Geometric Mechanics
Some Recent Progress

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Control and Dynamical Systems,
Caltech

Oberwolfach, July 21, 2008

Collaborators: as we go along
Reduction Theory
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  ★ Symplectic reduction (Meyer, JM, Weinstein, 1972)
  ★ Cotangent bundle (in, eg, Foundations of Mechanics)
  ★ Poisson reduction (various + JM and Ratiu, 1986)
  ★ Lagrangian reduction (Scheurle, JM Cendra, Ratiu).
  ★ Dirac structures and reduction (Bloch, Crouch, van der Schaft, Blankenstein, Ratiu, Weinstein and “oids” folks, Cendra, JM, Ratiu, Yoshimura...).
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• Lets talk about reduction by stages to be specific.
Brief Stages Comments
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- **Hamiltonian reduction by stages** ...
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• The “hand calculation” of the resulting Hamiltonian and variational structure is difficult—Shashikanth et al and Borisov and Mamaev. Reduction by stages gives this (and more general) and how they are related.
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• Work of Cendra, Etchechoury, JM on Dirac-Gotay-Nester constraints in the Dirac Structure Context. Confused?
Discrete Mechanics
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Lagrangian $L(q, \dot{q})$
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Hamilton’s Principle

$$\delta \int_{a}^{b} L(q, \dot{q}) \, dt = 0$$
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\[ \delta \int_a^b L(q, \dot{q}) \, dt = 0 \]

Gives the Euler-Lagrange equations

\[ \frac{d}{dt} \frac{\partial L}{\partial \dot{q}^i} - \frac{\partial L}{\partial q^i} = 0 \]
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\( L_d(q_0, q_1, h) \approx \int_{0}^{h} L(q(t), \dot{q}(t)) \, dt \)

Discrete Hamilton’s Principle

\[ \delta \sum_{k=0}^{N-1} L_d(q_k, q_{k+1}, h_k) = 0 \]
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Gives discrete E-L equations

$$D_h L_d(q_{i-1}, q_i, h_{i-1}) + D_1 L_d(q_i, q_{i+1}, h_i) = 0$$
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Add forces to this formalism, such as control forces
Simple Example

- Let $M$ be a positive definite symmetric $n$ by $n$ matrix and $V$ be a given potential. Choose the Lagrangian:

$$L(q, \dot{q}) = \frac{1}{2} \dot{q}^T M \dot{q} - V(q).$$

- Discrete Lagrangian chosen by using the rectangle rule on the action integral together with a naive finite difference approximation of the velocity (more sophisticated quadrature rules give higher order accurate algorithms, such as SPARK).

$$L_d(q_0, q_1, h) = h \left[ \frac{1}{2} \left( \frac{q_1 - q_0}{h} \right)^T M \left( \frac{q_1 - q_0}{h} \right) - V(q_0) \right]$$

- Resulting DEL equations (the algorithm):

$$M \left( \frac{q_{k+1} - 2q_k + q_{k-1}}{h^2} \right) = -\nabla V(q_k)$$
Noether’s theorem

A nice thing about the variational formulation is that such algorithms are naturally symplectic and results such as Noether’s theorem remain valid.

The proofs of these things are basically the same as in the continuous theory.
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250,000 years into the future

Runge-Kutta Method (RK4)  
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Structured perturbations induce flipping with low energy.
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Philip DuToit, Igor Mezic, JM
Stochastic Example (Bou-Rabee)

- Variational simulation of the 3 mass system with Brownian excitation at constant temperature
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Mass effects important for the most probable conformation change.
Variational simulation of the 3 mass system with Brownian excitation at constant temperature.
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Discrete Mechanics also Applies to Field Theories, such as E and M

standard “synchronized” method

our new “asynchronous” method
More on Dirac Structures?
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• Reduction for discrete mechanics is also troublesome; perhaps the Dirac setting helps there too.
Discrete Mechanics and Optimal Control (DMOC)

• DMOC recasts the problem in discrete time as

\[
\text{Minimize } J_d(q_d, u_d) = \sum_{k=0}^{N-1} F_d(q_k, q_{k+1}, u_k, u_{k+1}),
\]

subject to boundary conditions (prescribed or periodic), and the appropriate forced DEL equations of motion.

• This formulation is an equality constrained, nonlinear optimization problem that can be solved with sequential quadratic programming (SQP).

Sina Ober-Blöbaum, Oliver Junge, JM
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- Some examples...
Falling Cats, Divers, Swimming

James Martin, Eva Kanso
Falling Cats, Divers, Swimming

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Satellite Reorientation

Sigrid Leyendecker, Sina Ober-Blöbaum, Michael Ortiz, JM
Satellite Reorientation

Handling Constraints: variational discrete null space method

• Easy formulation of discrete Lagrangian with constant mass matrix.
• Symplectic-momentum conserving.
• Exact constraint fulfillment.
• Minimal dimension of resulting system.
• Condition number independent of time step.

Unlike

• Lagrange multiplier method: larger dimension of system than necessary and conditioning problems.
• Generalized coordinates: higher nonlinearity difficult for large multibody systems.
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Optimal Helicopter Flight

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DMOC Primitives, Roadmap Strategies (Marin Kobilarov)

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Future Directions: Combine with trend optimization techniques for charting efficient roadmaps. Make use of this technique in the surveillance problem.
Global Strategies using DMOC Primitives
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Roadmap, dynamic programming strategies for rapid search methods drawing from the DMOC primitives library for the component pieces. Dynamics is faithfully represented.
Another Examples
Multiscale Trend Optimization (MTO)
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The Idea of MT Optimization

Evaluations of $J$ given $\alpha$ are assumed to be expensive, and thus gradient based methods would be inefficient.

Good for situations in which $J$ is noisy

Trend optimization samples a small number of points,...

...fits a trend to the results using prescribed basis functions,...
The Idea of MT Optimization

...rapidly determines the global minimum of the trend, ...

...evaluates new points around the trend minimizer â, ...

...updates the trend, ...

...and repeats until convergence.
Multiscale Trend Optimization—More
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Developed in the self assembly problem, useful for optimization when the cost function is multiscale, noisy, lots of local minima and expensive to evaluate.

We plan to use this tool in surveillance
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**Test Case—Robotic Walker (David Pekarek):**
- DMOC and inner-outer loop strategy—more on DMOC shortly
- What is the best place for the knees?
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Design of Dynamics

Example Systems to Optimize:

- Bipedal Robots
Design of Dynamics

Inner/Outer Loop Architecture

- Inner loop: DMOC determines optimal trajectories and controls
- Outer loop efficiently searches for optimal design parameters (trend optimization)
- Scheme yields optimal mechanical system design for specified tasks

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Initial Design → MTO → Optimal Design

Optimize given Design

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DMOC

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Future: Stochastic DMOC
Trend Optimization’s minimizer
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Global solution for optimal control problems

Formation of hovercraft

- relative arrangement on target manifold
- minimize control effort
- many local minima

Sampling-based Roadmap

- graph of DMOC primitives
- dynamic programming
- global state space exploration
- near globally optimal solution
Hurricane Nabi (Philip DuToit)

www.digital-typhoon.org

Typhoon 200514 : 2005-08-29 00:00 UTC
LCS for Hurricane Nabi

Sun Sep 4 04:00:00 2005

NCEP/NCAR Reanalysis Data at the 850mb pressure level.
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Free Ride (Dellnitz)
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• Invariant manifolds are very efficient highways for navigating in the solar system
Free Ride (Dellnitz)

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Nature was there first (naturally)
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Here DMOC and LCS come together
(Ashley Moore, Evan Gawlick)
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<table>
<thead>
<tr>
<th>Delta V (m/s)</th>
<th>Initial Guess</th>
<th>DMOC</th>
</tr>
</thead>
<tbody>
<tr>
<td>case 1</td>
<td>175.8273</td>
<td>0.2331</td>
</tr>
<tr>
<td>case 2</td>
<td>178.5763</td>
<td>0.4452</td>
</tr>
<tr>
<td>case 3</td>
<td>172.7951</td>
<td>0.0672</td>
</tr>
<tr>
<td>case 4</td>
<td>171.3516</td>
<td>0.0902</td>
</tr>
<tr>
<td>case 5</td>
<td>177.8498</td>
<td>0.4386</td>
</tr>
</tbody>
</table>

Initial Guess

DMOC Result
Coherent Structures Everywhere
Coherent Structures Everywhere

- Hurricanes
Coherent Structures Everywhere

- Hurricanes
Coherent Structures Everywhere

- Hurricanes
- Jupiter’s red spot
Coherent Structures Everywhere

• Hurricanes

• Jupiter’s red spot
Coherent Structures Everywhere

- Hurricanes
- Jupiter’s red spot
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Coherent Structures Everywhere

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Features of Invariant Manifolds
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- Invariant manifolds—usually thought of for autonomous or periodic systems. Ocean and atmosphere are not!
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• Invariant manifolds are usually “attached” to fixed points, periodic orbits, or other invariant sets—not required by LCS the way we do them today.

• George Haller idea—use FTLE (Finite Time Liapunov Exponent Fields) and look for ridges—this was developed in the PhD Theses of Lekien, Shadden.
Invariant Manifolds: Standard View
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- Start with the simple pendulum—a swing!
Invariant Manifolds: Standard View

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  \[ \ddot{x} + \sin x = 0 \]
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- Periodically perturb the simple pendulum with forcing
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• Velocity field—hard to tell what is going on:
Homoclinic Chaos: Standard View

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Standard way around this
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• Use of the Poincaré map (1880) to get a homoclinic tangle: excellent way to view for periodic systems.
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Transient Chaos

- Poincaré’s homoclinic tangle corresponds to transient chaos—dynamic events over intermediate time scales.
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• LCS will reveal the tangle in hurricane dynamics !!
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- First, a bit more about the tangle
Smale Horseshoe
Smale Horseshoe

- Smale (in the 1960s) abstracted what was going on in the tangle to produce the horseshoe map.
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- Proved lots of nice things—eg, an invariant Cantor set.
Smale Horseshoe in the Tangle
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Poincaré, one of the creators of modern dynamical systems, 1890
Lagrangian Coherent Structures
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- Generalizes invariant manifolds to the case of time dependent dynamical systems
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- For time varying systems, LCS move in time
Lagrangian Coherent Structures

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Look at lobes, mixing, dynamically
Back to Nabi
Nabi has horseshoes in it
Nabi has horseshoes in it
Nabi has horseshoes in it
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Even Shades of a Cantor Set
Even Shades of a Cantor Set
Even Shades of a Cantor Set
3D Structures
3D Structures
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LCS Key Features
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• Useful for computing mixing and transport via lobe dynamics
• In fluids, particles move in a dynamical system given by the velocity field of the flow
• LCS divide particles with different dynamical fates
• Clear example in ocean dynamics
Lagrangian Coherent Structures in Monterey bay
Other Uses of LCS
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- Drifter deployment strategies
Other Uses of LCS

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• Also important in AOSN and ASAP projects (Naomi Leonard, Steve Ramp, other oceanographers)
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- Cardiovascular studies
LCS for flow over an Airfoil
Two Types of LCS: Attracting and Repelling
LCS for Ozone Hole Breakup
The ocean plays a major role in the distribution of the planet’s heat through deep sea circulation. This simplified illustration shows this “conveyor belt” circulation which is driven by differences in heat and salinity. Records of past climate suggest that there is some chance that this circulation could be altered by the changes projected in many climate models, with impacts to climate throughout lands bordering the North Atlantic.
Vortex Rings
Vortex Rings
Vortex Rings
Vortex Rings
Vortex Rings
Laboratory Vortex Rings
LCS for the vortex ring

Vortex velocity field

Shawn Shadden, Stanford
LCS for the vortex ring

Vortex velocity field

Shawn Shadden, Stanford
LCS for the vortex ring

Vortex velocity field

Shawn Shadden, Stanford
LCS and Vortex Ring Boundaries

LCS gives much sharper boundaries than vorticity
LCS gives much sharper boundaries than vorticity.
Lobes in the vortex ring
Lobes, Mixing, Transport
Jellyfish
Jellyfish
Jellyfish one more time

Lobes determine which fluid is entrained

LCS
Lobes, Mixing, Transport
3D LCS-Meddies
3D LCS-Meddies

Ellipsoid of vorticity
3D LCS-Meddies

Ellipsoid of vorticity
3D LCS-Meddies

Ellipsoid of vorticity

Particle trajectories
3D LCS

- Mediterranean Salt Lenses in the Atlantic
3D LCS

- Mediterranean Salt Lenses in the Atlantic
3D LCS

- Mediterranean Salt Lenses in the Atlantic

Side view
3D LCS

• Mediterranean Salt Lenses in the Atlantic

Side view
3D LCS

- Mediterranean Salt Lenses in the Atlantic

Side view

Top view
Hairpin vortices in near wall turbulent flow

- Melissa Green,
  George Haller,
  Clancy Rowley
Hairpin vortices in near wall turbulent flow

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A Word on Computing LCS
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• Approximate the fastest separation rate of nearby particles by finding the maximum eigenvalues of a $2 \times 2$ or $3 \times 3$ symmetric matrix—the Cauchy-Green tensor (pull-back of the metric tensor under the flow map)
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• Compute **ridges** in the FTLE field. Those are the LCS!

• Run time backwards for the **attracting LCS.**

• Computations in 2d can be done on a laptop, but in 3d it requires a hefty computer.
Ridges can be complicated
Robustness against uncertainty
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- Uncertainty can be in the data itself, the mathematical model, the computational resolution, or noise
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Attracting LCS low res
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Attracting LCS low res

Attracting LCS high res
Cardiovascular Applications
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• Emerging field of virtual surgery-Charlie Taylor, Stanford
Cardiovascular Applications

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