldi

University of Manchester

Oberwolfach, July 2008

Context

Another degeneracy for RE:
Reduced spaces are $\mathbf{J}^{-1}(\mu)/G_\mu$

Context

Another degeneracy for RE:
Reduced spaces are $\mathbf{J}^{-1}(\mu)/G_\mu$ ... depends on G_μ.

Context

Another degeneracy for RE:
Reduced spaces are $\mathbf{J}^{-1}(\mu)/G_\mu$ ... depends on G_μ.

For $\mathbf{SO}(3)$,
$\mu \neq 0 \Rightarrow G_\mu = S^1$
$\mu = 0 \Rightarrow G_\mu = \mathbf{SO}(3)$

Context

Another degeneracy for RE:
Reduced spaces are $\mathbf{J}^{-1}(\mu)/G_\mu$ ... depends on G_μ.

For $\mathbf{SO}(3)$,
$\mu \neq 0 \Rightarrow G_\mu = S^1$
$\mu = 0 \Rightarrow G_\mu = \mathbf{SO}(3)$

Consider $\mathbf{SO}(3)$-symmetric Hamiltonian systems with *free* action ...

Context

Another degeneracy for RE:
Reduced spaces are $\mathbf{J}^{-1}(\mu)/G_\mu$ ... depends on G_μ.

For $\mathbf{SO}(3)$,
$\mu \neq 0 \Rightarrow G_\mu = S^1$
$\mu = 0 \Rightarrow G_\mu = \mathbf{SO}(3)$

Consider $\mathbf{SO}(3)$-symmetric Hamiltonian systems with *free* action ...

Two questions ...

1. generic REs

G. Patrick and M. Roberts (2000) define the notion of *transverse RE*, and show that for *generic* systems with symmetry all RE are transverse.

In particular,

> *for generic Hamiltonian systems with* **SO**(3) *symmetry, when* $\mu = 0$ *then* $\xi \neq 0$.

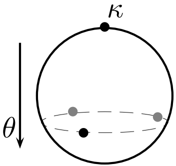
(μ =angular momentum, ξ=angular velocity)

This is not what we see in "simple mechanical systems"

2. Stability

Bifurcation diagram for 4 point vortices on the sphere:

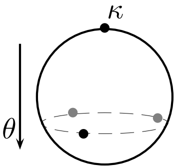
ring + pole



2. Stability

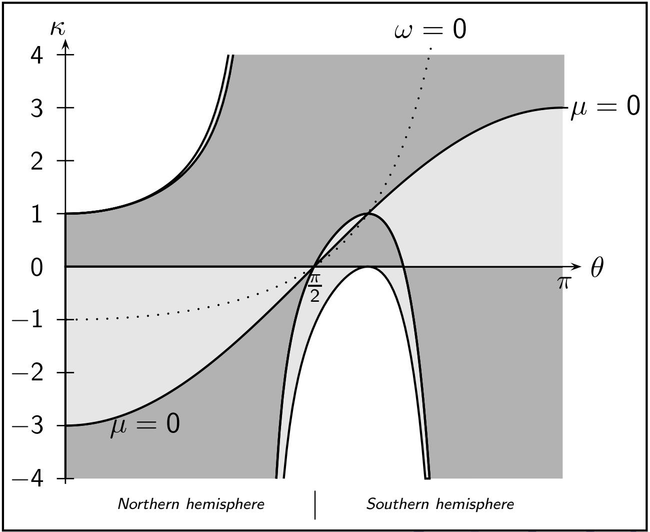
Bifurcation diagram for 4 point vortices on the sphere:

ring + pole



Key:

- Lyapounov stable
- elliptic
- linearly unstable



3. Reduction

Orbit reduction for a free action: write locally

$$\mathcal{P}/G \;\simeq\; \mathcal{P}_0 \times \mathfrak{g}^*$$

Then with $s \in \mathcal{P}_0,\; \mu \in \mathfrak{g}^*$, $H(s,\mu)$ is Hamiltonian on orbit space.

3. Reduction

Orbit reduction for a free action: write locally

$$\mathcal{P}/G \ \simeq\ \mathcal{P}_0 \times \mathfrak{g}^*$$

Then with $s \in \mathcal{P}_0,\ \mu \in \mathfrak{g}^*$, $H(s,\mu)$ is Hamiltonian on orbit space.

Theorem (JM, 1997): *If* $(0,0)$ *is a non-degenerate* RE *in* $\mathcal{P}_0$ *then there is a map* $\phi : \mathfrak{g}^* \to \mathcal{P}_0$ *such that*

$$(s,\mu) \textit{ is a } \mathrm{RE} \textit{ of } H \iff \begin{cases} s = \phi(\mu) \\ d(h_{|\mathcal{O}_\mu})(\mu) = 0, \end{cases}$$

where $h : \mathfrak{g}^* \to \mathbf{R}$ *is just* $h(\mu) := H(\phi(\mu),\mu)$.

3. Reduction

Orbit reduction for a free action: write locally

$$\mathcal{P}/G \;\simeq\; \mathcal{P}_0 \times \mathfrak{g}^*$$

Then with $s \in \mathcal{P}_0$, $\mu \in \mathfrak{g}^*$, $H(s,\mu)$ is Hamiltonian on orbit space.

Theorem (JM, 1997): *If* $(0,0)$ *is a non-degenerate* RE *in* $\mathcal{P}_0$ *then there is a map* $\phi : \mathfrak{g}^* \to \mathcal{P}_0$ *such that*

$$(s,\mu) \text{ is a RE of } H \iff \begin{cases} s = \phi(\mu) \\ d(h_{|\mathcal{O}_\mu})(\mu) = 0, \end{cases}$$

where $h : \mathfrak{g}^* \to \mathbf{R}$ *is just* $h(\mu) := H(\phi(\mu), \mu)$.

Moreover at such a RE, $\xi = \mathrm{d}h$.

4. Generic case

Interested in critical points of a function h on $\mathfrak{g}^*$, when restricted to spheres (=energy-Casimir method).

Generically, $\mathrm{d}h(0) \neq 0$. In that case near 0 there is a smooth curve of REs, and at 0, $\xi = \mathrm{d}h \neq 0$. (These are the transverse REs from before).

4. Generic case

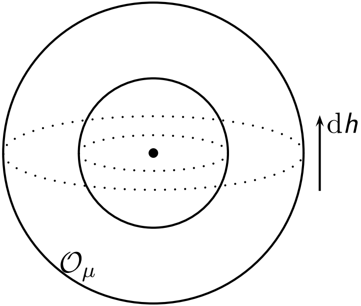
Interested in critical points of a function h on $\mathfrak{g}^*$, when restricted to spheres (=energy-Casimir method).

Generically, $\mathrm{d}h(0) \neq 0$. In that case near 0 there is a smooth curve of REs, and at 0, $\xi = \mathrm{d}h \neq 0$. (These are the transverse REs from before).

Indeed,
locally can choose coords so that

$$h(x, y, z) = z,$$

then REs form the z-axis.



4. Generic case

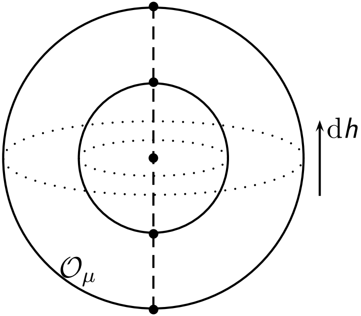
Interested in critical points of a function h on $\mathfrak{g}^*$, when restricted to spheres (=energy-Casimir method).

Generically, $\mathrm{d}h(0) \neq 0$. In that case near 0 there is a smooth curve of RES, and at 0, $\xi = \mathrm{d}h \neq 0$. (These are the transverse RES from before).

Indeed,
locally can choose coords so that

$$h(x, y, z) = z,$$

then RES form the z-axis.



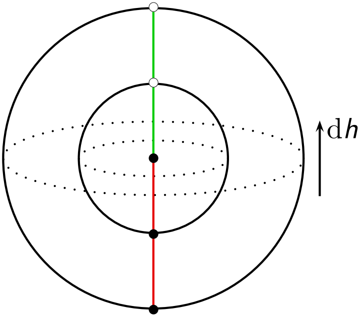
5. Stabilities

Assume $(0,0)$ is a local minimum of $H(s,0)$ (so Lyapounov stable RE in $\mathcal{P}_0$) then

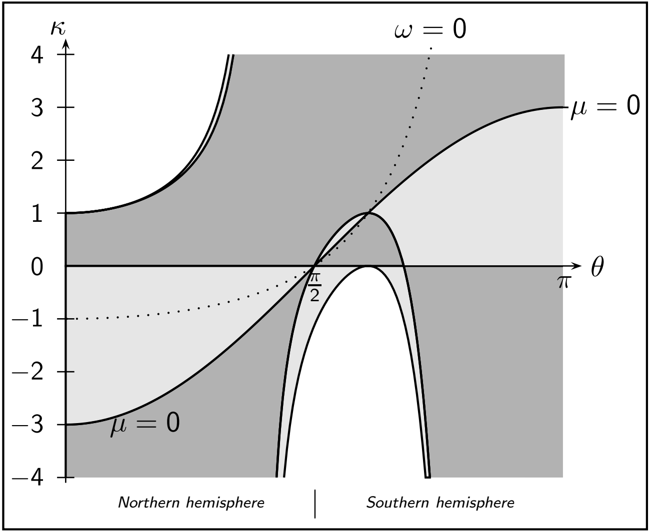
with $h(x,y,z) = z$

h restricted to sphere has
• minimum at $(0,0,z)$ with $z<0$, and
• a maximum at $(0,0,z)$ with $z>0$.

Thus:
• overall Lyapounov stable RE at points $(0,0,z)$ with $z<0$
• and only elliptic at points with $z>0$ (because of coupling with $\mathcal{P}_0$).



Example revisited



Key:

- Lyapounov stable
- elliptic
- linearly unstable

6. Non-generic RE

Now suppose $\mathrm{d}h(0) = 0$, — eg time reversible system or simple mechanical system.

Then (Taylor series) $h(x, y, z) = ax^2 + by^2 + cz^2 + \cdots$

If a, b, c distinct, can show (Singularity Theory) that '$\cdots$' are irrelevant.

cf. rigid body

6. Non-generic RE

Now suppose $\mathrm{d}h(0) = 0$, — eg time reversible system or simple mechanical system.

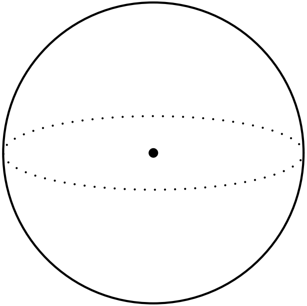
Then (Taylor series) $h(x, y, z) = ax^2 + by^2 + cz^2 + \cdots$

If a, b, c distinct, can show (Singularity Theory) that '$\cdots$' are irrelevant.

cf. rigid body

Suppose $a > b > c$. Then

- $(0, 0, \pm z)$ is minimum (Lyapounov)
- $(0, \pm y, 0)$ is saddle (unstable)
- $(\pm x, 0, 0)$ is maximum (elliptic)



6. Non-generic RE

Now suppose $\mathrm{d}h(0) = 0$, — eg time reversible system or simple mechanical system.

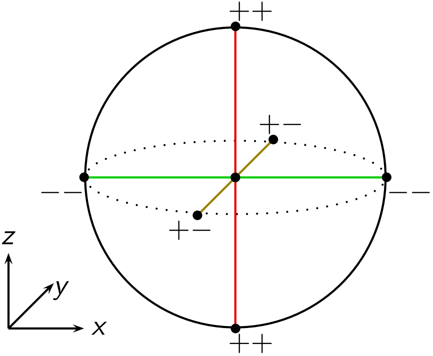
Then (Taylor series) $h(x, y, z) = ax^2 + by^2 + cz^2 + \cdots$

If a, b, c distinct, can show (Singularity Theory) that '$\cdots$' are irrelevant.

cf. rigid body

Suppose $a > b > c$. Then

- $(0, 0, \pm z)$ is minimum (Lyapounov)
- $(0, \pm y, 0)$ is saddle (unstable)
- $(\pm x, 0, 0)$ is maximum (elliptic)



7. Unfolding

The condition $\mathrm{d}h(0) = 0$ is really 3 conditions, so it's a codimension-3 singularity.

3 unfolding parameters α, β, γ :

$$h(x, y, z) = ax^2 + by^2 + cz^2 + \alpha x + \beta y + \gamma z$$

7. Unfolding

The condition $\mathrm{d}h(0) = 0$ is really 3 conditions, so it's a codimension-3 singularity.

3 unfolding parameters α, β, γ :

$$h(x, y, z) = ax^2 + by^2 + cz^2 + \ \alpha x + \beta y + \gamma z$$

Condition for RE is $\mathrm{d}h \ - \ \lambda \mathrm{d}(x^2 + y^2 + z^2) \ = \ 0,$

or

$$\mathrm{rank} \begin{bmatrix} x & y & z \\ h_x & h_y & h_z \end{bmatrix} < 2$$

7. Unfolding

The condition $\mathrm{d}h(0) = 0$ is really 3 conditions, so it's a codimension-3 singularity.

3 unfolding parameters α, β, γ :

$$h(x, y, z) = ax^2 + by^2 + cz^2 + \ \alpha x + \beta y + \gamma z$$

Condition for RE is $\mathrm{d}h \ - \ \lambda \mathrm{d}(x^2 + y^2 + z^2) \ = \ 0$,

or

$$\mathrm{rank} \begin{bmatrix} x & y & z \\ h_x & h_y & h_z \end{bmatrix} < 2$$

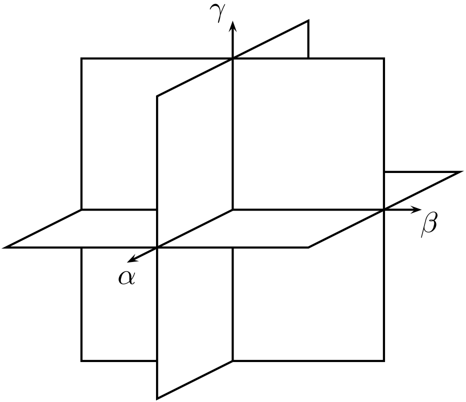
3 equations in 3 unknowns, but solution set is a curve!

Unfolding

Equations are (taking $a = 1, b = 0, c = -1$):

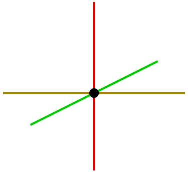
$$\begin{aligned}(x+\alpha)(y-\beta) &= -\alpha\beta \\ (y+\beta)(z-\gamma) &= -\beta\gamma \\ (2x+\alpha)(2z-\gamma) &= -\alpha\gamma\end{aligned}$$

Unfolding

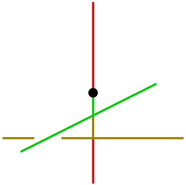


Discriminant in unfolding space

Unfolding — with stabilities



(i) $\alpha = \beta = \gamma = 0$



(ii) $\alpha = \beta = 0$
$\gamma > 0$

Key:

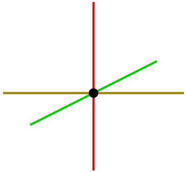
Lyapounov stable

elliptic

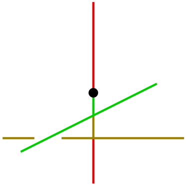
linearly unstable

• $\mu = 0$

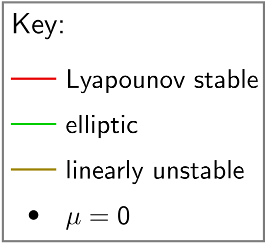
Unfolding — with stabilities

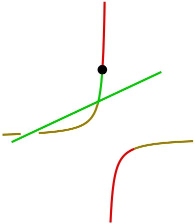


(i) $\alpha = \beta = \gamma = 0$

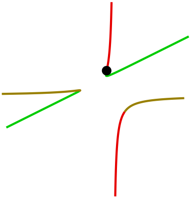


(ii) $\alpha = \beta = 0$
$\gamma > 0$





(iii) $\alpha = 0,\ \beta, \gamma > 0$



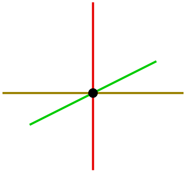
(iv) $\alpha, \beta, \gamma > 0$

the 3 degenerate deformations

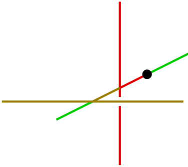
Along the five axes of the discriminant —

Key:

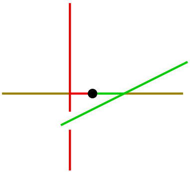
- Lyapounov stable
- elliptic
- linearly unstable
- $\mu = 0$



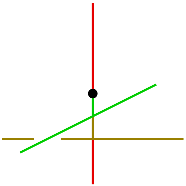
$\alpha = \beta = \gamma = 0$



$\alpha > 0$



$\beta > 0$



$\gamma > 0$