Degenerate relative equilibria

— and the concept of criticality in hydrodynamics —

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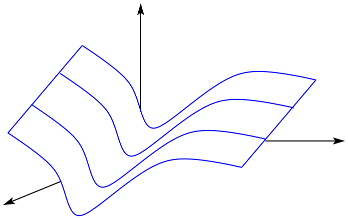

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Table of Contents

1. The concept of criticality in shallow-water hydrodynamics
2. Correspondence between criticality and degenerate RE
3. Degenerate relative equilibria and homoclinic bifurcation
   - Saddle-center transverse to the group orbit
   - Role of geometry of momentum map
4. Criticality and degenerate RE: creating new families of solitary waves
5. Degenerate conservation laws

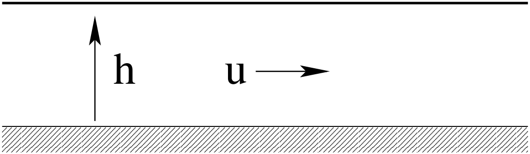
$$\mathbf{U}_t + \mathbf{F}(\mathbf{U})_x = \mathbf{0}\,, \quad \mathbf{U} \in \mathbb{R}^n\,.$$

   Why is the flux vector $\mathbf{F}(\mathbf{U})$ like a momentum map?

6. Degenerate conservation laws with regularization

Criticality in shallow water hydrodynamics

In shallow water, a uniform flow with velocity u and depth h,



is said to be critical when $u^2 = gh$ (Froude number unity).

Another characterization: the flow is critical when the speed of a plane wave in the linearization about uniform flow equals speed of uniform flow:

$$\text{speed of plane waves} = \pm\sqrt{gh}$$

Criticality is a bifurcation point for solitary waves.

In this case, Froude number unity is a bifurcation point for the KdV solitary wave (add dispersion to SWE to see this).

Criticality in shallow water hydrodynamics

Shallow water equations

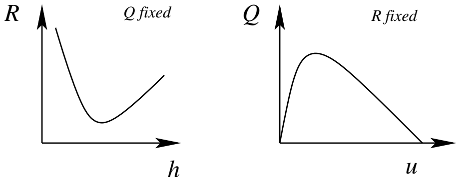
$$u_t + uu_x + gh_x = 0 \quad \text{and} \quad h_t + uh_x + hu_x = 0\,.$$

Steady solutions realize constant values of Q and R

$$R = gh + \tfrac{1}{2}u^2 \quad \text{and} \quad Q = hu\,.$$

Given Q and R find values of h and u. Criticality:

- the state at which R is a minimum for fixed $Q \neq 0$
- the state at which Q is a maximum for fixed $R > 0$ ($u > 0$)



Generalize criticality to non-trivial states?

Use quasi-static approximation, or consider the flow to be slowly-varying in the $x-$direction and use WKB theory

- GILL (1977) *J. Fluid Mech.*
- KILLWORTH (1992) *J. Fluid Mech.*
- JOHNSON & CLARKE (2001) *Ann. Rev. Fluid Mech.*

Define criticality to be when eigenvalues of the linearization pass through zero.

- BENJAMIN (1971) A unified theory of conjugate flows, *Phil. Trans. Royal Soc. London A*

But restricted to parallel flows (independent of x), and criticality is treated as a one-parameter problem.

New observation: uniform flows are relative equilibria

Criticality is an $n-$parameter problem with $n = \dim(\mathfrak{g})$

- (h, u) are coordinates for a Lie algebra
- (R, Q) are coordinates for a momentum map

Criticality of uniform flows corresponds to degeneracy of RE

Degenerate RE generate zero eigenvalues: a saddle-center bifurcation transverse to the group orbit

Saddle-center leads to homoclinic bifurcation (SW)

- Role of curvature of the momentum map
- Geometric phase along group
- Thom-Boardman classification of singularities

Generalize criticality: can define criticality for any flow which can be characterized as a RE !

$G-$equivariant Hamiltonian systems

Consider a Hamiltonian system with symmetry. For example, take

$$\mathbb{J}\mathbf{u}_t = \nabla H(\mathbf{u})\,, \quad \mathbf{u} \in M = \mathbb{R}^{2n+2}\,,$$

and suppose that it is equivariant with respect to an $n-$dimensional abelian Lie group G (subgroup of the Euclidean group) with Lie algebra $\mathfrak{g}$, action $\Phi_g(\mathbf{u})$ and generator

$$\xi_M(\mathbf{u}) := \frac{d}{ds}\bigg|_{s=0} \Phi_{\exp(t\xi)}(\mathbf{u})\,, \quad \xi \in \mathfrak{g}\,.$$

Suppose G is symplectic and the Hamiltonian function is G$-$invariant, etc., and momentum map

$$\mathbf{J} \;:\; M \to \mathfrak{g}^*\,.$$

Symplectic relative equilibria

Relative equilibria are solutions which travel along a group orbit at constant speed. An RE is of the form

$$\mathbf{u}(t) = \Phi_{\exp(t\xi)}(\varphi) \quad \text{for some} \quad \xi \in \mathfrak{g}\,,$$

where $\varphi : \mathfrak{g} \to M$ is a critical point of the augmented Hamiltonian

$$H_\xi(\mathbf{u}) := H(\mathbf{u}) - \langle \mathbf{J}(\mathbf{u}) - \mu, \xi \rangle\,.$$

A critical point, φ, of H_ξ is a mapping from $\mathfrak{g}$ into M. Substitution into the momentum gives

$$\mu = \mathbf{J} \circ \varphi(\xi)\,.$$

a mapping from $\mathfrak{g}$ into $\mathfrak{g}^*$.
The equation $DH_\xi = 0$ can also be interpreted as the Lagrange necessary condition for a constrained variational principle: find critical points of H restricted to level sets of the momenta.

(cf. MARSDEN (1992), MARSDEN & RATIU (1994))

Degenerate relative equilibria

A RE is non-degenerate when the second variation of H_ξ at a critical point is a non-degenerate quadratic form on the subspace consisting of vectors tangent to $\mathbf{J}^{-1}(\mu)$ and transverse to the group orbit.

Four types of degeneracy

• Singularity of the momentum map.

Assume throughout that μ is a regular value of the momentum map.

• Failure of G-Morse: the dimension of the kernel of the second variation of H_ξ is greater than the dimension of the group. Related to $\mathrm{D}\mathbf{c}(\mu)$ singular.

• Failure of G-Morse: μ fixed, but external parameters varied; example: PALMORE, Measure of degenerate relative equilibria I, *Ann. Math.* (1976).

• $\det[\mathrm{D}\mathbf{P}(\mathbf{c})] = 0$, $\mathbf{P}(\mathbf{c}) := \mathbf{J} \circ \varphi$ and $\mathbf{c}$ are coordinates for $\mathfrak{g}$.

Degenerate RE and the Jacobian

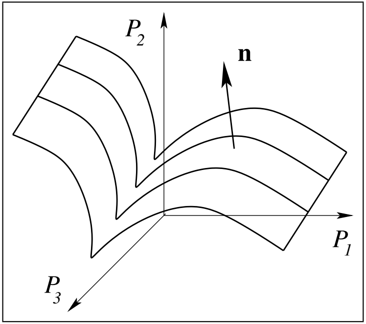
The key to the study of the nonlinear behaviour transverse to the group orbit near degenerate RE is the geometry of

$$\mathbf{P} \; : \; \mathfrak{g} \to \mathfrak{g}^* \, .$$

The condition

$$\det[\mathrm{D}\mathbf{P}(\mathbf{c})] = 0 \, ,$$

defines a hypersurface in $\mathfrak{g}$ with image in $\mathfrak{g}^*$



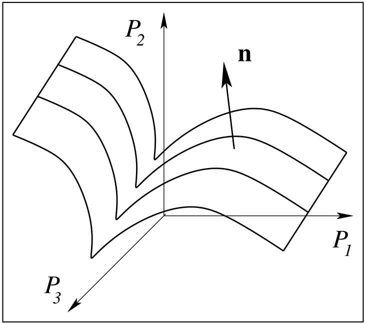
Geometry of $\det[\mathrm{D}\mathbf{P}(\mathbf{c})] = 0$

When $\mathrm{D}\mathbf{P}(\mathbf{c})$ has rank $n-1$ there exists $\mathbf{n} \in T_{\mathbf{c}}\,\mathfrak{g}$ with

$$[\mathrm{D}\mathbf{P}(\mathbf{c})]\mathbf{n} = 0\,.$$

The image of the hypersurface in $\mathfrak{g}^*$ can have singularities. By introducing a metric, $\mathbf{n}$ can be interpreted as a normal vector to the surface in $\mathfrak{g}^*$, at regular points.

The surface in $\mathfrak{g}^*$ is a barrier to the existence of RE.



Thom-Boardman singularity theory

For a mapping $P : \mathbb{X} \to \mathbb{Y}$, with $\mathbb{X}, \mathbb{Y}$ $n-$dimensional vector spaces, the subsets

$$\Sigma^k(\mathbf{P}) = \{\mathbf{c} \in \mathbb{X} \ : \ \mathrm{rank}(\mathrm{Jac}(\mathbf{c})) = n - k\}$$

are known in singularity theory as the Thom-Boardman singularities. Restrict to the case $k = 1$. There is a hierarchy of singular sets, for example

$$\Sigma^{11}(\mathbf{P}) = \Sigma^1 \left(\mathbf{P}\Big|_{\Sigma^1(\mathbf{P})} \right)$$

is the set where Jacobian of the kernel of **P** restricted to $\Sigma^1(\mathbf{P})$ drops in rank by one. The classification continues until the dimension is exhausted. The connection with degenerate RE:

- Momentum map $\mathbf{P}(\mathbf{c}) \in \Sigma^1(P) \Rightarrow$ saddle-center bifurcation
- Nonlinearity: $\mathbf{P}(\mathbf{c}) \notin \Sigma^{11}(P) \Rightarrow$ homoclinic bifurcation

Degeneracy of $\mathrm{D}\mathbf{P}(\mathbf{c})$ and saddle-center

Linearize about a degenerate relative equilibrium

- 0 is an eigenvalue of geometric multiplicity n
- 0 is an eigenvalue of algebraic multiplicity $2n$
- 0 is an eigenvalue of (at least) algebraic multiplicity $2n+2$ if and only if $\det[\mathrm{D}\mathbf{P}(\mathbf{c})] = 0$ (invoking the $G-$Morse hypothesis).

Saddle-center bifurcation of eigenvalues in the linearization transverse to the group orbit corresponds to $\mathbf{P}(\mathbf{c}) \in \Sigma^1(\mathbf{P})$.

Transform linearization to Williamson normal form.

Leading order nonlinear normal form

For values of the momenta in a neighbourhood of a degenerate point,there exists coordinates $(\phi_1, \ldots, \phi_n, u, I_1, \ldots, I_n, v) \in \mathbb{R}^{2n+2}$ satisfying

$$
\begin{aligned}
-\frac{dv}{dt} &= I_1 - \tfrac{1}{2}\kappa u^2 + \cdots\,, \\
\frac{du}{dt} &= s_1 v + \cdots\,, \\
-\frac{dI_j}{dt} &= 0\,, && j = 1, \ldots, n \\
\frac{d\phi_1}{dt} &= u + \cdots \\
\frac{d\phi_j}{dt} &= s_j\, I_j + \cdots\,, && j = 2, \ldots, n\,.
\end{aligned}
$$

The coordinates $(I_1, \ldots, I_n)$ are local coordinates near a point on the criticality hypersurface in $P-$space. The coordinate I_1 is associated with the direction transverse to the hypersurface, and $I_2, \ldots, I_n$ are associated with directions tangent to the image of the hypersurface $\det[\mathrm{D}\mathbf{P}(\mathbf{c})] = 0$.

Formula for κ and the symplectic signs

The coefficient of the nonlinear term in the normal form, κ, can be expressed in terms of the generalized eigenvectors,

$$\begin{aligned}\kappa &= -\langle \xi_{n+1}, D^3H(\xi_{n+1},\xi_{n+1})\rangle - 3\langle \xi_1, D^3H(\xi_1,\xi_{2n+2})\rangle \\ &\quad +3\langle \xi_1, D^3H(\xi_{n+1},\xi_{2n+1})\rangle \,.\end{aligned}$$

The sign $s_1 = \pm 1$ is a symplectic invariant associated with the symplectic Jordan theory.

The signs s_j for $j = 2, \ldots, n$ are the signs of the nonzero eigenvalues of $\mathrm{D}\mathbf{P}(\mathbf{c})$.

κ and the momentum map

The coefficient κ has a characterization in terms of the geometry of the momentum map $\mathbf{P}$

$$\kappa = a_0^3 \langle \mathrm{d}f(\mathbf{c}), \mathbf{n} \rangle \,, \qquad f(\mathbf{c}) := \det[\mathrm{D}\mathbf{P}(\mathbf{c})] \,,$$

where a_0 is a positive constant.

Remark: κ can also be characterized as the intrinsic second derivative[1] of the mapping $\mathbf{P}(\mathbf{c})$ (e.g. PORTEOUS 1971, GOLUBITSKY & GUILLEMIN 1973):

$$\langle \mathrm{d}f(\mathbf{c}), \mathbf{n} \rangle = \text{Constant} \, \langle D^2\mathbf{P}(\mathbf{c})(\mathbf{n}, \mathbf{n}), \mathbf{n} \rangle \,.$$

[1] Thanks to James Montaldi (Manchester) for this observation.

Curvature of the momentum map

Let

$$\begin{aligned} \mathfrak{g} \quad &\cong \quad T_{\mathbf{c}}\,\mathfrak{g} = \mathfrak{h} \oplus \mathbb{X}\,, \quad \mathfrak{h} = \mathrm{Ker}(\mathrm{D}\mathbf{P}(\mathbf{c})) \\ \mathfrak{g}^* \quad &\cong \quad T_{\mathbf{P}(\mathbf{c})}\,\mathfrak{g}^* = \mathbb{Y} \oplus \mathfrak{h}^*\,, \end{aligned}$$

It is the curvature of the graph of the function

$$\mathscr{K}(\mathbf{c}, s) = \langle \mathbf{n}, \mathbf{P}(\mathbf{c} + s\mathbf{n}) \rangle\,,$$

on $\mathfrak{h} \times \mathfrak{h}^*$ that appears in the normal form

$$\kappa = \text{Constant}\, \frac{d^2}{ds^2}\bigg|_{s=0} \mathscr{K}(\mathbf{c}, s)\,,$$

for some positive constant.

(cf. TJB, J. Diff. Eqns, 2008)

Leading order nonlinear normal form

Normal form transverse to the group

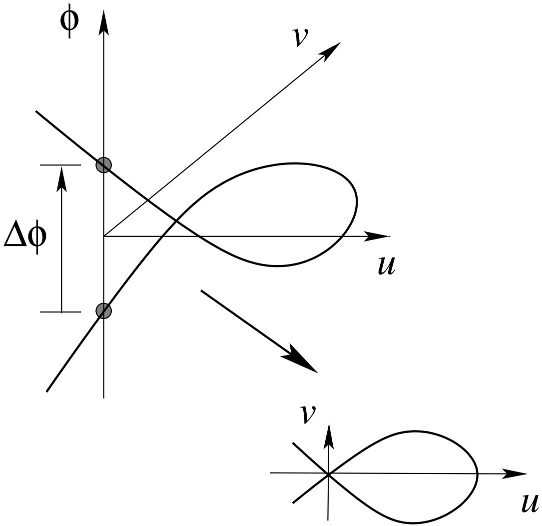
$$
\begin{aligned}
-\frac{dv}{dt} &= I_1 - \tfrac{1}{2}\kappa u^2 + \cdots\,, \\
\frac{du}{dt} &= s_1 v + \cdots\,.
\end{aligned}
$$

Normal form tangent to the group

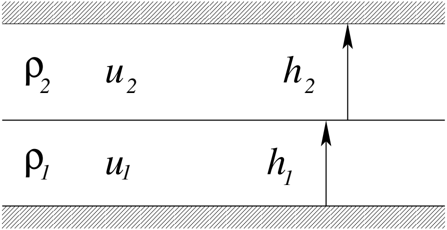
$$
\begin{aligned}
-\frac{dI_j}{dt} &= 0\,, && j = 1, \ldots, n \\
\frac{d\phi_1}{dt} &= u + \cdots \\
\frac{d\phi_j}{dt} &= s_j\, I_j + \cdots\,, && j = 2, \ldots, n
\end{aligned}
$$

Directional geometric phase, plus dynamic drift along the group.

Schematic of the geometric phase



Bifurcation of internal solitary waves



Taking a Boussinesq model for internal waves (e.g. CHOI & CAMASSA (1999) *J. Fluid Mech.*), can formulate the steady part as a Hamiltonian system on $\mathbb{R}^8$ with a three-dimensional group of affine translations.

The Lie algebra can be coordinatized by the parameters associated with the uniform flow (h_1, u_1, u_2), and the momentum map can be coordinatized by (R, Q_1, Q_2) where R is the Bernoulli energy and Q_j are the mass flux in each layer. Rigid lid implies $h_1 + h_2 = d$.

Criticality and geometry of $\mathbf{P} : \mathfrak{g} \longrightarrow \mathfrak{g}^*$

$$\mathbf{P}(\mathbf{c}) := \mathbf{J} \circ \varphi = R(\mathbf{c})\xi_1^* + Q_1(\mathbf{c})\xi_2^* + Q_2(\mathbf{c})\xi_3^* \,,$$

with $\mathbf{c} = (h_1, u_1, u_2)$ and

$$\begin{aligned} R(\mathbf{c}) &= \tfrac{1}{2}\rho_1 u_1^2 - \tfrac{1}{2}\rho_2 u_2^2 + (\rho_1 - \rho_2) g h_1 \\ Q_1(\mathbf{c}) &= \rho_1 h_1 u_1 \\ Q_2(\mathbf{c}) &= \rho_2 (d - h_1) u_2 \,. \end{aligned}$$

$$\mathrm{D}\mathbf{P}(\mathbf{c}) = \begin{bmatrix} (\rho_1 - \rho_2) g & \rho_1 u_1 & -\rho_2 u_2 \\ \rho_1 u_1 & \rho_1 h_1 & 0 \\ -\rho_2 u_2 & 0 & \rho_2 (d - h_1) \end{bmatrix} ,$$

and there exists $\mathbf{n}$ satisfying $\mathrm{Jac}(\mathbf{c})\mathbf{n} = 0$ when $f(\mathbf{c}) = 0$ where

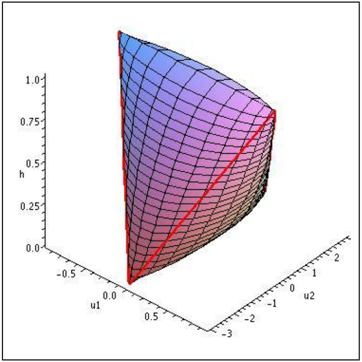
$$f(\mathbf{c}) = \det(\mathrm{Jac}(\mathbf{c})) = \rho_1\rho_2(\rho_1 - \rho_2) g h_1 (d - h_1) \left[1 - F_1^2 - r F_2^2\right] \,,$$

where $F_j^2 = u_j^2/((1 - r) g h_j)$ and $r = \rho_2/\rho_1$.

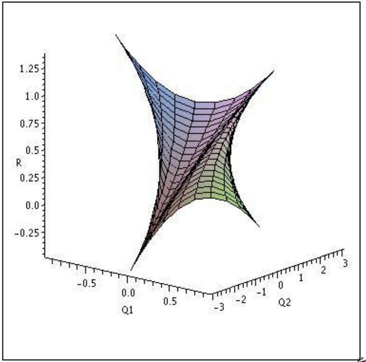
Plot the surface $f(\mathbf{c}) = 0$ and its image in the (R, Q_1, Q_2) plane.

Criticality surfaces for two-layer flow

$\mathfrak{g}$



$\mathfrak{g}^*$



Criticality and $\mathrm{d}f(\mathbf{c}) \cdot \mathbf{n}$

Now

$$f(\mathbf{c}) := \det[\mathrm{Jac}(\mathbf{c})] = \mathrm{C}\left[(1-r) - \frac{u_1^2}{gh_1} - r\frac{u_2^2}{gh_2}\right], \quad \mathrm{C} = \rho_1^2\rho_2 g h_1 h_2 \,.$$

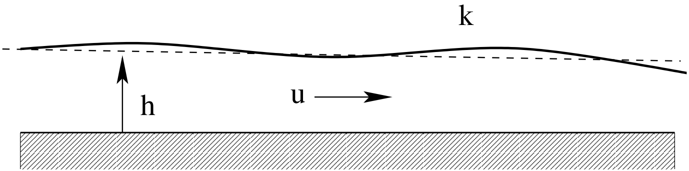
The criticality surface in (h_1, u_1, u_2) space is defined by $f^{-1}(0)$ and a vector $\mathbf{v}$ is tangent to this surface if $\mathrm{d}f \cdot \mathbf{v} = 0$. Now,

$$\mathrm{d}f = \frac{\mathrm{C}}{g}\left(\frac{u_1^2}{h_1^2} - \frac{ru_2^2}{h_2^2}, -\frac{2u_1}{h_1}, -\frac{2ru_2}{h_2}\right),$$

and so

$$\langle \mathrm{d}f, \mathbf{n}\rangle = \frac{3\mathrm{C}}{\rho_1 g}\left(\rho_1\frac{u_1^2}{h_1^2} - \rho_2\frac{u_2^2}{h_2^2}\right).$$

(cf. TJB & Donaldson, Phys. Fluids, 2007)



Stokes waves in shallow water coupled to a mean flow are RE associated with $G = \mathbb{R}^2 \times S^1$ with $\mathbb{R}^2$ associated with mean flow, and S^1 associated with the periodic wave (the Stokes wave):

$$(h, u, k) \to (R, Q, B)$$

When these RE are degenerate,

$$\det \left[\frac{\partial(R, Q, B)}{\partial(h, u, k)} \right] = 0 \,,$$

the flow is critical and a class of solitary waves is generated: steady "dark solitary waves".

(cf. TJB & DONALDSON, J. Fluid Mech. 2006)

Model Hamiltonian system with $\mathbb{S}^1 \times \mathbb{R}^2$ symmetry

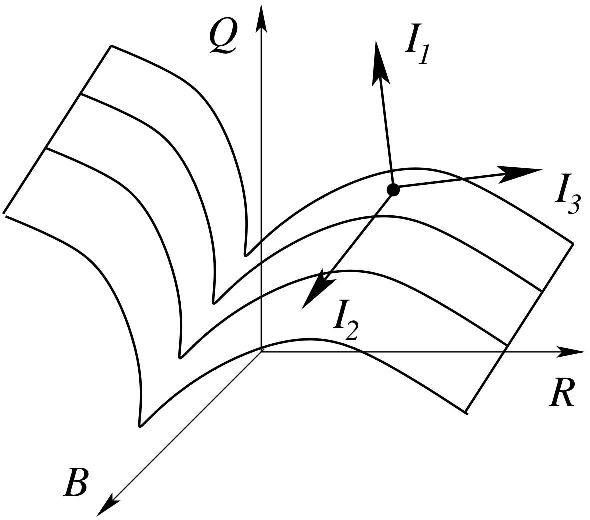
$$\begin{aligned} a\,A_{xx} + 2\mathrm{i}b\,A_x + \beta|A|^2 A &= -2\,(\ell h_x + m u_x)A \\ r\,h_{xx} + c\,u_{xx} &= \ell\,(|A|^2)_x \\ c\,h_{xx} + s\,u_{xx} &= m\,(|A|^2)_x\,, \end{aligned}$$

where $a, b, \beta, \ell, m, r, s$ and c are given (in general nonzero) real parameters with $rs - c^2 \neq 0$. (For water waves $gh_0 - c_g^2 \neq 0$.)

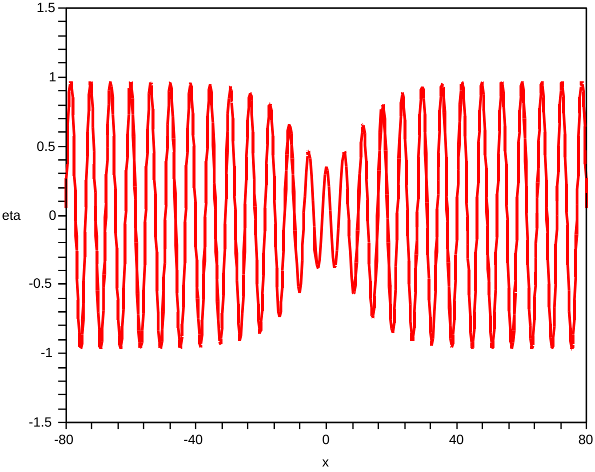
$$\mathbb{J}u_x = \nabla H(u)\,, \quad u \in \mathbb{R}^8\,.$$

When RE associated with the group $\mathbb{S}^1 \times \mathbb{R}^2$ are degenerate, a homoclinic bifurcation occurs which corresponds to a form of steady dark solitary wave. Found also in full water wave problem (cf. TJB & Donaldson J. Fluid Mech. 2006).

Schematic of the image of $\Sigma^1(P)$ for degenerate Stokes waves



Schematic of steady dark solitary waves



κ calc required a few days – versus months/year for direct calc!

Degenerate RE and internal solitary waves

- Two-layer flow with a rigid lid
  - uniform flows = 3D RE, critical surface is 2D
  - $\langle \mathrm{d}f, \mathbf{n} \rangle = 0$ separates solitary waves of elevation from solitary waves of depression.
  - 3D mean flow (uniform flow) coupled to a periodic wave = 4D RE, 3D critical surface, bif. to internal steady DSWs
- Two-layer flow with a free surface
  - uniform flow = 4D RE, critical surface is 3D
  - $\langle \mathrm{d}f, \mathbf{n} \rangle = 0$ is a 2D manifold
  - uniform flow (mean flow) coupled to a periodic wave = 5D RE, 4D critical surface, bif. to internal steady DSWs

Theory predicts manifold of bifurcating solitary waves from each family of degenerate RE. The bifurcating SWs may have exponentially small tails in the case of two layers with free surface.

(cf. TJB & DONALDSON, Eur J Mech B/Fluids (2008))

Remarks on dimension and group action

When the group has dimension n, $2n+2$ is the lowest dimension phase space in which the phenomena can occur.

— Dimension N with $N > 2n+2$: complementary dimensions hyperbolic, can use center-manifold reduction.

– Dimension N with $N > 2n+2$: complementary dimensions elliptic, will get persistence issues and exponentially-small tails, as in the case without symmetry (e.g. Iooss & Lombardi, J. Diff. Eqns 2006)

– When the group is non-abelian need to bring in more theory to do the tangent/transverse splitting of the vectorfield (e.g. ROBERTS, WULFF & LAMB J. Diff. Eqns 2002), but one expects the basic idea to persist (geometry of momentum map on RE determining the nonlinear normal form transverse to group).

Degenerate conservation laws – and criticality

$n-$layer models of stratified flow, in the shallow water approximation, lead to conservation laws of the form

$$\mathbf{U}_t + \mathbf{F}(\mathbf{U})_x = 0\,, \quad \mathbf{U} \in \mathbb{R}^{2n}\,,$$

where $\mathbf{F} : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ is the flux vector.
The conservation law is said to be degenerate at $\mathbf{U}_0$ if the Jacobian $D\mathbf{F}(\mathbf{U}_0)$ is singular

$$\det[D\mathbf{F}(\mathbf{U}_0)] = 0 \quad \text{criticality!}$$

For the $n-$layer models the flux vector is of the form

$$\mathbf{F}(\mathbf{U}) = \mathbf{M}\nabla E(\mathbf{U})\,,$$

where $\mathbf{M}$ is a symmetric invertible – but indefinite – matrix. For this class of systems, the flux vector can be related to a momentum map.

Conservation laws – with dispersive regularization

Consider

$$\mathbf{U}_t + \mathbf{F}(\mathbf{U})_x = \mathbf{D}\mathbf{U}_{xxx}\,, \quad \mathbf{U} \in \mathbb{R}^{2n}\,,$$

with $\mathbf{F}(\mathbf{U}) = \mathbf{M}\nabla E(\mathbf{U})$, $\mathbf{D}$ invertible, and $\mathbf{M}^{-1}\mathbf{D}$ symmetric.
Steady solutions satisfy

$$\mathbf{D}\mathbf{U}_{xxx} = \mathbf{M}\nabla E(\mathbf{U})_x\,,$$

which is Hamiltonian:

$$\begin{aligned} -\mathbf{R}_x &= 0 \\ -\mathbf{P}_x &= \mathbf{R} - \nabla E(\mathbf{U}) \\ \phi_x &= \mathbf{U} \\ \mathbf{U}_x &= \mathbf{D}^{-1}\mathbf{M}\mathbf{P}\,. \end{aligned}$$

$$H(\phi, \mathbf{U}, \mathbf{R}, \mathbf{P}) = \tfrac{1}{2}\langle \mathbf{D}^{-1}\mathbf{M}\mathbf{P}, \mathbf{P}\rangle + \langle \mathbf{R}, \mathbf{U}\rangle - E(\mathbf{U})\,.$$

Conservation laws – degenerate RE

$$
\begin{aligned}
-\mathbf{R}_x &= 0 \\
-\mathbf{P}_x &= \mathbf{R} - \nabla E(\mathbf{U}) \\
\phi_x &= \mathbf{U} \\
\mathbf{U}_x &= \mathbf{D}^{-1}\mathbf{M}\mathbf{P}\,.
\end{aligned}
$$

$2n-$dimensional affine symmetry: invariant under $\epsilon \mapsto \phi + \epsilon$ for all $\epsilon \in \mathbb{R}^{2n}$. $\mathbf{R} \in \mathbb{R}^{2n}$ defines the momentum map. Look at RE

$$\phi(x) = \mathbf{c}x + \phi_0\,,$$

Then

$$\mathbf{U} = \mathbf{c}\,, \quad \mathbf{P} = \mathbf{0}\,,$$

and

$$\mathbf{R} = \nabla E(\mathbf{c})\,.$$

The RE are non-degenerate precisely when

$$\det[D^2 E(\mathbf{c})] \neq 0\,.$$

Conservation laws, dispersion and degenerate RE
– summary –

$$\mathbf{U}_t + \mathbf{F}(\mathbf{U})_x = \mathbf{D}\mathbf{U}_{xxx}\,, \quad \mathbf{U} \in \mathbb{R}^{2n}\,,$$

$$\mathbf{F}(\mathbf{U}) = \mathbf{M}\nabla E(\mathbf{U})\,, \quad \mathbf{M} \text{ invertible}\,, \quad \mathbf{M}^{-1}\mathbf{D} \text{ symmetric}\,.$$

- Introduce a potential $\mathbf{U} = \phi_x$, creating a symmetry.
- Homogeneous (constant) states $\mathbf{U}_0 \in \mathbb{R}^{2n}$ can be characterized as relative equilibria.
- These RE are degenerate precisely when the flux vector is degenerate

$$\det[D\mathbf{F}(\mathbf{c})] = 0 \quad (\text{equivalent to } \det[D^2E(\mathbf{c})] = 0)\,.$$

- A mechanism for generating solitary waves.

Degenerate conservation laws

Consider conservation laws with regularization

$$\mathbf{U}_t + \mathbf{F}(\mathbf{U})_x = \mathbf{D}\mathbf{U}_{xx} \quad \text{or} \quad \mathbf{U}_t + \mathbf{F}(\mathbf{U})_x = \mathbf{D}\mathbf{U}_{xxx}\,,$$

where $\mathbf{U} \in \mathbb{R}^n$, $\mathbf{F} : \mathbb{R}^n \to \mathbb{R}^n$ is a given smooth mapping (the flux vector), and $\mathbf{D}$ is an $n \times n$ matrix.

Let $\mathbf{U}_0 \in \mathbb{R}^n$ be any constant vector. The conservation law is *degenerate* at $\mathbf{U}_0$ if the Jacobian $D\mathbf{F}(\mathbf{U}_0)$ is singular

$$\det[D\mathbf{F}(\mathbf{U}_0)] = 0 \quad \text{criticality!}$$

Assume simple degeneracy, then there exists eigenvectors

$$D\mathbf{F}(\mathbf{U}_0)\xi = 0 \quad \text{and} \quad \eta^T D\mathbf{F}(\mathbf{U}_0) = 0\,.$$

Appropriate model near criticality?

KdV or Burger's model near criticality

Let $X = \varepsilon x$ and $T = \varepsilon^2 t$ and decompose

$$\mathbf{U}(x,t) = \mathbf{U}_0 + \varepsilon A(X,T,\varepsilon)\boldsymbol{\xi} + \varepsilon^2 V(X,T,\varepsilon)\,, \quad \boldsymbol{\eta}^T V = 0\,.$$

Formally,

$$\begin{aligned} & A_T + \kappa\, A A_X - \nu A_{XX} = \varepsilon\, \mathbf{R}_1 \\ & \tfrac{d}{dX}\left(\mathbf{P}D\mathbf{F}(\mathbf{U}_0)V + \tfrac{1}{2}\mathbf{P}D^2\mathbf{F}(\mathbf{U}_0)u^2 - \mathbf{P}\mathbf{D}(\mathbf{U}_0)u_X\right) = \varepsilon \mathbf{R}_2\,, \end{aligned}$$

where

$$\kappa = \frac{d^2}{ds^2}\bigg|_{s=0} \langle \boldsymbol{\eta}, F(\mathbf{U}_0 + s\boldsymbol{\xi})\rangle\,,$$

suggesting

$$V = \mathbf{L}_1 u^2 + \mathbf{L}_2 u_X + \cdots\,.$$

where $\mathbf{L}_1$ and $\mathbf{L}_2$ are constant matrices.

KdV or Burger's model near criticality

Remarks on the formal construction

- The reduction for V has a form similar to a center-manifold reduction.
- $D\mathbf{F}(\mathbf{U}_0)$ is not required to have real eigenvalues.
- No special requirements on $\mathbf{D}$ except that $\nu = \langle \boldsymbol{\eta}, \mathbf{D}\boldsymbol{\xi} \rangle \neq 0$.
- Dispersive regularization: same argument but with $T = \varepsilon^3 t$, and reduced equation is KdV
- The hypersurface defined by $\det[D\mathbf{F}(\mathbf{U}_0)] = 0$ is not in general connected.

Formally, the dynamics near the $\boldsymbol{\xi}$ direction in $\mathbb{R}^n$ is governed by

$$A_T + \kappa A A_X = \nu A_{XX} \quad \text{or} \quad A_T + \kappa A A_X = \nu A_{XXX} \,.$$

Validity of the reduced models

What can one say rigorously about these reduced models?

Suppose the conservation law is hyperbolic ($D\mathbf{F}(\mathbf{U}_0)$ diagonalizable with real eigenvalues), the regularization dissipative, and suppose $\mathbf{D}$ is symmetric and positive. Then there exists $T_0 > 0$ such that

$$\|\mathbf{U} - \varepsilon A^{\mathrm{Burgers}}\xi\|_{W_b^{1,2}(\mathbb{R})} \leq C\varepsilon^{3/2}\mathrm{e}^{KT_0}\,,$$

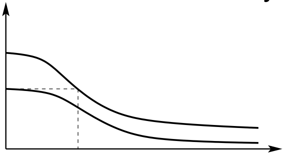
where C and K depend on the norm of the initial data (initial data in $W_b^{2,2}(\mathbb{R})$), but are independent of ε. Here, $W_b^{1,2}(\mathbb{R})$ is the Sobolev space based on the uniformly local space $L_b^p(\mathbb{R})$ with norm

$$\|u\|_{L_b^p(\mathbb{R})} := \sup_{s\in\mathbb{R}} \|u\|_{L^p([s,s+1])}\,.$$

(TJB & Zelik, in preparation).

Comments on reductions to KdV or Burgers

- Results in the literature on the steady problem (validity of steady Burgers, then Evans function analysis of the reduced stability problem): see FREISTÜHLER & SZMOLYAN, ARMA (2002), PLAZA & ZUMBRUN, DCDS (2004).
- Does there is exist an invariant manifold decomposition in $\mathbb{R}^n$ near $\mathrm{span}\{\xi\}$?
- In the dispersive case, validity is trickier due to resonances. Example: longwave-shortwave resonance in two-layer model (disconnected criticality surface)



Multiple zero eigenvalues of $D\mathbf{F}(\mathbf{c})$

In the dissipative case, multiple (semisimple) zero leads to coupled Burgers

$$
\begin{aligned}
\frac{\partial u}{\partial t} + \Gamma^1_{11}\, u\frac{\partial u}{\partial x} + \Gamma^1_{12}\,\frac{\partial (uv)}{\partial x} + \Gamma^1_{22} v\frac{\partial v}{\partial x} &= \nu_{11}\,\frac{\partial^2 u}{\partial x^2} + \nu_{12}\,\frac{\partial^2 v}{\partial x^2} \\
\frac{\partial v}{\partial t} + \Gamma^2_{11}\, u\frac{\partial u}{\partial x} + \Gamma^2_{12}\,\frac{\partial (uv)}{\partial x} + \Gamma^2_{22} v\frac{\partial v}{\partial x} &= \nu_{21}\,\frac{\partial^2 u}{\partial x^2} + \nu_{22}\,\frac{\partial^2 v}{\partial x^2}\,,
\end{aligned}
$$

$$
\Gamma^k_{ij} := \langle \boldsymbol{\eta}_k, D^2\mathbf{F}(\mathbf{U}_0)(\xi_i,\xi_j)\rangle\,.
$$

Under appropriate hypotheses, coupled Burger's appears to be valid (TJB & Zelik, work in progress).
For the dispersive case, one finds coupled KdV equations – validity open.

Summary

– Generalization of criticality in fluid mechanics –

- Hamiltonian formulation
- Any flow that can be characterized as a RE has a concept of criticality: degeneracy of the RE
- Criticality generates solitary waves
- Properties of the bifurcating solitary wave (homoclinic orbit) encoded in the geometry of the momentum map evaluated on a family of RE
- Used to find new families of solitary waves in shallow water hydrodynamics

- New observations in dynamical systems.
- Connections with hyperbolic and mixed conservation laws.

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Homoclinic bifurcation from invariant tori

For illustration, consider a $\mathbb{T}^2-$equivariant Hamiltonian system on $\mathbb{R}^6$,

$$\mathbb{J}u_t = \nabla H(u)\,, \quad u \in \mathbb{R}^6\,,$$

with momentum map **J**.

RE associated with this group are tori,

$$u(t) = \Phi_{g(t)}(\varphi)\,.$$

Take coordinates ω_1 and ω_2 for the Lie algebra. The family of RE is non-degenerate when

$$\det \begin{bmatrix} \frac{\partial P_1}{\partial \omega_1} & \frac{\partial P_1}{\partial \omega_2} \\ \frac{\partial P_2}{\partial \omega_1} & \frac{\partial P_2}{\partial \omega_2} \end{bmatrix} \neq 0 \quad \text{equivalently} \quad \det \begin{bmatrix} \frac{\partial \omega_1}{\partial I_1} & \frac{\partial \omega_1}{\partial I_2} \\ \frac{\partial \omega_2}{\partial I_1} & \frac{\partial \omega_2}{\partial I_2} \end{bmatrix} \neq 0\,,$$

where (P_1, P_2) are the momenta evaluated on an RE, and (I_1, I_2) can be interpreted as values of level sets.

Degenerate invariant tori and homoclinic bifurcation

Near degeneracy, there exists new coordinates $(\phi_1, \phi_2, u, I_1, I_2, v)$ satisfying

$$\begin{aligned} -\frac{dv}{dt} &= I_1 - \tfrac{1}{2}\kappa u^2 + \cdots \\ \frac{du}{dt} &= s_1 v + \cdots \\ -\frac{dI_j}{dt} &= 0 \quad j = 1, 2 \\ \frac{d\phi_1}{dt} &= u + \cdots \\ \frac{d\phi_2}{dt} &= s_2 I_2 + \cdots \end{aligned}$$

with

$$\kappa = a_0^3 \langle \mathrm{d}f(\boldsymbol{\omega}), \mathbf{n} \rangle, \quad f(\boldsymbol{\omega}) := \det \begin{bmatrix} \frac{\partial P_1}{\partial \omega_1} & \frac{\partial P_1}{\partial \omega_2} \\ \frac{\partial P_2}{\partial \omega_1} & \frac{\partial P_2}{\partial \omega_2} \end{bmatrix}.$$

– There is a geometric phase shift on the invariant torus.
– A new mechanism for saddle-center bifurcation of tori?
– Even the case $n = 1$ is new!
(TJB & DONALDSON (2005) *Phys. Rev. Lett.*)

HANSSMANN (1998) takes a saddle-center bifurcation in the plane, and adds an integrable $n-$torus.

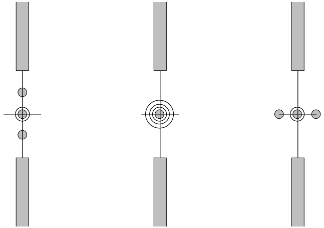
$$
\begin{aligned}
-\frac{dv}{dt} &= \lambda + b(\omega)u^2 \\
\frac{du}{dt} &= a(\omega)v \\
-\frac{dI_j}{dt} &= 0 \\
\frac{d\theta_j}{dt} &= \omega_j\,, \quad j = 1, \ldots, n\,.
\end{aligned}
$$

Then perturbation terms are added which break the symmetry (integrability) and persistence of the bifurcation on Cantor subsets of parameter space is proved.

See also BROER, HANSSMANN & YOU (2005).

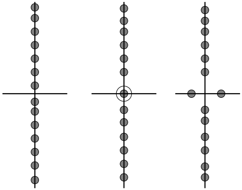
Infinite dimensions – with continuous spectrum

Suppose there is a continuous spectrum on the imaginary axis and a saddle-center bifurcation



– Normal form theory goes through to leading order, but the continuous spectrum will be an obstacle to persistence of the homoclinic orbit – open problem.
– This example arises in nonlinear Schrödinger equation with non-Kerr nonlinearity where the RE is a solitary wave.
– Need a nonlinear version of the linear theory of GRILLAKIS, SHATAH & STRAUSS (1987).

Saddle-center with infinite dimensional center



Formal normal form theory goes through to leading order for the saddle-center coupled to an infinite number of pure imaginary eigenvalues. But the center modes will be an obstacle to persistence.

This example arises in the time-dependent water-wave problem. There is a sequence (possibly infinite) of saddle-center bifurcations, and the attendant homoclinic bifurcations – have been found to be associated with a form of wave breaking – *micro-breakers*.

– TJB, J. Fluid Mech. (2004)

Whitham modulation theory

The Whitham modulation equations can be written in the form

$$\begin{aligned} -A_t - B_x &= 0 \\ \theta_t &= \omega \\ \theta_x &= \kappa \end{aligned}$$

where A is the *wave action* and B is the *wave action flux*. Substituting the second and third equations into the first results in the PDE

$$A_\omega \theta_{tt} + (A_\kappa + B_\omega)\theta_{xt} + B_\kappa \theta_{xx} = 0\,,$$

which is hyperbolic if

$$\det \begin{bmatrix} A_\omega & A_\kappa \\ B_\omega & B_\kappa \end{bmatrix} < 0\,,$$

and elliptic when the sign is positive.
What is the appropriate modulation equation when the determinant vanishes or is near vanishing? Spatio-temporal homoclinic bifurcation?

Spatio-temporal RE and homoclinic bifurcation

Replace symplectic relative equilibria with multi-symplectic relative equilibria. Consider a multi-symplectic PDE in canonical form

$$\mathbf{J}\mathbf{u}_t + \mathbf{K}\mathbf{u}_x = \nabla S(\mathbf{u})\,,$$

that is equivariant with respect to a Lie group G.

$$(P(\mathbf{u}), Q(\mathbf{u})) = (\langle \mathbf{J}\xi_M, \mathbf{u}\rangle, \langle \mathbf{K}\xi_M, \mathbf{u}\rangle)\,,$$

is the multi-momentum map, with ξ_M the generator of the group. An RE is of the form

$$\mathbf{u}(t) = \Phi_{g(t,x)}(\mathbf{u}_0)\,, \quad (g_t, g_x) = (\omega, \kappa)\,,$$

and $\mathbf{u}_0$ satisfies $\nabla H(\mathbf{u}_0) = \omega\,\nabla P(\mathbf{u}_0) + \kappa\,\nabla Q(\mathbf{u}_0)$. This variational principle is non-degenerate when

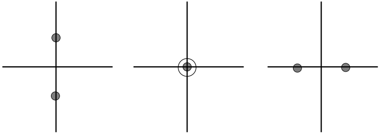
$$\det \begin{bmatrix} \frac{\partial P}{\partial \omega} & \frac{\partial P}{\partial \kappa} \\ \frac{\partial Q}{\partial \omega} & \frac{\partial Q}{\partial \kappa} \end{bmatrix} \neq 0\,.$$

Application: relation between degenerate MS-RE and the Benjamin-Feir instability, (TJB & DONALDSON, J. Fluid Mech. 2006) – may also lead to a new spatio-temporal homoclinic bifurcation.

Planar homoclinic bifurcation – Hamiltonian systems

$$-\dot{p} = \frac{\partial H}{\partial q}\,, \quad \dot{q} = \frac{\partial H}{\partial p}\,, \quad \text{or} \quad \mathbf{J}\mathbf{u}_t = \nabla H(\mathbf{u})\,, \quad \mathbf{u} = \begin{pmatrix} q \\ p \end{pmatrix}\,.$$

- Suppose there is an equilibrium point $\mathbf{u}_0 = (q_0, p_0)$.
- Let $\mathbf{L} = D^2 H(\mathbf{u}_0)$; spectrum of $\mathbf{J}^{-1}\mathbf{L}$ is of the form



- Introduce a parameter μ so $H(\mathbf{u}, \mu)$ and a double zero eigenvalue occurs in the linearization when $\mu = \mu_0$
- Generically the nonlinear problem has a homoclinic bifurcation for μ near μ_0.

Planar homoclinic bifurcation – nonlinear theory

At the double zero eigenvalue, the eigenvectors satisfy

$$\mathbf{L}\xi_1 = 0 \quad \text{and} \quad \mathbf{L}\xi_2 = \mathbf{J}\xi_1$$

Normalise the eigenvectors and introduce a transformation,

$$\mathbf{u}(x) = \widetilde{q}(t)\xi_1 + s\,\widetilde{p}(t)\xi_2 + h.o.t.$$

Then locally using normal form theory, $\widetilde{q}(t)$ and $\widetilde{p}(t)$ satisfy

$$\begin{array}{rcl} -\widetilde{p}_t & = & \tilde{\mu} - \frac{1}{2}\kappa\widetilde{q}^2 + \cdots \\ \widetilde{q}_t & = & s\,\widetilde{p} + \cdots \end{array}$$

where $\tilde{\mu} = C\,(\mu - \mu_0)$, $s = \pm 1$ (symplectic sign) and

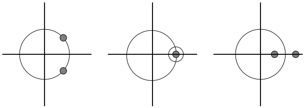
$$\kappa = \langle \xi_1, D^3 H(\mathbf{u}_0)(\xi_1, \xi_1)\rangle$$

Homoclinic bifurcation from periodic orbits
— with a transition of Floquet multipliers at $+1$

Consider a standard autonomous Hamiltonian system on $\mathbb{R}^4$ with a branch of periodic solutions.

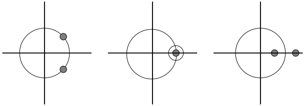
$$\mathbf{J}\mathbf{u}_t = \nabla H(\mathbf{u}) \,, \quad \mathbf{u} \in \mathbb{R}^4 \,.$$

Linearise about the periodic orbit and suppose that as a parameter is varied, a pair of Floquet exponents passes through $+1$.



What happens in the nonlinear problem?

Homoclinic bifurcation from periodic orbits



Standard theory: use a Floquet transformation to transform the system to a constant coefficient system, apply normal form theory to the constant coefficient problem ...

$$U(t,\theta) = \widetilde{q}(t)\xi_1(\theta) + s\,\widetilde{p}(t)\xi_2(\theta) + h.o.t.$$

with nonlinear normal form to leading order

$$\begin{array}{rcl} -\widetilde{p}_t & = & \mu - \frac{1}{2}\kappa\widetilde{q}^2 + \cdots \\ \widetilde{q}_t & = & s\,\widetilde{p} + \cdots \end{array}$$

– reduced system is the same as the saddle-center bifurcation of an equilibrium;

e.g. Arnold, Kozlov & Neishtadt (1993), Chapter 7.

What about the phase shift?

When the system is autonomous, the Floquet multiplier at $+1$ has algebraic multiplicity four and geometric multiplicity one.

The Jordan chain has length four, not two.

Let

$$\mathbf{L} = D^2 H(\hat{\mathbf{u}}(\theta)) - \omega \mathbf{J} \frac{d}{d\theta}\,,$$

then

$$\begin{aligned}
\mathbf{L}\xi_1 &= 0\,,\\
\mathbf{L}\xi_2 &= \mathbf{J}\xi_1\,,\\
\mathbf{L}\xi_3 &= \mathbf{J}\xi_2\,,\\
\mathbf{L}\xi_4 &= \mathbf{J}\xi_3\,.
\end{aligned}$$

Normal form with phase shift

Let

$$U(t,\theta) = \phi(t)\xi_1(\theta) + u(t)\xi_2(\theta) - s\,I(t)\xi_4(\theta) + s\,v(t)\xi_3(\theta) + \cdots ,$$

where $s = \pm 1$ is a symplectic sign. Then using nonlinear normal form theory[1] one can show to leading order,

$$\begin{aligned} -I_t &= 0 \\ -v_t &= I - \tfrac{1}{2}\kappa u^2 + \cdots \\ \phi_t &= u + \cdots \\ u_t &= s\,v + \cdots \end{aligned}$$

where

$$\kappa = -\langle \xi_2, D^3H(\xi_2,\xi_2)\rangle - 3\langle \xi_1, D^3H(\xi_1,\xi_4)\rangle + 3\langle \xi_1, D^3H(\xi_2,\xi_3)\rangle .$$

and $H = \frac{1}{2}sv^2 + Iu - \frac{1}{6}\kappa u^3 + \cdots$.

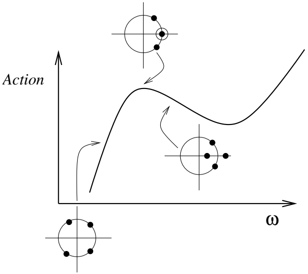
[1] Cushman-Sanders, Iooss

Action and periodic orbits

The action of a periodic orbit is defined by

$$A(\mathbf{q}, \mathbf{p}) = \oint \mathbf{p} \cdot \mathbf{q}_\theta \mathrm{d}\theta = \frac{1}{2\pi} \int_0^{2\pi} \tfrac{1}{2} \langle \mathbf{J} U_\theta, U \rangle \mathrm{d}\theta$$

When Floquet multipliers pass through $+1$ there is a stationary point of the action.



See POINCARÉ (1892), DEPRIT & HENRARD (1968) and SEPULCHRE & MACKAY (1997).

Curvature and geometric phase

$$
\begin{aligned}
-I_x &= 0 \\
-v_x &= I - \tfrac{1}{2}\kappa u^2 + \cdots \\
\phi_x &= u + \cdots \\
u_x &= s\, v + \cdots
\end{aligned}
$$

The coefficient κ can be expressed in terms of the curvature of the action-frequency curve

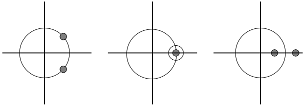
$$\kappa = a_0^3 \frac{d^2 A}{d\omega^2}\,, \quad a_0 = |\langle\!\langle \mathbf{J}\widehat{\xi}_4, \widehat{\xi}_1 \rangle\!\rangle|^{-1/2}\,.$$

(cf. TJB & DONALDSON, Phys. Rev. Lett. 2005). The geometric phase is then determined from the third equation

$$\Delta\phi = \int_{-\infty}^{+\infty} (u(x) - u_0)\,\mathrm{d}x\,, \quad u_0 = \pm\sqrt{\frac{2I}{\kappa}}\,.$$

Summary: nonlinearity near saddle-center bifurcation

$A'(\omega) = 0$: bifurcation of Floquet multipliers



Reduced normal form (after scaling u, v, I, t),

$$\begin{aligned} -v_t &= I - \tfrac{1}{2}A''(\omega)u^2 + \cdots \\ u_t &= s\,v + \cdots\,, \quad s = \pm 1\,, \end{aligned}$$

– Nonlinear term in normal form determined by curvature of frequency map
– I is a measure of the distance from bifurcation point in action space
– flow along the group has a geometric phase

$$I_t = 0 \quad \text{and} \quad \phi_t = u + \cdots\,.$$