Reduction of Dirac Structures and Dirac Systems

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CONTENTS

1. Linear Dirac Structures on Vector Spaces

- V, n-dimensional vector space, V^* dual space.
- Symmetric paring $\langle\!\langle .\,,. \rangle\!\rangle$ on $V \oplus V^*$ defined by

  $$\langle\!\langle (v_1, \alpha_1), (v_2, \alpha_2) \rangle\!\rangle = \langle \alpha_1, v_2 \rangle + \langle \alpha_2, v_1 \rangle \,,$$

  where $(v_1, \alpha_1), (v_2, \alpha_2) \in V \oplus V^*$, and $\langle \cdot\,, \cdot \rangle$ natural paring between V^* and V.
- A Dirac structure on V is a subspace $D \subseteq V \oplus V^*$ such that $D = D^\perp$, where $D^\perp$ is the orthogonal of D relative to the pairing $\langle\!\langle .\,,. \rangle\!\rangle$.

Example

A presymplectic vector space (V, ω) has an associated Dirac structure

$$D_\omega = \{(v, \alpha) \in V \oplus V^* \mid \alpha = \omega^\flat(v)\}.$$

A skew-symmetric form $\pi : V^* \times V^* \to \mathbb{R}$ on V defines a Dirac structure D_π on V

$$D_\pi = \{(v, \alpha) \in V \oplus V^* \mid v = \pi^\sharp(\alpha)\}.$$

Lemma 1

A vector subspace $D \subseteq V \oplus V^*$ is a Dirac structure on V if and only if it is maximally isotropic with respect to the symmetric pairing $\langle\!\langle . , . \rangle\!\rangle$. A further equivalent condition is given by $\dim D = n$ and $\langle \alpha_1, v_2 \rangle + \langle \alpha_2, v_1 \rangle = 0$ for all $(v_1, \alpha_1), (v_2, \alpha_2) \in D$.

For a given subset $W \subseteq V$ we define the ω-orthogonal complement W^ω by

$$W^\omega = \{v \in V \mid \omega(v, w) = 0 \text{ for all } w \in W\}.$$

Lemma 2

Let ω be a presymplectic form on a vector space V and let $W \subseteq V$ be any vector subspace. Then $W^\flat := \omega^\flat(W) = (W^\omega)^\circ$, where the right hand side denotes the annihilator of W^ω.

From Courant [1990] one easily deduces

Theorem 1

Let a given Dirac structure $D \subseteq V \oplus V^*$ and define the subspace $E_D \subseteq V$ to be the projection of D on V.

Then one can uniquely define a skew form ω_D on E_D by $\omega_D(v, w) = \alpha(w)$, where $v \oplus \alpha \in D$. (One checks that the definition of ω_D is independent of the choice of α.)

Conversely, given a vector space V, a subspace $E \subseteq V$ and a skew form ω on E, one sees that $D_\omega = \{(v, \alpha) \mid v \in E, \alpha(w) = \omega(v, w) \text{ for all } w \in E\}$ is the unique Dirac structure D on V such that $E_D = E$ and $\omega_D = \omega$.

2. (Almost-) Dirac Manifolds

- By definition an almost Dirac structure (or, in this talk, simply a Dirac structure) D on a manifold M is a subbundle of the Whitney sum $D \subseteq TM \oplus T^*M$ such that for each $x \in M$, $D_x \subseteq T_xM \oplus T_x^*M$ is a Dirac structure on the vector space T_xM. A Dirac manifold is a manifold with a Dirac structure on it.

  For a given 2-form ω on M there is a naturally associated Dirac structure D_ω on M, where $D_{\omega x} = D_{\omega(x)}$.

  For a given bivector π on M there is a naturally associated Dirac structure D_π on M, where $D_{\pi x} = D_{\pi(x)}$.

- Facts that can be proven

  A Dirac structure D on M yields a distribution $E_{Dx} \subseteq T_xM$ which carries a presymplectic form $\omega_D(x) : E_{Dx} \times E_{Dx} \to \mathbb{R}$, for all $x \in M$.

  An interesting way of building Dirac structures is the following

Theorem 1. *Let M be a manifold and let ω be a 2-form on M. Given a regular distribution E on M, define the skew-symmetric bilinear form ω_E on E by restricting ω to $E \times E$. For each $x \in M$ let*

$$D_{\omega_E}(x) = \{(v_x, \alpha_x) \in T_xM \oplus T_x^*M \mid v_x \in E(x) \\ \text{and } \alpha_x(w_x) = \omega_E(x)(v_x, w_x) \text{ for all } w_x \in E(x)\}\,.$$

Then $D_{\omega_E} \subseteq TM \oplus T^*M$ is a Dirac structure on M. It is the only Dirac structure D on M satisfying $E(x) = E_D(x)$ and $\omega_E(x) = \omega_D(x)$, for all $x \in M$.

By definition a Dirac structure D on M is called integrable if the following condition is satisfied

$$\langle L_{X_1}\alpha_2, X_3\rangle + \langle L_{X_2}\alpha_3, X_1\rangle + \langle L_{X_3}\alpha_1, X_2\rangle = 0,$$

for all pairs of vector fields and one-forms (X_1, α_1), (X_2, α_2), (X_3, α_3) that take values in D and where L_X denotes the Lie derivative along the vector field X on M.
This definition encompasses the notion of closedness for presymplectic form and Jacobi identity for brackets.

- The following fundamental theorem was proven in Courant[1990]

  Theorem 2. *Let D be an integrable Dirac structure on a manifold M. Then the distribution E_D is integrable, that is, for each $x \in M$ there exists a uniquely determined embedded submanifold S of M such that $T_y S = E_{Dy}$ for all $y \in S$. Each leaf S carries a presymplectic form $\omega_{D,S}$ defined by $\omega_{D,S}(x) = \omega_D(x)$, for each $x \in S$.*

  We will be mainly concerned with Dirac structures that need not be integrable, since this is the situation for nonholonomic systems.

3. The Dirac Equation

Let M be a given manifold, D a given Dirac structure on M. Let $\mathcal{E} : M \to \mathbb{R}$ be a given function, called the energy function. By definition the Dirac equation is the following equation

$$(x, \dot{x}) \oplus d\mathcal{E}(x) \in D_x. \tag{1}$$

This equation generalizes the equation considered in the Gotay-Nester algorithm,

$$\omega(x)(\dot{x},\ ) = dH(x),$$

and also the Poisson equation

$$\dot{f}(x) = \{f, H\}(x).$$

In fact, it is enough to take, respectively, $D = D_\omega$ and $D = D_\pi$ while $\mathcal{E} = H$.

Example: Nonholonomic Systems

A nonholonomic system is given by a configuration space Q, a distribution $\Delta \subseteq TQ$, called the nonholonomic constraint and a Lagrangian $L : TQ \to \mathbb{R}$. Equations of motion are given by Lagrange-d'Alembert's principle.

An equivalent form of this principle is the following

$$\delta \int_{t_0}^{t_1} (p\dot{q} - \mathcal{E}(q,v,p))dt = 0,$$

where $\mathcal{E} : TQ \oplus T^*Q \to \mathbb{R}$ is defined by $\mathcal{E}(q,v,p) = pv - L(q,v)$, and with the restriction on variations $\delta q \in \Delta$, $\delta q(t_i) = 0$ for $i = 1,2$, along with the kinematic restriction $v \in \Delta$. The resulting equations are the following

$$\dot{p} - \frac{\partial L}{\partial q} \in \Delta^\circ \tag{2}$$

$$\dot{q} = v \tag{3}$$

$$p - \frac{\partial L}{\partial v} = 0 \tag{4}$$

$$v \in \Delta \tag{5}$$

We are going to show that equations (2)-(5) can be written in the form of Dirac equation (1). For this purpose we must construct an appropriate Dirac structure associated to the nonholonomic constraint.

Define the following Dirac structure
$\bar{D}_\Delta \subseteq TM \oplus T^*M$ on $M = TQ \oplus T^*Q$ associated
to a given distribution $\Delta \subseteq TQ$ on a manifold Q
by the following local expression

$$\bar{D}_\Delta(q,v,p) = \{(q,v,p,\dot{q},\dot{v},\dot{p},\alpha,\gamma,\beta) \mid \dot{q} \in \Delta(q),\ \alpha + \dot{p} \in \Delta^\circ(q),\\ \beta = \dot{q},\ \gamma = 0\}.$$

Note: We shall accept both equivalent notations

$$(q,v,p,\dot{q},\dot{v},\dot{p},\alpha,\gamma,\beta) \equiv (q,v,p,\dot{q},\dot{v},\dot{p}) \oplus (q,v,p,\alpha,\gamma,\beta),$$

for an element of $TM \oplus T^*M$.

The following assertion establishes, in particular,
that $\bar{D}_\Delta$ is well defined globally: it does not
depend on the choice of a local chart.

Let $\bar{\tau} : TQ \oplus T^*Q \to Q$ and $\bar{\pi} : TQ \oplus T^*Q \to T^*Q$ be the natural maps that in local coordinates are given by $\bar{\tau}(q, v, p) = q$ and $\bar{\pi}(q, v, p) = (q, p)$. For a given distribution $\Delta \subseteq TQ$ consider the distribution $\bar{\Delta} = (T\bar{\tau})^{-1}(\Delta)$ and also the 2-form $\bar{\omega} = \bar{\pi}^*\omega$, on the manifold $TQ \oplus T^*Q$, where ω is the canonical 2-form on T^*Q. We have the local expressions $\bar{\Delta} = \{(q, v, p, \dot{q}, \dot{v}, \dot{p}) : \dot{q} \in \Delta\}$ and $\bar{\omega}(q, v, p) = dq \wedge dp$. Now we can apply Theorem 1 replacing M by $TQ \oplus T^*Q$, E by $\bar{\Delta}$ and ω by $\bar{\omega}$ and then we can easily check that the Dirac structure D_{ω_E} coincides with $\bar{D}_\Delta$.

The following assertion can be checked directly:

The condition

$$(x, \dot{x}) \oplus d\mathcal{E}(x) \in \bar{D}_{\Delta}, \tag{6}$$

where $x = (q, v, p)$, is equivalent to

$$\dot{p} - \frac{\partial L}{\partial q} \in \Delta^{\circ} \tag{7}$$

$$\dot{q} = v \tag{8}$$

$$p = \frac{\partial L}{\partial v} \tag{9}$$

$$\dot{q} \in \Delta, \tag{10}$$

which is clearly equivalent to equations (2)-(5).

4. Solving the Dirac Equation.

By definition, a solution to the Dirac equation (1)
on the Dirac manifold (M, D),

$$(x, \dot{x}) \oplus d\mathcal{E}(x) \in D_x.$$

is a curve $x(t)$ such that

$$(x(t), \dot{x(t)}) \oplus d\mathcal{E}\left(x(t)\right) \in D_{x(t)},$$

for all t.

<u>The Gotay-Nester Algorithm can be</u>
<u>generalized for the general Dirac equation.</u>

Define recursively $M \supseteq M_1 \supseteq M_2 \supseteq, ...$

$M_1 = \{x \in M \ \mid \exists (x, v) \in T_x M$ such that $(x, v) \oplus d\mathcal{E}(x) \in D_x\}$.

$M_{k+1} = \{x \in M_k \ \mid \exists (x, v) \in T_x M_k$ such that $(x, v) \oplus d\mathcal{E}(x) \in D_x\}$.

One assumes for simplicity that each M_k is a submanifold and that the sequence stops, say, $M_c = M_{c+1}$, and, moreover, that the dimension of the affine subspace

$$\{(x,v) \in T_xM_c \mid (x,v) \oplus d\mathcal{E}(x) \in D_x\}$$

is locally constant. Then the Dirac equation defines, at least locally, an ODE depending on parameters. Existence of solution and smooth dependence of the parameter is then guaranteed.

The previous algorithm, for the special case in which $D = D_\omega$ is the Dirac structure associated to a presymplectic form ω has been written in terms of the operator ${}^{\omega}$, defined on subsets of TM, by

$$W^\omega = \{(x,v) \in TM \mid \omega(x)(v,w) = 0, \text{ for all}(x,w) \in W\},$$

obtaining what is called the Gotay-Nester algorithm, namely

$$M_1 = \{x \in M \mid \langle d\mathcal{E}(x), E^\omega_{Dx}\rangle = 0, \}$$

$$M_{k+1} = \{x \in M_k \mid \langle d\mathcal{E}(x), (E_{Dx} \cap T_xM_k)^\omega\rangle = 0\}.$$

For any Dirac structure on M and any $W_x \subseteq E_{Dx}$
define

$W_x^D = \{(x,v) \in T_xM \mid D^\flat(x,w)(x,v) = \{0\},\ \mathrm{all}(x,w) \in W\}$,

where

$$D^\flat((x,v)) = \{\alpha \in T_x^*M \mid (x,v) \oplus \alpha \in D\}$$

and

$$D^\flat(W_x) \bigcup_{(x,v)\in W_x} D^\flat((x,v)).$$

Then, for any Dirac structure D one can
generalize the Gotay-Nester algorithm as follows,

$$M_1 = \{x \in M \mid \left\langle d\mathcal{E}(x), E_{Dx}^D \right\rangle = 0, \}$$

$$M_{k+1} = \{x \in M_k \mid \left\langle d\mathcal{E}(x), (E_{Dx} \cap T_xM_k)^D \right\rangle = 0\}.$$

- The previous algorithm gives a method to solve the Dirac equation (1) which generalizes the Gotay-Nester method. Namely, determine M_c, assume that the dimension of the affine space $\{(x,\dot{x}) \in T_xM_c \mid (1) \text{ is satisfied}\}$ is a locally constant function of x, then one can simply restrict (1) to M_c to obtain a vector field on M_c depending on a parameter.

More precisely, one can use any local parametrization of M_c, say $x = x(\lambda_1, ..., \lambda_r)$, where r is the dimension of M_c, and then substitute this expression for x in (1) to obtain an IDE in $\lambda = (\lambda_1, ..., \lambda_r)$, namely

$$\left(x(\lambda), D_\lambda x(\lambda) \cdot \dot{\lambda}\right) \oplus d\mathcal{E}\left(x(\lambda)\right) \in D\left(x(\lambda)\right).$$

With the method just described, one can deal with many examples of interest, such as nonholonomic systems and L-C circuits, provided that one chooses the manifold M and the Dirac structure D properly. One chooses the manifold M to be the Pontryagin bundle $TQ \oplus T^*Q$ and a canonically constructed Dirac structure $\bar{D}_\Delta$ on M.

- The constraint algorithm so described is naturally adapted for reduction of Dirac structures.
- A refinement of the algorithm gives a representantion of the final equation in terms of brackets, generalizing the Dirac theory of constranits for nonholonomic systema and L-C circuits.

5. Reduction

Dirac Anchored Vector Bundles *Think of an **anchored vector bundle** as being a generalization of the tangent bundle. Examples should be reduced tangent bundles $TM/G \to M/G$, where M is a principal bundle with structure group G.*

Define

(a) An anchored vector bundle is a pair $(\pi_{(E,M)}, \rho_E)$ where $\pi_{(E,M)} : E \to M$ is a given vector bundle and $\rho_E : E \to TM$ is a given vector bundle map over the identity (that is, $\rho_E(e) \in T_{\pi_{(E,M)}(e)}M$ for $e \in E$) called the anchor. For $x \in M$, we denote by $\rho_x : E_x \to T_xM$ the restriction of ρ to the fiber E_x. A morphism from the anchored vector bundle $(\pi_{(E,M)}, \rho_E)$ to the anchored vector bundle $(\pi_{(F,N)}, \rho_F)$ is a vector bundle map $f : E \to F$, covering a map $\underline{f} : M \to N$, such that $\rho_F \circ f = T\underline{f} \circ \rho_E$. This defines the category of anchored vector bundles . The morphism f is called an isomorphism if it has an inverse.

(b) Let $\pi_{(E,M)} : E \to M$ be a given vector bundle and let $\pi_{(E^*,M)} : E^* \to M$ be the dual vector bundle of E. A fiberwise Dirac structure on $\pi_{(E,M)}$, or simply a Dirac structure on $\pi_{(E,M)}$, is a vector subbundle $D_E \subseteq E \oplus E^*$ such that, for each $x \in M$, $(D_E)_x \subseteq E_x \oplus E_x^*$ is a linear Dirac structure on the vector space E_x. A Dirac anchored vector bundle is a triple $(\pi_{(E,M)}, \rho_E, D_E)$ where $(\pi_{(E,M)}, \rho_E)$ is an anchored vector bundle and D_E is a Dirac structure on $\pi_{(E,M)}$.

(c) A morphism (particular case, see below) from the Dirac anchored vector bundle $(\pi_{(E,M)}, \rho_E, D_E)$ to another Dirac anchored vector bundle $(\pi_{(F,N)}, \rho_F, D_F)$ is a pair of vector bundle maps $f : E \to F$, $\tilde{f} : E^* \to F^*$, each covering a map $\underline{f} : M \to N$, such that one is the dual of the other, $\rho_F \circ f = T\underline{f} \circ \rho_E$ and $(f \oplus \tilde{f})(D_E) = D_F$. (can define the notion of backward and forward morphism and the category of Dirac anchored vector bundles).

(d) Let $(\pi_{(E,M)}, \rho_E)$ be an anchored vector bundle. A curve $e(t)$, $t \in (a,b)$ on E is called admissible if the following condition is satisfied

$$\rho_E\,(e(t)) = \frac{d}{dt}\pi_{(E,M)}(e(t)), \qquad (11)$$

for all $t \in (a,b)$.

Remarks.

- Any Lie algebroid, and even the more general algebroid structure introduced by Grabowski has an underlying anchored vector bundle structure. For our version of Dirac reduction theory, it is sufficient to have the structure of a Dirac anchored vector bundle; that is, the algebroid structure is not needed. (Bursztyn, Crainic, Grabowski, Martinez).
- Note that we *do not* include any integrability conditions in the definition of a Dirac structure, as integrability does not hold in examples such as nonholonomic systems.

Dirac Equations of Motion. Let $(\pi_{(E,M)}, \rho_E, D_E)$ be a given Dirac anchored vector bundle.

Let $\varphi \in \Gamma(E^*)$ given section of the dual bundle of E, called the energy form. In many cases $\varphi = d\mathcal{E}$, where $\mathcal{E}$ represents energy.

By definition, the associated Dirac system is defined as follows

$$e \oplus \varphi(\pi_{(E,M)}(e)) \in (D_E)_{\pi_{(E,M)}(e)}. \qquad (12)$$

A solution to the Dirac system (12) is an admissible curve $e = e(t) \in E$, $t \in (a, b)$, such that (12) is satisfied for each $t \in (a, b)$.

By definition, a Dirac dynamical system is a pair $(\varphi, \mathcal{D})$ where $\mathcal{D} = \big(\pi_{(E,M)}, \rho_E, D_E\big)$ is an Dirac anchored vector bundle and φ is an energy form.

Reduced Anchored Vector Bundles. Let $\mathcal{D} = (\pi_{(E,M)}, \rho_E, D_E)$ be a given Dirac anchored vector bundle. Assume that M is a principal bundle with group G acting on M on the left, and let $\pi_{(M,M/G)} : M \to M/G$ be the natural projection. We denote the action of an element $g \in G$ on M by $\underline{f}_g : M \to M$.

Assume, in addition, that that G acts on $E \oplus E^*$ by isomorphisms $f_g \oplus \tilde{f}_g : E \oplus E^* \to E \oplus E^*$ of Dirac anchored vector bundles, covering the action of G on M and satisfies the condition $(f_g)^* = (\tilde{f}_g)^{-1}$, for each $g \in G$. Recall that this also means that this action leaves D_E invariant. Then we say we have an action of G on $\mathcal{D}$, or that G is a symmetry group of $\mathcal{D}$.

One can show that E/G is a vector bundle over M/G with a well defined vector bundle projection $\pi_{(E/G,M/G)} : E/G \to M/G$.

There is also a natural anchor for this vector bundle, which we will denote, following our general notation for the anchor, by $\rho_{E/G}$. It is straightforward to check that $\rho_{E/G} := [T\pi_{(M,M/G)}]_G \circ [\rho_E]_G$ does the job. By definition,

$$[T\pi_{(M,M/G)}]_G : (TM)/G \to T(M/G)$$

is the vector bundle map defined by

$$[T\pi_{(M,M/G)}]_G([v_m]_G) = T\pi_{(M,M/G)}(v_m),$$

which one can easily check is well defined. A common alternative notation for this map is $(T\pi_{(M,M/G)})/G = [T\pi_{(M,M/G)}]_G$.

Likewise, the map $[\rho_E]_G : E/G \to (TM)/G$ is defined by $[\rho_E]_G([e]_G) = [\rho_E(e)]_G$. The map $\rho_{E/G}$ so defined is an anchor because it is easy to check that it is a vector bundle map $\rho_{E/G} : E/G \to T(M/G)$ over the identity.

Theorem 3. *The quotient space $(E \oplus E^*)/G$ is isomorphic to $E/G \oplus (E/G)^*$ as vector bundles over M/G.*

Proof. One checks that a natural vector bundle isomorphism $\Lambda : (E \oplus E^*)/G \to E/G \oplus (E/G)^*$ covering the identity is given by

$$[e_m \oplus \alpha_m]_G \mapsto [e_m]_G \oplus [\alpha_m]_G$$

where $e_m \in E_m$ and $\alpha_m \in E_m^*$. One checks that this is well defined because if $[e_m \oplus \alpha_m]_G = [e'_{gm} \oplus \alpha'_{gm}]_G$, then $[e_m]_G = [e'_{gm}]_G$ and $[\alpha_m]_G = [\alpha'_{gm}]_G$ since the action of G, while moving the common base points, acts componentwise on the fibers of $E \oplus E^*$. The inverse map is similarly shown to be well defined as follows. First, choose a point $[e_m]_G \oplus [\alpha_{gm}]_G$ in $E/G \oplus (E^*/G)$. Then write $[\alpha_{gm}]_G = [\alpha'_m]_G$ which is possible as the action by G on the fibers is by linear isomorphisms. Now define a map by $[e_m]_G \oplus [\alpha'_m]_G \mapsto [e_m \oplus \alpha'_m]_G$, which is, as above, checked to be well defined and is the inverse of Λ. Thus, Λ is a bundle isomorphism. $\square$

Recall that $D_E \subseteq E \oplus E^*$. From the definition of the action of G, we can then form the quotient subbundle

$$D_E/G \subseteq (E \oplus E^*)/G \cong E/G \oplus (E/G)^*$$

Theorem 4. *Under the above assumptions and constructions, the triple*

$$\mathcal{D}/G = \left(\pi_{(E/G,M/G)}, \rho_{E/G}, D_E/G\right), \qquad (13)$$

is a Dirac anchored vector bundle, called the reduced Dirac anchored vector bundle. Moreover, there is a natural morphism of Dirac anchored vector bundles $\mathcal{P}_G : \mathcal{D} \to \mathcal{D}/G$ covering the projection $\pi_{M,M/G} : M \to M/G$. Restricted to each fiber of $E \oplus E^*$, the associated map of $E \oplus E^*$ to $(E \oplus E^*)/G$ is an isomorphism.

Proof. In the notation of the definition of a morphism of Dirac anchored vector bundles, the map $f : E \to F$ in this case is $f = \pi_{(E,E/G)} : E \to E/G$. The map $\tilde{f} : E^* \to F^*$ is in this case, is $\tilde{f} : E^* \to (E/G)^*$ the fiberwise inverse dual of $\tilde{f}$. Note again, that in our case, the fibers of E and E/G have the same dimension. These two maps clearly cover the quotient map $\underline{f} = \pi_{(M,M/G)} : M \to M/G$.

The requirement $\rho_F \circ f = T\underline{f} \circ \rho_E$ in this case becomes

$$\rho_{E/G} \circ \pi_{(E,E/G)} = T\pi_{(M,M/G)} \circ \rho_E$$

That is,

$$[T\pi_{(M,M/G)}]_G \circ [\rho_E]_G \circ \pi_{(E,E/G)} = T\pi_{(M,M/G)} \circ \rho_E$$

which is readily checked. Each side evaluated on a point $e_m \in E_m$ gives the tangent vector $[(T\pi_{(M,M/G)} \circ \rho_E)(e_m)]_G$.

The only thing we have not established yet is that D_E/G is a Dirac structure on the reduced bundle E/G. Indeed, for each $m \in M$ we have a linear isomorphism

$$f_m \oplus \tilde{f}_m : E_m \oplus E_m^* \to (E/G)_{[m]_G} \oplus (E/G)^*_{[m]_G}.$$

where $[m]_G = \pi_{(M,M/G)}(m)$ is the class of m.

This isomorphism clearly transforms the linear Dirac structure $D_m \subseteq E_m \oplus E_m^*$ into a linear Dirac structure, which by definition is $(D/G)_{[m]_G} \subseteq (E/G)_{[m]_G} \oplus (E/G)^*_{[m]_G}$. $\square$

Notice that in this method of Dirac reduction, the fibers of the original underlying bundle E, become, after reduction, the fibers of E/G, which are not smaller in dimension, although the base M/G of course is smaller. This will be shown to be consistent with what one has in examples, such as Lie-Poisson or Suslov reduction.

We shall introduce the following notation, often more convenient in the categorical language, namely

$$\mathfrak{R}_G(E) := [E]_G := E/G$$

$$\mathfrak{R}_G(M) := [M]_G := M/G$$

$$\mathfrak{R}_G(\pi_{(E,M)}) := \pi_{(E/G,M/G)}$$

$$\mathfrak{R}_G(\rho_E) := \rho_{E/G} = [T\pi_{(M,M/G)}]_G \circ [\rho_E]_G$$

$$\mathfrak{R}_G(D_E) := [D_E]_G := D_{E/G} := D_E/G$$

$$\mathfrak{R}_G(\mathcal{D}) := [\mathcal{D}]_G := \mathcal{D}/G$$

The various maps involved in the reduction process are shown in Figure 1.

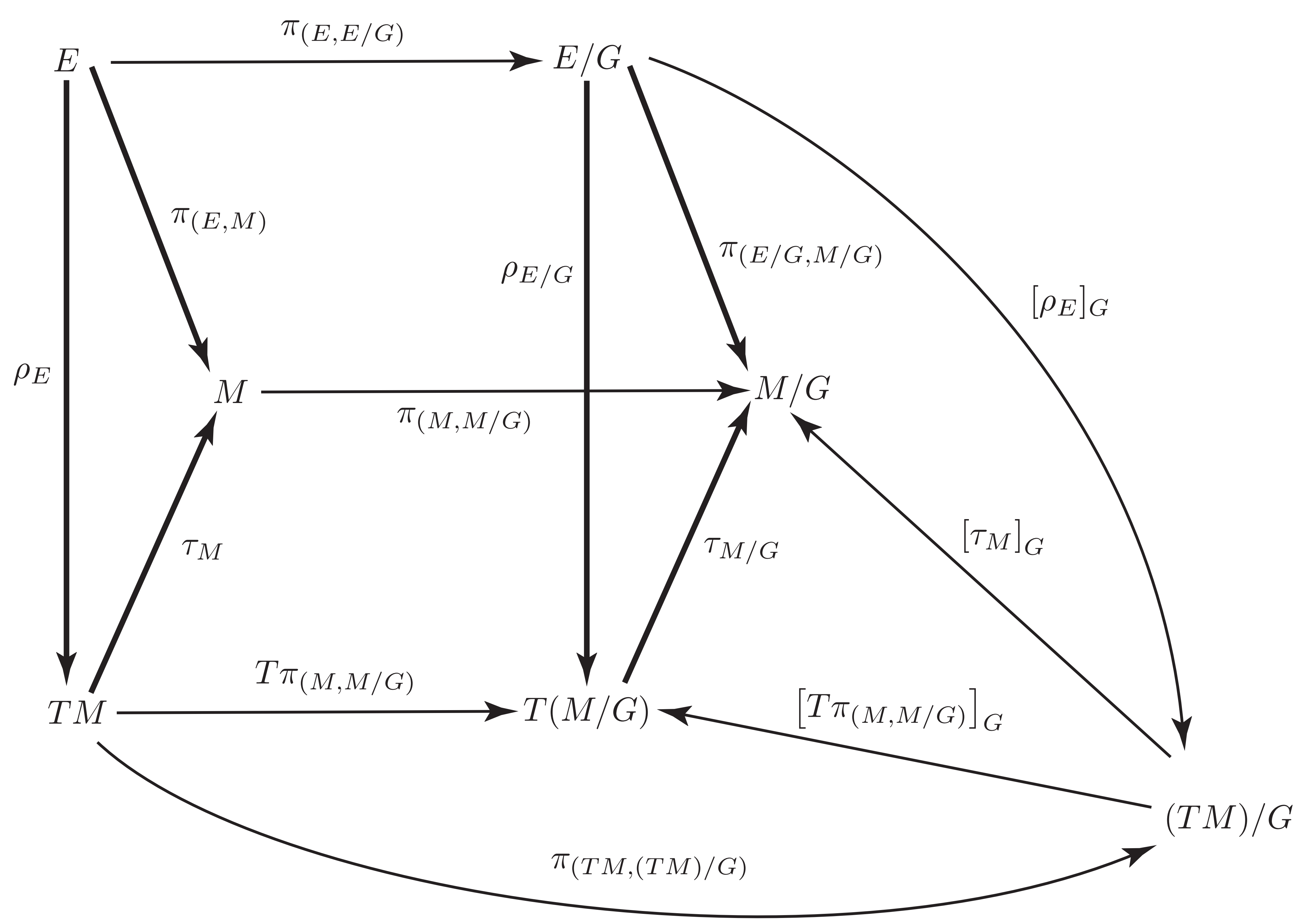

Figure 1: The dark commutative triangles show the original and reduced anchored vector bundles, alongwith maps connecting them.

Reduced Dirac Dynamical Systems. Now let $(\varphi, \mathcal{D})$ be a given Dirac dinamical system and assume that the group G acts on $\mathcal{D}$ and also that this action leaves φ invariant; that is, $\varphi(gm) = g\varphi(m)$. We also say that G is a symmetry for φ. In this case, there is an associated section $\varphi/G : M/G \to (E/G)^*$ defined by $(\varphi/G)([m]_G) = [\varphi(m)]_G$. Thus, we obtain a naturally defined reduced Dirac dynamical system $(\varphi/G, \mathcal{D}/G)$ which we also write as $\mathfrak{R}_G(\varphi, \mathcal{D}) = (\mathfrak{R}_G(\varphi), \mathfrak{R}_G(\mathcal{D})$, where $\mathfrak{R}_G(\varphi) = \varphi/G := [\varphi]_G$.

Reduction Theorem. We have the following theorem, whose proof will be a consequence of the above developments.

Theorem 5. *Let* $(\varphi, \mathcal{D}) = \big(\varphi, (\pi_{(E,M)}, \rho_E, D_E)\big)$ *be a given Dirac dynamical system and assume that the group* G *is a symmetry of* $(\varphi, \mathcal{D})$*. If a curve* $e(t)$*,* $t \in (a, b)$ *is a solution of the Dirac equations of motion; that is,*

$$e \oplus \varphi(m) \in (D_E)_m \tag{14}$$

where $m = \pi_{(E,M)}(e)$*, then the reduced curve* $[e]_G(t)$*,* $t \in (a, b)$ *is a solution of the reduced Dirac equations of motion; that is,*

$$[e]_G \oplus [\varphi]_G([m]_G) \in ([D_E]_G)_{[m]_G}. \tag{15}$$

Proof. Let $e(t)$, $t \in (a,b)$ be a solution of the Dirac equation of motion (14) that is, by definition it is an admissible curve satisfying

$$e(t) \oplus \varphi(m(t)) \in (D_E)_{m(t)} \tag{16}$$

for all $t \in (a,b)$, where $m(t) = \pi_{(E,M)}\left(e(t)\right)$. Taking equivalence classes of both sides of this relation, it is easy to see using the definition of the reduced Dirac structure that $[e(t)]_G$ satisfies the reduced Dirac equation of motion (15), that is,

$$[e(t)]_G \oplus [\varphi]_G([m(t)]_G) \in ([D_E]_G)_{[m(t)]_G} \tag{17}$$

for all $t \in (a,b)$. It only remains to show that $[e(t)]_G$ is an admissible curve, in other words,

$$\rho_{E/G}[e(t)]_G = \frac{d}{dt}\pi_{(E/G,M/G)}[e(t)]_G,$$

for all $t \in (a,b)$. Using the definitions of $\rho_{E/G}$ and of $[T\pi_{(M,M/G)}]_G$ and the commutativity of the diagram in Figure 1, we have

$$
\begin{aligned}
\rho_{E/G}[e(t)]_G &= \left([T\pi_{(M,M/G)}]_G \circ [\rho_E]_G\right)[e(t)]_G \\
&= [T\pi_{(M,M/G)}]_G \left([\rho_E\left(e(t)\right)]_G\right) \\
&= T\pi_{(M,M/G)}\left(\rho_E\left(e(t)\right)\right) \\
&= T\pi_{(M,M/G)} \cdot \frac{d}{dt} m(t) \\
&= \frac{d}{dt}\pi_{(M,M/G)}\left(m(t)\right) \\
&= \frac{d}{dt}[m(t)]_G \\
&= \frac{d}{dt}\pi_{(E/G,M/G)}[e(t)]_G,
\end{aligned}
$$

which shows that $[e(t)]_G$ is admissible. $\square$

Example: The Generalized Suslov Problem. We next consider the particular case of Example 1, where the manifold Q is a Lie group while the distribution Δ and the Lagrangian L are left invariant. Let G be a Lie group, $\Delta \subseteq TG$ a given left invariant distribution and $L : TG \to \mathbb{R}$ a given left invariant Lagrangian. Define $M = TG \oplus T^*G$ and also $\mathcal{D}_\Delta$ as in Example 1, then the Dirac equations of motion (12) will be the Lagrange-d'Alembert equations, which we now calculate in detail.

Let $TG \equiv G \times \mathfrak{g}$ be the body coordinate representation (that is, the left trivialization of TG) and let $\mathfrak{s} \subseteq \mathfrak{g}$ be a subspace such that $\Delta = G \times \mathfrak{s}$. We can identify $TM \equiv (G \times \mathfrak{g} \times \mathfrak{g}^*) \times (\mathfrak{g} \times \mathfrak{g} \times \mathfrak{g}^*)$. A given element of TM is written $(g, v, \alpha, \bar{v}, \dot{v}, \dot{\alpha})$, where $\bar{v} = g^{-1}\dot{g}$. The presymplectic form $\bar{\omega}$ has the following expression

$$\bar{\omega}(g,v,\alpha)\,((\bar{v},\dot{v},\dot{\alpha}),(\bar{w},\delta w,\delta\alpha)) = -\langle\dot{\alpha},\bar{w}\rangle + \langle\delta\alpha,\bar{v}\rangle + \langle\alpha,[\bar{v},\bar{w}]\rangle. \tag{18}$$

Then we obtain the following expression for $\mathcal{D}_\Delta$,

$$\begin{aligned}\mathcal{D}_\Delta = \{(g,v,\alpha,\dot{g},\dot{v},\dot{\alpha},\bar{p},p,u) \mid \bar{v}\in\mathfrak{s}, \bar{\omega}(g,v,\alpha)\,((\bar{v},\dot{v},\dot{\alpha}),(\bar{w},\delta w,\delta\alpha))\\ = \langle\bar{p},\bar{w}\rangle + \langle p,\delta w\rangle + \langle u,\delta\alpha\rangle, \text{for all } \bar{w}\in\mathfrak{s},\ \delta w\in\mathfrak{g},\ \delta\alpha\in\mathfrak{g}^*\}.\end{aligned}$$

One can check that $(g,v,\alpha,\dot{g},\dot{v},\dot{\alpha},\bar{p},p,u)\in\mathcal{D}_\Delta$ if and only if the following equations are satisfied

$$-\dot{\alpha} + \mathrm{ad}^*_{\bar{v}}\,\alpha - \bar{p} \in \mathfrak{s}^\circ, \quad p = 0, \quad \bar{v} = u, \quad \bar{v}\in\mathfrak{s}. \tag{19}$$

Define $l : \mathfrak{g} \to \mathbb{R}$ and $\mathfrak{e} : \mathfrak{g} \times \mathfrak{g}^* \to \mathbb{R}$, where $l(v) = L(e, v)$ and $\mathfrak{e}(v, \alpha) = \langle \alpha, v \rangle - l(v)$. Recall that one defines $\mathcal{E}(g, v, p) = \langle (g, p), (g, v) \rangle - L(g, v)$. Since L is invariant we clearly have $\mathcal{E}(g, v, \alpha) = \mathfrak{e}(v, \alpha)$, which does not depend on g.

To write equations of motion we must calculate $(g, v, \alpha, \bar{p}, p, u) = d\mathcal{E}(g, v, \alpha)$, and we obtain

$$\bar{p} = 0, \quad p = \alpha - \frac{\partial l}{\partial v}, \quad v = u. \tag{20}$$

From equations (19) and (20), we obtain

$$-\frac{d}{dt}\frac{\partial l}{\partial v} + \mathrm{ad}_v^* \frac{\partial l}{\partial v} \in \mathfrak{s}^\circ, \quad v \in \mathfrak{s}. \tag{21}$$

We have obtained the generalized Suslov equations by simply calculating equation (12) using a left trivialization.

Now we will show that the same equations can be obtained by reduction.

First of all, we are going to reduce the Dirac anchored vector bundle $(\pi_{TM,M}, \rho_{TM}) = (\tau_M, 1_{TM})$.

It is easy to see that

$TM/G = (\mathfrak{g} \times \mathfrak{g}^*) \times (\mathfrak{g} \times \mathfrak{g} \times \mathfrak{g}^*)$.

We have the following expression for the reduced Dirac structure $\mathcal{D}_\Delta/G$,

$$\begin{aligned}[\mathcal{D}_\Delta]_G = \{&(v, \alpha, \dot{g}, \dot{v}, \dot{\alpha}, \bar{p}, p, u) \mid \bar{v} \in \mathfrak{s}, [\bar{\omega}]_G(v, \alpha)\left((\bar{v}, \dot{v}, \dot{\alpha}), (\bar{w}, \delta w, \delta\alpha)\right)\\ &= \langle \bar{p}, \bar{w}\rangle + \langle p, \delta w\rangle + \langle u, \delta\alpha\rangle, \text{for all } \bar{w} \in \mathfrak{s},\ \delta w \in \mathfrak{g},\ \delta\alpha \in \mathfrak{g}^*\},\end{aligned}$$

where we have used the fact that

$$[\bar{\omega}]_G(v, \alpha)\left((\bar{v}, \dot{v}, \dot{\alpha}), (\bar{w}, \delta w, \delta\alpha)\right) = -\langle \dot{\alpha}, \bar{w}\rangle + \langle \delta\alpha, \bar{v}\rangle + \langle \alpha, [\bar{v}, \bar{w}]\rangle. \tag{22}$$

The reduced energy function is the function $\mathfrak{e} : \mathfrak{g} \times \mathfrak{g}^* \to \mathbb{R}$. It is very easy to see that the reduced Dirac equation reproduces the previous equations of motion.

Reduction by Stages: An Example Let $(\pi_{E,M}, \rho_E)$ be a given Dirac anchored vector bundle. Assume that M is a principal bundle with structure group G, acting on the left. Now assume that there is a normal subgroup $N \subseteq G$. Then we can reduce *by stages*. One can check that the natural isomorphism

$$E/G \equiv (E/N)/(G/N)$$

$[e]_G \equiv [[e]_N]_{G/N}$ is an isomorphism of Dirac anchored vector bundles.

Let us recall some facts from Lagrangian reduction by stages, (Cendra, Marsden, Ratiu [2001]). Let $\pi : Q \to Q/G$ principal bundle with structure group G acting on the left.

A principal connection. The curvature B is given by Cartan's structure equation
$\mathbf{d}A(u, v) = B(u, v) + [A(u), A(v)]$.

$\tilde{\mathfrak{g}}$ adjoint bundle.

Define the $\tilde{\mathfrak{g}}$-valued 2-form $\tilde{B}$ on the base Q/G by $\tilde{B}([q]_G)(X, Y) = [q, B(X^h(q), Y^h(q))]_G$, where X^h and Y^h are the horizontal lifts of the vector fields X and Y on Q/G. Denote $\tilde{\nabla}^A$ the affine connection naturally induced on the vector bundle $\tilde{\mathfrak{g}}$ by the principal connection A. Let $\bar{X}_i$, $i = 1, 2$ be given invariant vector fields on Q.

Recall that one has an isomorphism
$\alpha_A : TQ/G \to T(Q/G) \oplus \tilde{\mathfrak{g}}$, given by
$\alpha_A(q, \dot{q}) = T\pi(q, \dot{q}) \oplus [q, A(q, \dot{q})]_G$.

Let $\alpha_A\left([\bar{X}_i]_G\right) = X_i \oplus \bar{\xi}_i$, $i = 1, 2$. Then one can prove the following formula for the Lie bracket on sections of Lagrange-Poincaré bundles:

$$[X_1 \oplus \bar{\xi_1}, X_2 \oplus \bar{\xi_2}] = [X_1, X_2] \oplus \tilde{\nabla}^A_{X_1}\bar{\xi}_2 - \tilde{\nabla}^A_{X_2}\bar{\xi}_1 - \tilde{B}(X_1, X_2) + [\bar{\xi}_1, \bar{\xi}_2],$$

where, by definition,

$$[X_1 \oplus \xi_1, X_2 \oplus \xi_2] = \alpha_A\left(\left[[\bar{X}_i, \bar{X}_i]\right]_G\right).$$

One would like to perform reduction *by stages*, which has been done in Cendra, Marsden, Ratiu [2001] carefully for two stages. We are going to adapt that formula to show how to reduce in two stages the Suslov problem, omitting the technical aspects.

Let a Lie group G, N a normal subgroup of G and $K = G/N$. Let $\mathcal{G}$, $\mathcal{N}$ and $\mathcal{K}$ be the Lie algebras of G, N and K respectively. We choose an identification $\mathcal{G} \equiv \mathcal{K} \oplus \mathcal{N}$ as linear spaces. Let $\mathcal{A}_N$ be a principal conexion on the principal bundle G with structure group N having the property that $\mathcal{A}_N(gv_q) = \mathrm{Ad}_g \mathcal{A}_N(v_q)$, for every $g, q \in G$, $v_q \in T_qG$. Then $\mathcal{G} = \mathcal{K}^{\mathcal{A}_N} + \mathcal{N}$ where $\mathcal{K}^{\mathcal{A}_N}$ is the horizontal lift of $\mathcal{K}$ in the bundle $G \to G/N$. Note that $\mathcal{K}^{\mathcal{A}_N} \cap \mathcal{N} = \{0\}$.

Then, the Lie bracket in the Lie algebra $\mathcal{G}$ can be written in terms of the brackets of the Lie algebra $\mathcal{N}$ and the Lie algebra $\mathcal{K}$, and also in terms of $\nabla^{(\mathcal{A}_N, V)}$ and $\widetilde{B}^{\mathcal{A}_N}$ as follows:

$$[\kappa_1 \oplus \eta_1, \kappa_2 \oplus \eta_2] = [\kappa_1, \kappa_2] \oplus [\nabla^{(\mathcal{A}_N, V)}]_{G/N, \kappa_1} \eta_2 \\ - [\nabla^{(\mathcal{A}_N, V)}]_{G/N, \kappa_2} \eta_1 - [\widetilde{B}^{\mathcal{A}_N}]_{G/N}(\kappa_1, \kappa_2) + [\eta_1, \eta_2] \, .$$

Using formulas for reduced covariant derivatives
in the previous reference we can calculate
explicitly $[\widetilde{B}^{\mathcal{A}_N}]_{G/N}(\kappa_1, \kappa_2)$ and $[\nabla^{(\mathcal{A}_N, V)}]_{G/N, \kappa}\eta$.

If we define the bilinear forms
$b_N : \mathcal{K} \times \mathcal{N} \to \widetilde{\mathcal{N}}/K \equiv \mathcal{N}$ and
$a_N : \mathcal{K} \times \mathcal{K} \to \widetilde{\mathcal{N}}/K \equiv \mathcal{N}$

$b_N(\kappa, \eta) := \left[\left[e, [\kappa^{\mathcal{A}_N}, \eta] \right]_N \right]_K$ and
$a_N(\kappa, \overline{\kappa}) := \left[\left[e, -\mathcal{A}_N(e) \left([\kappa^{\mathcal{A}_N}, \overline{\kappa}^{\mathcal{A}_N}] \right) \right]_N \right]_K$

Then we can write the Lie bracket as follows:

$[\kappa \oplus \eta, \overline{\kappa} \oplus \overline{\eta}] = [\kappa, \overline{\kappa}] \oplus b_N(\kappa, \overline{\eta}) - b_N(\overline{\kappa}, \eta) - a_N(\kappa, \overline{\kappa}) + [\eta, \overline{\eta}].$

Using this we can reduce in two stages. We are not going to go through the details, but simply observe that a key technical point of the calculation will be to introduce the previous expression of the Lie bracket in the expression of the presymplectic form

$[\bar{\omega}]_G(v, \alpha)\,((\bar{v}, \dot{v}, \dot{\alpha}), (\bar{w}, \delta w, \delta\alpha)) = -\langle \dot{\alpha}, \bar{w}\rangle + \langle \delta\alpha, \bar{v}\rangle + \langle \alpha, [\bar{v}, \bar{w}]\rangle.$

The final expression of the equations is the following

$$\left\{ \begin{array}{rcl} \dot{\alpha} & = & \mathrm{ad}^*_\kappa \alpha \\ \dot{\beta} & = & \beta\,([\eta,\ .\ ] + b_N(\kappa,\ .\ )) \\ 0 & = & \beta\,(b_N\,(\ .\ , \eta) + a_N(\kappa,\ .\ )) \\ \alpha & = & \dfrac{\partial l}{\partial \kappa} \\ \beta & = & \dfrac{\partial l}{\partial \eta} \end{array} \right.$$

THANK YOU FOR YOUR ATTENTION