# Variational and dissipative aspects of nonholonomic systems

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Work with Marsden, Zenkov, Rojo, Fernandez, Mestdag...

- **Double bracket dissipation**
- **Dissipation in nonholonomic systems**
- **Nonholonomic systems and fields**
- **Inverse Problems and nonholonomic systems**

**•Background**

**• Basic observation about Hamiltonian systems: satisfy Liouville's theorem, preserving volume in phase space, thus cannot exhibit asymptotic stability.**

**Reflection of this: spectrum of linearization about a fixed point symmetric about imaginary axis.**

**Class of energy preserving systems which can exhibit asymptotic stability: nonholonomic systems – systems with nonintegrable constraints. In the absence of external dissipative forces, are always energy preserving.**

**Do not necessarily preserve volume in the phase space – – see for example Zenkov, Bloch and Marsden [1998], Zenkov and Bloch [2002], Kozlov, Jovanovich,**

- **Infinite Dimensions – oscillators interacting with fields. Hagerty, Bloch and Weinstein. Bloch, Hagerty, Rojo and Weinstein. Radiation Damping. Sofer and Weinstein. Original model of Lamb. Overall system Hamiltonian but can induce dissipation locally in oscillator.**

**•Double Brackets and Dissipation Double bracket flows: dissipative mechanism in otherwise energy conserving mechanical systems, Bloch, Krishnaprasad, Marsden and Ratiu [1996].**

**• Simple example: rigid body equations:**

$$I\dot{\Omega} = (I\Omega) \times \Omega,$$

**or, in terms of the body angular momentum** $M = I\Omega$**,**

$$\dot{M} = M \times \Omega.$$

**Energy equals the Lagrangian:** $E(\Omega) = L(\Omega)$ **and energy is conserved.**

**Add a term cubic in the angular velocity:**

$$\dot{M} = M \times \Omega + \alpha M \times (M \times \Omega),$$

**where** $\alpha$ **is a positive constant.**

**• Related example is the Landau-Lifschitz equations for the magnetization vector $M$ in a given magnetic field $B$:**

$$\dot{M} = \gamma M \times B + \frac{\lambda}{\|M\|^2}(M \times (M \times B)),$$

**where $\gamma$ is the magneto-mechanical ratio (so that $\gamma\|B\|$ is the Larmour frequency) and $\lambda$ is the damping coefficient due to domain walls.**

**• The equations are Hamiltonian with the rigid body Poisson bracket:**

$$\{F, K\}_{\mathrm{rb}}(M) = -M \cdot [\nabla F(M) \times \nabla K(M)]$$

**with Hamiltonians given respectively by $H(M) = (M \cdot \Omega)/2$ and $H(M) = \gamma M \cdot B$.**

**Dissipation in these systems is not induced by *any* Rayleigh dissipation function in the *literal* sense**

**However, it is induced by a dissipation function in the following restricted sense: It is a gradient when restricted to each momentum sphere,**

**Have:**

$$\frac{d}{dt}\|M\|^2 = 0$$

$$\frac{d}{dt}E = -\alpha\|M \times \Omega\|^2,$$

**for the rigid body,**

**• Interesting feature of these dissipation terms is that they can be derived from a symmetric bracket. in much the same way that the Hamiltonian equations can be derived from a skew symmetric Poisson bracket. For the case of the rigid body, this bracket is**

$$\{\{F, K\}\} = \alpha(M \times \nabla F) \cdot (M \times \nabla K).$$

**(For more on symmetric brackets see Crouch [1981] and Lewis and Murray [1999].)**

## •The Chaplygin Sleigh

**Here we describe the Chaplygin sleigh, perhaps the simplest mechanical system which illustrates the possible dissipative nature of energy preserving nonholonomic systems.**

**Nonholonomic: subject to nonintegrable constraints – satisifes Lagrange D'Alembert equations.**

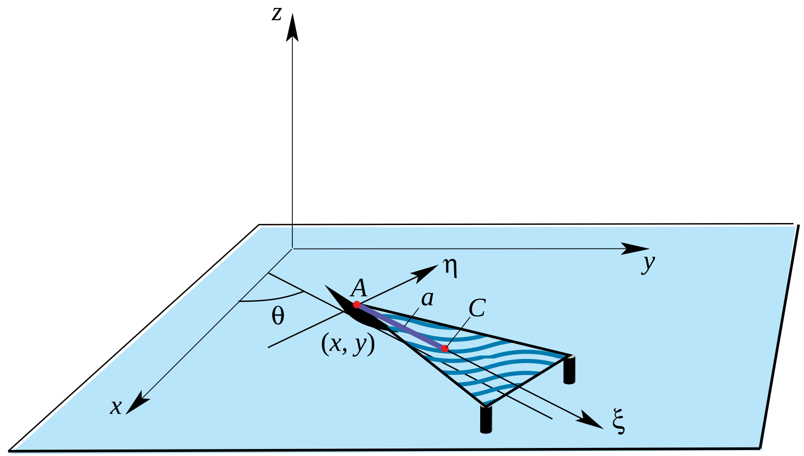


Figure 0.1: The Chaplygin sleigh is a rigid body moving on two sliding posts and one knife edge.

**Equations:**

$$\begin{aligned}\dot{v} &= a\omega^2\\ \dot{\omega} &= -\frac{ma}{I+ma^2}v\omega\end{aligned}$$

**Equations have a family of relative equilibria given by** $(v,\omega)|v = \text{const}, \quad \omega = 0$**.**

**Linearizing about any of these equilibria one finds one zero eigenvalue and one negative eigenvalue.**

**In fact the solution curves are ellipses in** $v-\omega$ **plane with the positive** $v$**-axis attracting all solutions.**


Figure 0.2: Chaplygin Sleigh/2d Toda phase portrait.

**•Euler-Poincaré-Suslov Equations**

**Important special case of the reduced nonholonomic equations.**

**•Example: Euler-Poincaré-Suslov Problem on $SO(3)$ In this case the problem can be formulated as the standard Euler equations**

$$I\dot{\omega} = I\omega \times \omega$$

**where $\omega = (\omega_1, \omega_2, \omega_3)$ are the system angular velocities in a frame where the inertia matrix is of the form $I = \mathrm{diag}(I_1, I_2, I_3)$ and the system is subject to the constraint**

$$a \cdot \omega = 0$$

**where $a = (a_1, a_2, a_3)$.**

**The nonholonomic equations of motion are then given by**

$$I\dot{\omega} = I\omega \times \omega + \lambda a$$

**subject to the constraint. Solve for $\lambda$:**

$$\lambda = -\frac{I^{-1}a \cdot (I\omega \times \omega)}{I^{-1}a \cdot a}\,.$$

**If $a$ is an eigenvector of the moment of inertia tensor flow is measure preserving.**

**Invariant Measures of the Euler-Poincaré-Suslov Equations** An important special case of the reduced nonholonomic equations is the case when there is no shape space. In this case the system is characterized by the Lagrangian $L = \frac{1}{2}\mathbb{I}_{AB}\Omega^A\Omega^B$ and the left-invariant constraint

$$\langle a, \Omega \rangle = a_A \Omega^A = 0. \tag{0.1}$$

Here $a = a_A e^A \in \mathfrak{g}^*$ and $\Omega = \Omega^A e_A$, where $e_A$, $A = 1, \ldots, k$, is a basis for $\mathfrak{g}$ and $e^A$ is its dual basis. Multiple constraints may be imposed as well. The two classical examples of such systems are the ***Chaplygin Sleigh*** and the ***Suslov problem.*** These problems were introduced by Chaplygin in 1895 and Suslov in 1902, respectively.

**We can consider the problem of when such systems exhibit asymptotic behavior. Following Kozlov [1988] it is convenient to consider the unconstrained case first. In the absence of constraints the dynamics is governed by the basic Euler-Poincaré equations**

$$\dot{p}_B = C^C_{AB}\mathbb{I}^{AD}p_C p_D = C^C_{AB}p_C\Omega^A \tag{0.2}$$

**where** $p_B = \mathbb{I}_{AB}\Omega^B$ **are the components of the momentum** $p \in \mathfrak{g}^*$**. One considers the question of whether the (unconstrained) equations** (0.2) **have an absolutely continuous integral invariant** $f d^k\Omega$ **with summable density** $\mathcal{M}$**. If** $\mathcal{M}$ **is a positive function of class** $C^1$ **one calls the integral invariant an invariant measure. Kozlov [1988] shows**

**Theorem 0.1** ***The Euler-Poincaré equations have an invariant measure if and only if the group*** $G$ ***is unimodular.***

**A group is said to be unimodular if it has a bilaterally invariant measure. A criterion for unimodularity is $C^C_{AC} = 0$ (using the Einstein summation convention). Now we know (Liouville's theorem) that the flow of a vector differential equation $\dot{x} = f(x)$ is phase volume preserving if and only if $\mathrm{Div}\, f = 0$. In this case the divergence of the right hand side of equation $(0.2)$ is $C^C_{AC}\mathbb{I}^{AD}p_D = 0$. The statement of the theorem now follows from the following theorem of Kozlov [1998]:** ***A flow due to a homogeneous vector field in $\mathbb{R}^n$ is measure-preserving if and only if this flow preserves the standard volume in $\mathbb{R}^n$.***

**Now, turning to the case where we have the constraint** $(0.1)$ **we obtain the *Euler-Poincaré-Suslov equations***

$$\dot{p}_B = C_{AB}^C \mathbb{I}^{AD} p_C p_D + \lambda a_B = C_{AB}^C p_C \Omega^A + \lambda a_B \tag{0.3}$$

**together with the constraint** $(0.1)$**. Here** $\lambda$ **is the Lagrange multiplier. This defines a system on the subspace of the dual Lie algebra defined by the constraint. Since the constraint is assumed to be nonholonomic, this subspace is not a subalgebra. One can then formulate a condition for the existence of an invariant measure of the Euler-Poincaré-Suslov equations.**

**Theorem 0.2** ***Equations*** $(0.3)$ ***have an invariant measure if and only if***

$$K \operatorname{ad}^*_{\mathbb{I}^{-1}a} a + T = \mu a, \qquad \mu \in \mathbb{R}, \tag{0.4}$$

***where*** $K = 1/\langle a, \mathbb{I}^{-1} a\rangle$ ***and*** $T \in \mathfrak{g}^*$ ***is defined by*** $\langle T, \xi\rangle = \operatorname{Trace}(\operatorname{ad}\xi)$***.***

**This theorem was proved by Kozlov [1988] for compact algebras and for arbitrary algebras by Jovanović [1998].**

**In coordinates, condition** $(0.4)$ **becomes**

$$KC^C_{AB}\mathbb{I}^{AD}a_Ca_D + C^C_{BC} = \mu a_B.$$

**For a compact algebra** $(0.4)$ **becomes**

$$[\mathbb{I}^{-1}a, a] = \mu a, \qquad \mu \in \mathbb{R}, \tag{0.5}$$

**where we identified** $\mathfrak{g}^*$ **with** $\mathfrak{g}$**.**

**The proof of theorem 0.2 reduces to the computation of the divergence of the vector field in** $(0.3)$**.**

**In the compact case only constraint vectors $a$ which commute with $\mathbb{I}^{-1}a$ allow the measure to be preserved. This means that $a$ and $\mathbb{I}^{-1}a$ must lie in the same maximal commuting subalgebra. In particular, if $a$ is an eigenstate of the inertia tensor, the reduced phase volume is preserved. When the maximal commuting subalgebra is one-dimensional this is a necessary condition. This is the case for groups such as $SO(3)$.**

**Bloch and Zenkov extend this to the case of internal variables.**

**•Radiation Damping**

**See Hagerty, Bloch and Weinstein [1999], [2002].**

**Important early work: Lamb [1900]. Related recent work may be found in Soffer and Weinstein [1998a,b] [1999] and Kirr and Weinstein [2001].**

**• Original Lamb model an oscillator is physically coupled to a string. The vibrations of the oscillator transmit waves into the string and are carried off to infinity. Hence the oscillator loses energy and is effectively damped by the string.**

**• Lamb model**

$w(x,t)$ **displacement of the string. with mass density** $\rho$**, tension** $T$**. Assuming a singular mass density at** $x = 0$**, we couple dynamics of an oscillator,** $q$**, of mass** $M$**:**

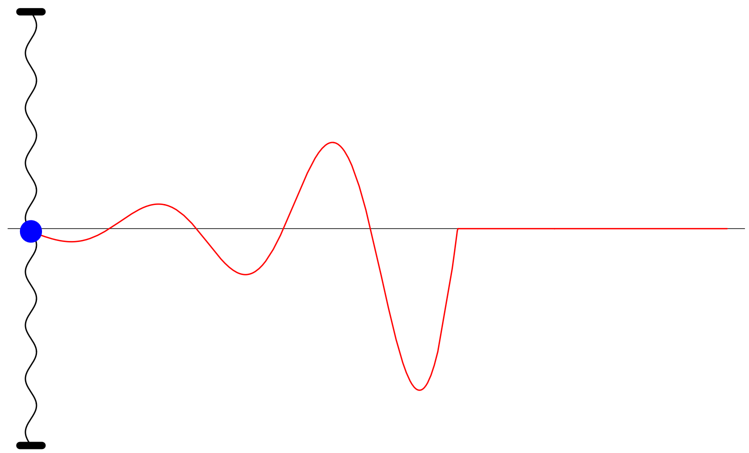

Figure 0.3: **Lamb model of an oscillator coupled to a string.**

$$\frac{\partial^2 w}{\partial t^2} = c^2 \frac{\partial^2 w}{\partial x^2}$$
$$M\ddot{q} + Vq = T[w_x]_{x=0}$$
$$q(t) = w(0,t).$$

$[w_x]_{x=0} = w_x(0+,t) - w_x(0-,t)$ **is the jump discontinuity of the slope of the string. Note that this is a Hamiltonian system.**

**Can solve for $w$ and reduce:**

• **Obtain a reduced form of the dynamics describing the explicit motion of the oscillator subsystem,**

$$M\ddot{q} + \frac{2T}{c}\dot{q} + Vq = 0.$$

**The coupling term arises explicitly as a Rayleigh dissipation term** $\frac{2T}{c}\dot{q}$ **in the dynamics of the oscillator.**

**Gyroscopic systems:**
**See Bloch, Krishnaprasad, Marsden and Ratiu [1994].**
**Linear systems of the form**

$$M\ddot{q} + S\dot{q} + \Lambda q = 0$$

**where** $q \in \mathbb{R}^n$**,** $M$ **is a positive definite symmetric** $n \times n$ **matrix,** $S$ **is skew, and** $\Lambda$ **is symmetric and indefinite.**

**This system Hamiltonian with** $p = M\dot{q}$**, energy function**

$$H(q,p) = \frac{1}{2} p M^{-1} p + \frac{1}{2} q \Lambda q$$

**and the bracket**

$$\{F, K\} = \frac{\partial F}{\partial q^i}\frac{\partial K}{\partial p_i} - \frac{\partial K}{\partial q^i}\frac{\partial F}{\partial p_i} - S_{ij}\frac{\partial F}{\partial p_i}\frac{\partial K}{\partial p_j}.$$

**Systems of this form arise from simple mechanical systems via reduction; normal form of the linearized equations when one has an** *abelian* **group.**

**Theorem 0.3 Dissipation induced instabilities—abelian case** ***Under the above conditions, if we modify the equation to***

$$M\ddot{q} + (S + \epsilon R)\dot{q} + \Lambda q = 0$$

***for small*** $\epsilon > 0$***, where*** $R$ ***is symmetric and positive definite, then the perturbed linearized equations***

$$\dot{z} = L_\epsilon z,$$

***where*** $z = (q, p)$ ***are spectrally unstable,*** i.e., ***at least one pair of eigenvalues of*** $L_\epsilon$ ***is in the right half plane.***

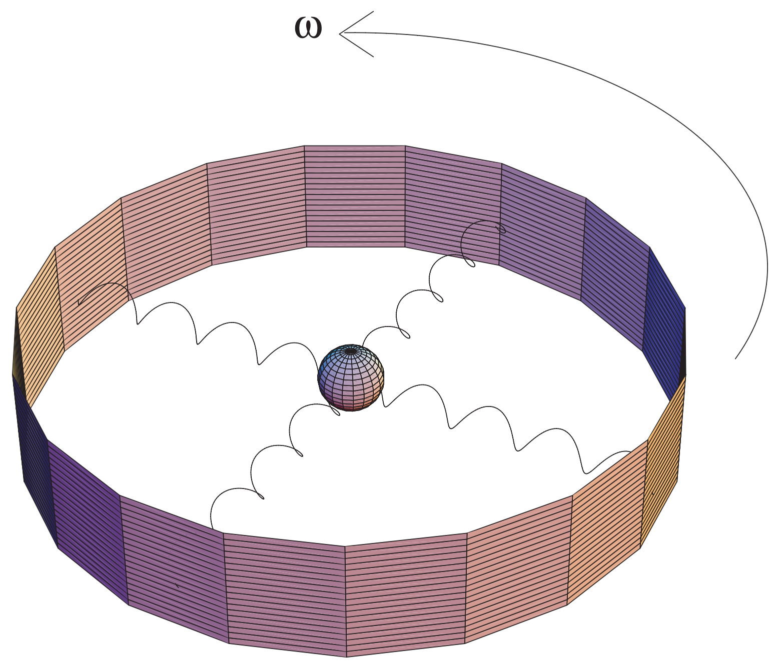


Figure 0.4: **Rotating plate with springs.**

- **Gyroscopic systens connected to wave fields.**

**In Hagerty, Bloch and Weinstein [2002] we describe a gyroscopic version of the Lamb model coupled to a standard non-dispersive wave equation and to a dispersive wave equation. Show that instabilities will arise in certain mechanical systems.**

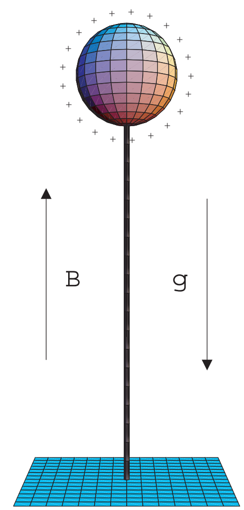


Figure 0.5: **Inverted spherical pendulum.**

**In the dispersionless case, the system is of the form**

$$\begin{aligned} \frac{\partial^2 \mathbf{w}}{\partial t^2}(z,t) &= c^2 \frac{\partial^2 \mathbf{w}}{\partial z^2}(z,t), \\ M\ddot{\mathbf{q}}(t) + S\dot{\mathbf{q}}(t) + V\mathbf{q}(t) &= T\left[\frac{\partial \mathbf{w}}{\partial z}\right]_{z=0} \\ \mathbf{w}(0,t) &= \mathbf{q}(t), \end{aligned}$$

$\mathbf{w} = \begin{bmatrix} w_1(z,t) & \cdots & w_n(z,t) \end{bmatrix}^T$ **is the displacement of the string in the first** $n$ **dimensions and** $[\frac{\partial \mathbf{w}}{\partial z}]_{z=0}$ **is the jump discontinuity in the slope of the string.**

- **Can reduce dynamics to essentially:**

$$M\ddot{\mathbf{q}}(t) = -\, S\dot{\mathbf{q}}(t) - V\mathbf{q}(t) - \frac{2T}{c}\dot{\mathbf{q}}(t),$$

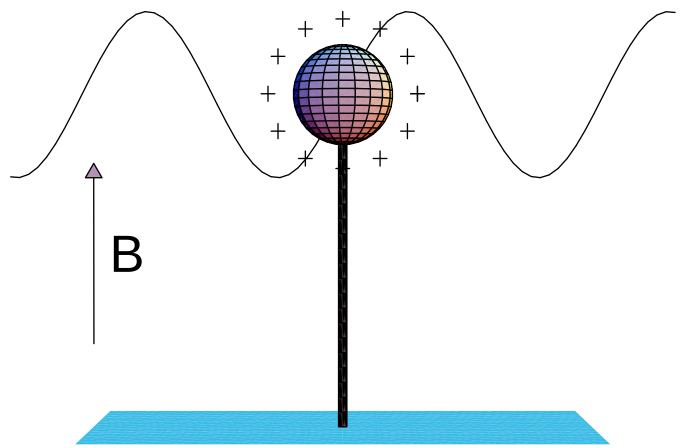


Figure 0.6: **Gyroscopic Lamb coupling to a spherical pendulum.**

**• Nonholonomic Systems as Limits**

**It has been known (see even Cartheodory –1933) that the Lagrange–d'Alembert equations can be obtained by starting with an unconstrained system subject to appropriately chosen dissipative forces, and then letting these forces go to infinity in an appropriate manner.**

**Kozlov showed that the variational nonholonomic equations too can be obtained as the result of another limiting process: He added a parameter-dependent "inertial term" to the Lagrangian of the constrained system, and then showed that the unconstrained equations approach the variational equations as the parameter approaches infinity.**

Nonholonomic constraints can be regarded in some sense as due to "infinite" friction. Several authors have asked if this can be quantified. Interestingly this goes back to the work at least of Caratheodory who asked if the limiting case of such friction could explain the motion of Chaplygin's sleigh. Caratheodory claimed this could not be done but Fufaev (1964) showed that this was indeed possible. The general case was considered by Kozlov (1983) and Karapetyan (1983).

The key idea it so take a nonlinear Rayleigh dissipation function of the form

$$F = -\frac{1}{2}k\sum_{j=1}^{m}\left(\sum_{i=1}^{n} a_i^{(j)}(\mathbf{q})\dot{q}_i\right)^2 \tag{0.6}$$

where $k >$ is a positive constant. Taking the limit as $k$ goes to zero and using Tikhonov's theorem yields the nonholonomic dynamics.

**However, the system in this setting is still not Hamiltonian. The goal here is to keep the system in the class of Hamiltonian systems by emulating the dissipation by coupling to an external field**

- **The Chaplygin Sleigh**

**This system consists of a rigid body moving on two sliding posts and one knife edge, and is perhaps the simplest n.h.s. containing the quasi-dissipative feature mentioned above. This mechanical system has three coordinates, two for the center of mass** $(x_C, y_C)$ **and one "internal" angular variable** $\theta$ **for the rotation with respect to the knife edge located at** $(x, y) = (x_C + a\cos\theta, y_C + a\sin\theta)$**. The system can rotate freely around** $(x, y)$ **but is only allowed to translate in the direction** $(\cos\theta, \sin\theta)$**: if we choose our coordinates as** $\mathbf{q} = (x, y, \theta)$ **there is a single constraint given by**

$$\dot{x}\sin\theta - \dot{y}\cos\theta = 0, \tag{0.7}$$

**or,** $\mathbf{a}^{(1)} = (\sin\theta, -\cos\theta, 0)$**.**

**The equations of motion can be obtained without resorting to the Lagrangian formalism, using simple balance of forces and can be expressed in the form:**

$$\begin{aligned} \dot{v} &= a\omega^2, \\ \dot{\omega} &= -\frac{ma}{I + ma^2} v\omega, \end{aligned} \tag{0.8}$$

**with** $v = \dot{x}\cos\theta + \dot{y}\sin\theta$ **the translational velocity,** $\omega = \dot{\theta}$**,** $m$ **the mass and** $I$ **the moment of inertia with respect to the center of mass.** $a$ **is the distance between the center of mass and the contact point of the knife edge. The solutions of the above equations are ellipses in the** $(v, \omega)$ **plane with equilibria given by** $\{v = \text{const}, \omega = 0\}$ **and which are asymptotically stable.**

**The above equations can be also obtained using the virtual force method starting with the unconstrained Lagrangian**

$$L_0 = \frac{m}{2}\left[\left(\dot{x} - a\dot{\theta}\sin\theta\right)^2 + \left(\dot{y} + a\dot{\theta}\cos\theta\right)^2\right] + \frac{I}{2}\dot{\theta}^2, \tag{0.9}$$

**and using a Lagrange multiplier in the equations of motion:**

$$\begin{aligned} m\frac{d}{dt}\left(\dot{x} - a\dot{\theta}\sin\theta\right) &= -\lambda\sin\theta, \\ m\frac{d}{dt}\left(\dot{y} + a\dot{\theta}\cos\theta\right) &= \lambda\cos\theta, \\ (I + ma^2)\ddot{\theta} + ma\dot{\theta}(\dot{x}\cos\theta + \dot{y}\sin\theta) &= 0. \end{aligned} \tag{0.10}$$

**Carathedory and Fufaev added a viscous friction force of the from**

$$R = -Nu \tag{0.11}$$

**to the sleigh equations.where $u$ is the velocity in the direction perpendicular to the blade. (Note that interchange $u$ and $v$ compared to the original paper of Fufaev.)**

**Setting**

$$k^2 = \frac{m}{I + ma^2}, \quad \epsilon = \frac{I}{Na^2} \tag{0.12}$$

**the equations with dissipation become**

$$u = \epsilon a \dot{\omega} \tag{0.13}$$

$$\dot{v} = a\omega^2 + \epsilon a \omega \dot{\omega} \tag{0.14}$$

$$ak^2\dot{\omega} + v\omega = -\epsilon a \ddot{\omega} \tag{0.15}$$

**It is clear that as $\epsilon$ goes to zero one recovers the original equations. Cartheodory incorrectly argued however that since no matter how small $\epsilon$ is these equations yield trejectories which**

**differ from that of the original system, dissipation cannot yield the nonholonomic constraints.**

**Fufaev realized this is not correct since the system degenerates from a system of three to two equations and thus there is a singularity. Setting $\mu = \epsilon a$ and $\sigma = \dot{\omega}$ we then get**

$$\dot{\omega} = \sigma \tag{0.16}$$

$$\dot{v} = a\omega^2 + \mu\omega\sigma \tag{0.17}$$

$$\mu\dot{\sigma} = -ak^2\sigma - v\omega \tag{0.18}$$

**Then as $\mu \to 0$ we get rapid motion except for the surface**

$$ak^2\sigma + \mu\omega = 0. \tag{0.19}$$

**The slow motion of this surface onto the $v$-$\omega$ plane then gives the correct equations of motion.**

• **The Chaplygin Sleigh as as Particle in a Radiation Field**

**We now show that the sleigh equations can be obtained from a variational principle as reduced equations of motion after the system is coupled to an environment described by an** $U(1)$ **infinite field of the form** $\mathbf{a}(\mathbf{z},t) \equiv [\cos\alpha(\mathbf{z},t), \sin\alpha(\mathbf{z},t)]$**. For the Lagrangian of the free field we choose**

$$L_{\mathrm{F}} = \frac{K}{2}\int d^2\mathbf{z}\;\dot{\mathbf{a}}^2, \tag{0.20}$$

**and we couple the sleigh and the field with a term of the form**

$$L_1 = \int d^2\mathbf{z}\;\delta(\mathbf{z}-\mathbf{x})\;[\gamma\dot{\mathbf{x}}\cdot\mathbf{a} + \mu\cos\left(\alpha(\mathbf{z},t)-\theta\right)]\,. \tag{0.21}$$

**The first term in square brackets corresponds to a minimal coupling that favors ẋ in the direction of a; the second has the form of a potential coupling that favors an alignment of the internal variable** $\theta$ **with the local direction of a.**

**The total action is $S = \int dt(L_0 + L_F + L_1)$ where $L_0$ is the Lagrangian of the free sleigh**

$$L_0 = \frac{m}{2}\left[\left(\dot{x} - a\dot{\theta}\sin\theta\right)^2 + \left(\dot{y} + a\dot{\theta}\cos\theta\right)^2\right] + \frac{I}{2}\dot{\theta}^2, \qquad (0.22)$$

**and can be regarded as a full "microscopic" theory of the sleigh coupled to an environment.**

**The equations of motion of the combined system are now obtained from a variational principle, $\delta S = 0$, and have the form**

$$
\begin{aligned}
& m\frac{d}{dt}\left(\dot{x} - a\dot{\theta}\sin\theta\right) \\
= \;& \gamma\left\{-\sin\alpha(\mathbf{x},t)\frac{\partial\alpha}{\partial t} + \left[\dot{x}\sin\alpha(\mathbf{x},t) - \dot{y}\cos\alpha(\mathbf{x},t)\right]\frac{\partial\alpha}{\partial x}\right\} \\
& -\mu\sin[\alpha(\mathbf{x},t)-\theta)]\frac{\partial\alpha}{\partial x}\,, \\
& m\frac{d}{dt}\left(\dot{y} + a\dot{\theta}\cos\theta\right) \\
= \;& \gamma\left\{\cos\alpha(\mathbf{x},t)\frac{\partial\alpha}{\partial t} + \left[\dot{x}\sin\alpha(\mathbf{x},t) - \dot{y}\cos\alpha(\mathbf{x},t)\right]\frac{\partial\alpha}{\partial y}\right\} \\
& -\mu\sin[\alpha(\mathbf{x},t)-\theta)]\frac{\partial\alpha}{\partial y}\,, \\
& (I+ma^2)\ddot{\theta} - ma\frac{d}{dt}(\dot{x}\sin\theta - \dot{y}\cos\theta) - ma\dot{\theta}(\dot{x}\cos\theta + \dot{y}\sin\theta) \\
= \;& \mu\sin\left[\alpha(\mathbf{x},t)-\theta\right]\,, \\
& K\frac{\partial^2\alpha(\mathbf{z},t)}{\partial t^2} \\
= \;& \delta(\mathbf{z}-\mathbf{x})\left\{\gamma\left[\dot{x}\sin\alpha(\mathbf{x},t) - \dot{y}\cos\alpha(\mathbf{x},t)\right] + \mu\sin\left[\alpha(\mathbf{x},t)-\theta\right]\right\} \qquad (0.23)
\end{aligned}
$$

**At this point we take the limit $\mu \to \infty$ in the third equation above. This limit can be understood from the singular perturbation theory, by dividing the left hand side of the equation by $\mu$, which amounts to rescaling the times in the derivatives by $\sqrt{\mu}$. (This is immediate by noting that the r.h.s. is homogeneous in the derivatives.) Therefore, for very large $\mu$ we have a very slow dynamics on the r.h.s., which amounts to setting $\sin[\alpha(\mathbf{x},t)-\theta]=0$. This is equivalent to saying that in the $\mu \to \infty$ limit the variables $\alpha(\mathbf{x},t)$ and $\theta$ are pinned to the same value. Next we integrate equation (0.23) over an infinitesimal region around x and obtain**

$$\dot{x}\sin\alpha(\mathbf{x},t) - \dot{y}\cos\alpha(\mathbf{x},t) = \dot{x}\sin\theta - \dot{y}\cos\theta = 0, \qquad (0.24)$$

**which means that the constraint is satisfied. Replacing the constraint (and $\sin[\alpha(\mathbf{x},t)-\theta]=0$) in the first three equations we obtain the same structure as (0.10) and therefore the same flow as in Eq. (0.8).**

**The calculation shows that we have succeeded in deriving the nonholonomic equations for a system with one internal (compact) variable from a pure Lagrangian formalism. The classical trayectories are obtained from a variational principle and quantization can be introduced through the Path integral formalism: the propagator is $e^{iS/\hbar}$, where $S$ is the complete action.**

**Intuitively, the sleigh is coupled to an infinite bath of rotors and, for $\mu \to \infty$, the internal variable and the rotors are locally the same. In the limit $K \to 0$ (vanishing moment of inertia for the rotors) the internal variable imposes its value on the local field instantaneously. Since the rotors are fixed in space they can still guide the motion imposing the velocity to be locally parallel to a. Also, since we are taking the limit $K \to 0$, the field does not take energy from the sleigh, and the nonholonomic motion conserves energy.**

**Quantum case for a=0:**
**The Hamiltonian in this limit has the form**

$$H = \frac{1}{2m}\left[p_x - \lambda\cos\alpha(\mathbf{x})\right]^2 + \frac{1}{2m}\left[p_y - \lambda\sin\alpha(\mathbf{x})\right]^2 + \frac{1}{2I}p_\theta^2 \quad (0.25)$$

$$+ \frac{1}{2K}\int d\mathbf{z}\,\Pi^2\left(\alpha(\mathbf{z})\right) + \mu\cos\left[\theta - \alpha(\mathbf{x})\right]. \quad (0.26)$$

**For the quantization of** $H$ **we procede with the usual replacements**

$$\mathbf{p} = -i\hbar(\partial_x, \partial_y), \quad p_\theta = -i\hbar\partial_\theta, \quad \Pi\left(\alpha(\mathbf{z})\right) = -i\hbar\partial_{\alpha(\mathbf{z})}. \quad (0.27)$$

**For the completely uncoupled case** ($\lambda = \mu = 0$) **the eigenstates are of the form**

$$\Psi_0 = e^{i\int d\mathbf{z}\; m(\mathbf{z})\alpha(\mathbf{z})}e^{i\mathbf{k}\cdot\mathbf{x}}e^{in\theta}, \quad (0.28)$$

**with** $m(\mathbf{z})$ **and** $n$ **integers and k the wave number of the translational degree of freedom.**

**The limit $\mu \to \infty$ ammounts to projecting the wave function and the Hamiltonian to states where $\alpha(\mathbf{x}) = \theta$, in such a way that the Hamiltonian becomes**

$$H = \frac{1}{2m}\left[p_x - \lambda\cos\theta\right]^2 + \frac{1}{2m}\left[p_y - \lambda\sin\theta\right]^2 + \frac{1}{2I'}p_\theta^2 \qquad (0.29)$$

$$+ \frac{1}{2K}\int d\mathbf{z}\,\Pi^2\left(\alpha(\mathbf{z})\right)\left[1 - \delta(\mathbf{x} - \mathbf{z})\right], \qquad (0.30)$$

**with $1/I' = 1/I + 1/K$. Without loss of generality we can take the quantum numbers $m[(\mathbf{z}) = 0$ for $\mathbf{z} \neq \mathbf{x}$ and the wave function depends only on the $\{\theta, \mathbf{x}\}$ degrees of freedom and obeys the following Schoedinger equation:**

$$\left\{\frac{1}{2m}\left[p_x - \lambda\cos\theta\right]^2 + \frac{1}{2m}\left[p_y - \lambda\sin\theta\right]^2 + \frac{1}{2I'}p_\theta^2\right\}\Psi = \epsilon\Psi. \qquad (0.31)$$

**The above equation can be solved by separation of variables $\Psi = e^{i\mathbf{k}\cdot\mathbf{x}}\psi_{\mathbf{k}}(\theta)$, with $\mathbf{k} = k(\cos\theta_{\mathbf{k}}, \sin\theta_{\mathbf{k}})$ a quasi-translational wave-**

vector. The reduced equation satisfied the the angular part of the wave function is

$$\left\{\frac{1}{2I'}p_\theta^2 - \frac{\lambda\hbar k}{m}\cos(\theta-\theta_{\mathbf{k}})\right\}\psi_{\mathbf{k}}(\theta) = \epsilon'\psi_{\mathbf{k}}(\theta), \tag{0.32}$$

with $\epsilon' = \epsilon - (\lambda^2 + \hbar^2 k^2)/2m$. This equation has well known solutions in terms of the Mathieu functions. One can gain insight on the structure of the solutions by looking at the fast limit ($k \to \infty$) which should exhibit features of the classical solution. In this limit the fluctiations of the angle are small and centered around $\theta = \theta_{\mathbf{k}}$. This means that, up to small quantum fluctuations, the knife edge is pointing in the direction of the plane wave propagation. Expanding for small values of the angle we find that the solutions in the fast limit are of the form

$$\Psi_{\mathbf{k}}(\mathbf{x},\theta) = e^{i\mathbf{k}\cdot\mathbf{x}}e^{-(\theta-\theta_{\mathbf{k}})^2/2\Delta_\theta^2}, \tag{0.33}$$

with

$$\Delta_\theta^2 = \frac{m\hbar}{\lambda k I'}. \tag{0.34}$$

**• Inverse Problems and Hamiltonization**

**• Idea: apply theory of inverse problem for Lagrangian systems (see e.g. Douglas, Crampin et. al.) to nonholonomic systems. Inspired by work of Aboud Filho at. al. Can we find a Lagrangian system whose dynamics restricts to the nonholonomic dynamics for the right initial data?**

**Consider example first: The vertical rolling disk is a homogeneous disk rolling without slipping on a horizontal plane, with configuration space $Q = \mathbb{R}^2 \times S^1 \times S^1$ and parameterized by the coordinates $(x, y, \theta, \varphi)$, where $(x, y)$ is the position of the center of mass of the disk, $\theta$ is the angle that a point fixed on the disk makes with respect to the vertical, and $\varphi$ is measured from the positive $x$-axis. The system has the Lagrangian and constraints given by**

$$\begin{aligned} L &= \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + \frac{1}{2}I\dot{\theta}^2 + \frac{1}{2}J\dot{\varphi}^2, \\ \dot{x} &= R\cos(\varphi)\dot{\theta}, \\ \dot{y} &= R\sin(\varphi)\dot{\theta}, \end{aligned} \tag{0.35}$$

**where $m$ is the mass of the disk, $R$ is its radius, and $I, J$ are the moments of inertia about the axis perpendicular to the plane of the disk, and about the axis in the plane of the disk, respectively. The constrained equations of motion are simply:**

$$\ddot{\theta} = 0, \quad \ddot{\varphi} = 0, \quad \dot{x} = R\cos(\varphi)\dot{\theta}, \quad \dot{y} = R\sin(\varphi)\dot{\theta}. \tag{0.36}$$

**The solutions of the first two equations are of course**

$$\theta(t) = u_\theta t + \theta_0, \qquad \varphi(t) = u_\varphi t + \varphi_0,$$

**and in the case where $u_\varphi \neq 0$, we get that the $x$- and $y$-solution is of the form**

$$\begin{aligned} x(t) &= \left(\frac{u_\theta}{u_\phi}\right) R\sin(\varphi(t)) + x_0, \\ y(t) &= -\left(\frac{u_\theta}{u_\phi}\right) R\cos(\varphi(t)) + y_0, \end{aligned} \tag{0.37}$$

**from which we can conclude that the disk follows a circular path. If $u_\varphi = 0$, we simply get the linear solutions**

$$x(t) = R\cos(\varphi_0)u_\theta t + x_0, \quad y(t) = R\sin(\varphi_0)u_\theta t + y_0. \tag{0.38}$$

**The situation in (0.38) corresponds to the case when $\varphi$ remains constant, i.e. when disk is rolling along a straight line. Solutions like this turn out to be problematic...**

**Consider the nonholonomic equations of motion (0.36). As a system of ordinary differential equations, these equations form a mixed set of coupled first- and second-order equations. It is well-known that these equations are never variational on their own, in the sense that we can never find a regular Lagrangian whose (unconstrained) Euler-Lagrange equations are equivalent to the nonholonomic equations of motion.**

**There are, however, infinitely many systems of second-order equations (only), whose solution set contains the solutions of the nonholonomic equations. We shall call these second-order systems *associated second-order systems*, and we wish to find out whether or not we can find a regular Lagrangian for one of those associated second-order systems. If so, we can use the Legendre transformation to get a full Hamiltonian system on the associated phase space.**

**There are infinitely many ways to arrive at an associated second-order system for a given nonholonomic system. We shall illustrating three choices below using the vertical rolling disk as an example.**

**Consider, for example, taking the time derivative of the constraint equations, so that a solution of the nonholonomic system (0.36) also satisfies the following complete set of second-order differential equations in all variables $(\theta, \varphi, x, y)$:**

$$\ddot{\theta} = 0, \quad \ddot{\varphi} = 0, \quad \ddot{x} = -R\sin(\varphi)\dot{\theta}\dot{\varphi}, \quad \ddot{y} = R\cos(\varphi)\dot{\theta}\dot{\varphi}. \tag{0.39}$$

**We shall call this associated second-order system the *first associated second-order system.* Excluding for a moment the case where $u_\phi = 0$, the solutions of equations (0.39) can be written**

**as**

$$
\begin{aligned}
\theta(t) &= u_\theta t + \theta_0 \\
\varphi(t) &= u_\varphi t + \varphi_0 \\
x(t) &= \left(\frac{u_\theta}{u_\phi}\right) R\sin(\varphi(t)) + u_x t + x_0, \\
y(t) &= -\left(\frac{u_\theta}{u_\phi}\right) R\cos(\varphi(t)) + u_y t + y_0.
\end{aligned}
$$

**By restricting the above solution set to those that also satisfy the constraints $\dot{x} = \cos(\varphi)\dot{\theta}$ and $\dot{y} = \sin(\varphi)\dot{\theta}$ (i.e. to those solutions above with $u_x = u_y = 0$), we get back the solutions (0.37) of the non-holonomic equations (0.36). A similar reasoning holds for the solutions of the form (0.38). The question we then wish to answer is whether the second-order equations (0.39) are equivalent to the Euler-Lagrange equations of some regular Lagrangian or not.**

**Now, taking note of the special structure of equations (0.39), we may use the constraints (0.36) to eliminate the $\dot{\theta}$ dependency. This yields another plausible choice for an associated system:**

$$\ddot{\theta} = 0, \quad \ddot{\varphi} = 0, \quad \ddot{x} = -\frac{\sin(\varphi)}{\cos(\varphi)}\dot{x}\dot{\varphi}, \quad \ddot{y} = \frac{\cos(\varphi)}{\sin(\varphi)}\dot{y}\dot{\varphi}. \tag{0.40}$$

**We shall refer to this choice later as the *second associated second-order system.***

Lastly, we may simply note that, given that on the constraint manifold the relation $\sin(\varphi)\dot{x} - \cos(\varphi)\dot{y} = 0$ is satisfied, we can easily add a multiple of this relation to some of the equations above. One way of doing so leads to the system

$$\begin{aligned} J\ddot{\varphi} &= -mR(\sin(\varphi)\dot{x} - \cos(\varphi)\dot{y})\dot{\theta}, \\ (I + mR^2)\ddot{\theta} &= mR(\sin(\varphi)\dot{x} - \cos(\varphi)\dot{y})\dot{\varphi}, \\ (I + mR^2)\ddot{x} &= -R(I + mR^2)\sin(\varphi)\dot{\theta}\dot{\varphi} + mR^2\cos(\varphi)(\sin(\varphi)\dot{x} - \cos(\varphi)\dot{y})\dot{\varphi}, \\ (I + mR^2)\ddot{y} &= R(I + mR^2)\cos(\varphi)\dot{\theta}\dot{\varphi} + mR^2\sin(\varphi)(\sin(\varphi)\dot{x} - \cos(\varphi)\dot{y})\dot{\varphi}. \end{aligned} \tag{0.41}$$

We shall refer to it as the ***third associated second-order system.*** Indeed this complicated looking system is indeed variational! The Euler-Lagrange equations for the regular Lagrangian

$$L = -\frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + \frac{1}{2}I\dot{\theta}^2 + \frac{1}{2}J\dot{\varphi}^2 + mR\dot{\theta}(\cos(\varphi)\dot{x} + \sin(\varphi)\dot{y}), \tag{0.42}$$

are indeed equivalent to equations (0.41), and, when restricted to the constraint distribution, its solutions are exactly those of the nonholonomic equations (0.36).

**Other examples include a nonholonomically constrained free particle with unit mass moving in $\mathbb{R}^3$ In this example one has a free particle with Lagrangian and constraint given by**

$$L = \frac{1}{2}\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right), \qquad \dot{z} + x\dot{y} = 0. \tag{0.43}$$

**and the constrained equations, which take the form**

$$\ddot{x} = 0, \quad \ddot{y} = -\frac{x\dot{x}\dot{y}}{1+x^2}, \quad \dot{z} = -x\dot{y}. \tag{0.44}$$

**Another example is the knife edge on a plane. It corresponds physically to a blade with mass $m$ moving in the $xy$ plane at an angle $\phi$ to the $x$-axis . The Lagrangian and constraints for the system are:**

$$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + \frac{1}{2}J\dot{\phi}^2, \qquad \dot{x}\sin(\phi) - \dot{y}\cos(\phi) = 0, \tag{0.45}$$

**from which we obtain the constrained equations:**

$$\ddot{\phi} = 0, \qquad \ddot{x} = -\tan(\phi)\dot{\phi}\dot{x}, \qquad \dot{y} = \tan(\phi)\dot{x}.$$

**Associated Second Order Systems in General:**

**Assume the configuration space $Q$ is locally just the Euclidean space $\mathbb{R}^n$ and that the base space of the fibre bundle is two dimensional, writing $(r_1, r_2; s_\alpha)$ for the coordinates. We will consider the class of nonholonomic systems where the Lagrangian is given by**

$$L = \frac{1}{2}(I_1\dot{r}_1^2 + I_2\dot{r}_2^2 + \sum_\alpha I_\alpha \dot{s}_\alpha^2), \tag{0.46}$$

**(with all $I_\alpha$ positive constants) and where the constraints take the following special form**

$$\dot{s}_\alpha = -A_\alpha(r_1)\dot{r}_2. \tag{0.47}$$

**Although this may seem to be a very thorough simplification, this interesting class of systems does include, for example, all the classical examples described above. We also remark that all of the above systems fall in the category of so-called Chaplygin systems.**

**In what follows, we will assume that none of the $A_\alpha$ are constant (in that case the constraints are, of course, holonomic). The nonholonomic equations of motion are now**

$$\ddot{r}_1 = 0, \quad \ddot{r}_2 = -N^2\Big(\sum_\beta I_\beta A_\beta A'_\beta\Big)\dot{r}_1\dot{r}_2, \quad \dot{s}_\alpha = -A_\alpha \dot{r}_2, \tag{0.48}$$

**where $N$ is shorthand for the function**

$$N(r_1) = \frac{1}{\sqrt{I_2 + \sum_\alpha I_\alpha A_\alpha^2}}. \tag{0.49}$$

**This function is directly related to the invariant measure of the system. Indeed, we have shown that for a two-degree of freedom system such as (0.48), we may compute the density $N$ of the invariant measure (if it exists) by integrating two first-order partial differential equations derived from the condition that the volume form be preserved along the nonholonomic**

**flow. In the present case, these two equations read:**

$$\frac{1}{N}\frac{\partial N}{\partial r_1} + \frac{\sum_\beta I_\beta A_\beta A'_\beta}{I_2 + \sum_\alpha I_\alpha A_\alpha^2} = 0, \qquad \frac{1}{N}\frac{\partial N}{\partial r_2} = 0, \tag{0.50}$$

**and obviously the expression for $N$ in (0.49) is its solution up to an irrelevant multiplicative constant. In case of the free nonholonomic particle and the knife edge the invariant measure density is $N \sim 1/\sqrt{1+x^2}$ and $N \sim 1/\sqrt{(1+\tan^2(\phi))} = \cos(\phi)$, respectively. In case of the vertically rolling disk it is a constant. We shall see later that systems with a constant invariant measure (or equivalently, with constant $\sum_\alpha I_\alpha A_\alpha^2$) always play a somehow special role.**

**We are now in a position to generalize the associated second-order systems to the more general class of nonholonomic systems above. In the set-up above, the first associated second-order system is, for the more general systems (0.48), the system**

$$\ddot{r}_1 = 0, \quad \ddot{r}_2 = -N^2\big(\sum_\beta I_\beta A_\beta A'_\beta\big)\dot{r}_1\dot{r}_2, \quad \ddot{s}_\alpha = -(A'_\alpha \dot{r}_1\dot{r}_2 + A_\alpha \ddot{r}_2),$$

**or equivalently, in normal form,**

$$\begin{aligned} \ddot{r}_1 &= 0, \qquad\qquad \ddot{r}_2 = -N^2\big(\sum_\beta I_\beta A_\beta A'_\beta\big)\dot{r}_1\dot{r}_2, \\ \ddot{s}_\alpha &= -\Big(A'_\alpha - N^2 A_\alpha\big(\sum_\beta A_\beta A'_\beta\big)\Big)\dot{r}_1\dot{r}_2. \end{aligned} \tag{0.51}$$

**For convenience, we will often simply write**

$$\ddot{r}_1 = 0, \quad \ddot{r}_2 = \Gamma_2(r_1)\dot{r}_1\dot{r}_2, \quad \ddot{s}_\alpha = \Gamma_\alpha(r_1)\dot{r}_1\dot{r}_2,$$

**for these types of second-order systems.**

**The second associated second-order system we encountered for the vertically rolling disk also translates to the more general setting. We get**

$$\ddot{r}_1 = 0, \qquad \ddot{r}_2 = -N^2\Big(\sum_\beta I_\beta A_\beta A'_\beta\Big)\dot{r}_1\dot{r}_2,$$
$$\ddot{s}_\alpha = \left(A'_\alpha - N^2 A_\alpha\Big(\sum_\beta I_\beta A_\beta A'_\beta\Big)\right)\dot{r}_1\left(\frac{\dot{s}_\alpha}{A_\alpha}\right), \tag{0.52}$$

**where in the right-hand side of the last equation, there is no sum over $\alpha$. A convenient byproduct of this way of associating a second-order system to (0.48) is that now all equations decouple except for the coupling with the $r_1$-equation. To highlight this, we will write this system as**

$$\ddot{r}_1 = 0, \quad \ddot{q}_a = \Xi_a(r_1)\dot{q}_a\dot{r}_1$$

**(no sum over $a$) where, from now on, $(q_a) = (r_2, s_\alpha)$ and $(q_i) = (r_1, q_a)$.**

**We can show essentially that it is possible to solve the inverse problem for the second associated systems, but not for the first. The third works in special cases.**

**We have:**

**Proposition 1** ***The function***

$$L = \rho(\dot{r}_1) + \frac{1}{2N}\left(C_2\frac{\dot{r}_2^2}{\dot{r}_1} + \sum_\beta C_\beta \frac{\dot{s}_\beta^2}{A_\beta \dot{r}_1}\right), \tag{0.53}$$

***with*** $d^2\rho/d\dot{r}_1^2 \neq 0$ ***and all*** $C_\alpha \neq 0$ ***is in any case a regular Lagrangian for the second associated systems (0.52). If the invariant measure density*** $N$ ***is a constant, then also***

$$L = \rho(\dot{r}_1) + \sigma(\dot{r}_2) + \frac{1}{2N}\sum_\beta a_\beta \frac{\dot{s}_\beta^2}{A_\beta \dot{r}_1}, \tag{0.54}$$

***where*** $d^2\rho/d\dot{r}_1^2 \neq 0$***,*** $d^2\sigma/d\dot{r}_1^2 \neq 0$ ***and all*** $C_\alpha \neq 0$ ***is a regular Lagrangian for the second associated systems (0.52)***

**A list of the Lagrangians for the nonholonomic free particle, the knife edge on a horizontal plane and the vertically rolling disk. The respective Lagrangians (0.53) for the first two examples are:**

$$L = \rho(\dot{x}) + \frac{1}{2}\sqrt{1+x^2}\left(C_2\frac{\dot{y}^2}{\dot{x}} + C_3\frac{\dot{z}^2}{x\dot{x}}\right), \tag{0.55}$$

**and**

$$\begin{aligned} L &= \rho(\dot{\phi}) + \frac{1}{2}\sqrt{m(1+\tan(\phi)^2)}\left(C_2\frac{\dot{x}^2}{\dot{\phi}} + C_3\frac{\dot{y}^2}{\tan(\phi)\dot{\phi}}\right), \\ &= \rho(\dot{\phi}) + \frac{1}{2}C_2\sqrt{m}\frac{\dot{x}^2}{\cos(\phi)\dot{\phi}} + \frac{1}{2}C_3\sqrt{m}\frac{\dot{y}^2}{\sin(\phi)\dot{\phi}}. \end{aligned} \tag{0.56}$$

**The vertically rolling disk in one of those systems with constant invariant measure. The first Lagrangian (0.53) is:**

$$L = \rho(\dot{\varphi}) + \frac{\sqrt{I+mR^2}}{2}\left(C_2\frac{\dot{\theta}^2}{\dot{\varphi}} + C_3\frac{\dot{x}^2}{\cos(\varphi)\dot{\varphi}} + C_4\frac{\dot{y}^2}{\sin(\varphi)\dot{\varphi}}\right) \tag{0.57}$$

**and the second Lagrangian (0.54) is:**

$$L = \rho(\dot{\varphi}) + \sigma(\dot{\theta}) - \frac{\sqrt{I + mR^2}}{2}\left(a_3 \frac{\dot{x}^2}{\cos(\varphi)\dot{\varphi}} + a_4 \frac{\dot{y}^2}{\sin(\varphi)\dot{\varphi}}\right). \tag{0.58}$$

**Let us put for convenience $\rho(\dot r_1) = \frac{1}{2} I_1 \dot r_1^2$ and $\sigma(\dot r_2) = \frac{1}{2} I_2 \dot r_2^2$ in the Lagrangians of the previous section.**

**Proposition 2** ***Given the second associated second-order system (0.52), the regular Lagrangian (0.53) (away from $\dot r_1 = 0$) and constraints (0.47) on $TQ$ are mapped by the Legendre transform to the Hamiltonian and constraints in $T^*Q$ given by:***

$$H = \frac{1}{2I_1}\left(p_1 + \frac{1}{2}N\left(\frac{p_2^2}{C_2} + \sum_\beta A_\beta \frac{p_\beta^2}{C_\beta}\right)\right)^2, \qquad C_2 p_\alpha = -C_\alpha p_2. \quad (0.59)$$

***In case $N$ is constant, the second Lagrangian (0.54) and constraints (0.47) are transformed into***

$$H = \frac{1}{2I_2}p_2^2 + \frac{1}{2I_1}\left(p_1 + \frac{1}{2}N\left(\sum_\beta \frac{A_\beta}{a_\beta} p_\beta^2\right)\right)^2, \quad I_2 N \dot r_1 p_\alpha + a_\alpha p_2 = 0, \quad (0.60)$$

***where*** $\dot{r}_1(r_1, p_1, p_\alpha) = (p_1 + \frac{1}{2}N \sum_\alpha A_\alpha p_\alpha^2 / a_\alpha)/I_1$**.**

- **Chaplygin Sleigh with Oscillator**

**Consider dynamics of the Chaplygin sleigh coupled with an oscillator. We show that the phase flow is integrable, and generic invariant manifolds are two-dimensional tori.**

**Consider the Chaplygin sleigh with a mass sliding along the direction of the blade. The mass is coupled to the sleigh through a spring. One end of the spring is attached to the sleigh at the contact point, the other end is attached to the mass. The spring force is zero when the mass is positioned above the contact point.**

**The configuration space for this system is $\mathbb{R} \times \mathrm{SE}(2)$. This system has one shape (the distance from the mass to the contact point, $r$) and three group degrees of freedom.**

**The reduced Lagrangian $l : T\mathbb{R} \times \mathfrak{se}(2) \to \mathbb{R}$ is given by the formula**

$$
\begin{aligned}
l(r, \dot r, \xi) &= \tfrac{1}{2} m\dot r^2 + m\dot r \xi^2 \\
&\quad + \tfrac{1}{2}\left((J + mr^2)(\xi^1)^2 + 2mr\xi^1\xi^3 + (M+m)((\xi^2)^2 + (\xi^3)^2)\right) - \tfrac{1}{2} k r^2,
\end{aligned}
$$

**where $\xi = g^{-1}\dot g \in \mathfrak{se}(2)$ and $k$ is the spring constant. The constrained reduced Lagrangian is**

$$
l_c(r, \dot r, \xi^1, \xi^2) = \tfrac{1}{2} m\dot r^2 + m\dot r\xi^2 + \tfrac{1}{2}\left((J + mr^2)(\xi^1)^2 + (M+m)(\xi^2)^2\right) - \tfrac{1}{2} k r^2,
$$

**The constrained reduced energy**

$$
\tfrac{1}{2} m\dot r^2 + m\dot r\xi^2 + \tfrac{1}{2}\left((J + mr^2)(\xi^1)^2 + (M+m)(\xi^2)^2\right) + \tfrac{1}{2} k r^2
$$

**is positive-definite, and thus the mass cannot move infinitely far from the sleigh throughout the motion.**

**The constraint is given by the formula**

$$
\Omega^3 = 0.
$$

**The reduced Lagrangian written as a function of $(r, \dot r, \Omega)$ be-**

**comes**

$$l(r,\dot r,\Omega) = \frac{1}{2}\frac{Mm}{M+m}\dot r^2 \\ + \frac{1}{2}\left((J+mr^2)(\Omega^1)^2 + 2mr\Omega^1\Omega^3 + (M+m)((\Omega^2)^2 + (\Omega^3)^2\right) - \frac{1}{2}kr^2.$$

**The constrained reduced Lagrangian written as a function of** $(r,\dot r,p)$ **is**

$$l_c(r,\dot r,p) = \frac{1}{2}\frac{Mm\dot r^2}{M+m} + \frac{1}{2}\left(\frac{p_1^2}{J+mr^2} + \frac{p_2^2}{M+m}\right) - \frac{kr^2}{2}.$$

**Can show the reduced dynamics becomes**

$$\frac{Mm}{M+m}\ddot r = \frac{Mmr}{(M+m)(J+mr^2)^{\frac{M}{M+m}+1}}p_1^2 - kr \qquad (0.61)$$

$$\dot p_1 = -\frac{mr}{(M+m)(J+mr^2)}p_1p_2, \qquad (0.62)$$

$$\dot p_2 = \frac{mr}{(J+mr^2)^{\frac{M}{M+m}+1}}p_1^2. \qquad (0.63)$$

**Relative Equilibria of the Sleigh-Mass System:**

**Assuming** $(r, p) = (r_0, p_0)$ **is a relative equilibrium, equation** $(0.63)$ **implies** $r_0 p_1^0 = 0$**. Thus, either** $r_0 = 0$ **and** $p_1^0$ **is an arbitrary constant, or, using** $(0.61)$**,** $p_1^0 = 0$ **and** $r_0 = 0$**. Thus the only relative equilibria of the sleigh-mass system are**

$$r = 0, \qquad p = p_0.$$

**The Discrete Symmetries and Integrability:**

**It is straightforward to see that equations** $(0.61)$–$(0.63)$ **are invariant with respect to the following transformations:**

**(i)** $(r, p_1, p_2) \to (r, -p_1, p_2)$**,**

**(ii)** $(r, p_1, p_2) \to (-r, p_1, -p_2)$**,**

**(iii)** $(t, r, p) \to (-t, -r, p)$**,**

**(iv)** $(t, r, p_1, p_2) \to (-t, r, p_1, -p_2)$**.**

**We now use these transformations to study some of the solutions of** $(0.61)$–$(0.63)$**. Consider an initial condition** $(r, \dot{r}, p) = (0, \dot{r}_0, p_0)$**. Then the** $r$**-component of the solution subject to this initial condition is odd, and the** $p$**-component is even. Indeed, let**

$$(r(t), p(t)), \qquad t > 0, \tag{0.64}$$

**be the part of this solution for** $t > 0$**. Then**

$$(-r(-t), p(-t)), \qquad t < 0, \tag{0.65}$$

**is also a solution. This follows from the invariance of equations** $(0.61)$–$(0.63)$ **with respect to transformation (iii). Using the formula**

$$\left.\frac{dr(t)}{dt}\right|_{t=0} = \left.\frac{d(-r(-t))}{dt}\right|_{t=0},$$

**we conclude that** $(0.64)$ **and** $(0.65)$ **satisfy the same initial condition and thus represent the forward in time and the backward in time branches of a the same solution. Thus,** $r(-t) = -r(t)$ **and** $p(-t) = p(t)$**.**

**Next,** $p_1(t) = 0$ **implies that** $p_2(t) =$ **const and that** $r(t)$ **satisfies the equation**

$$\frac{Mm}{M+m}\ddot{r} = -kr,$$

**and thus equations** (0.61)–(0.63) **have periodic solutions**

$$r(t) = A\cos\omega t + B\sin\omega t, \qquad p_1 = 0, \qquad p_2 = C,$$

**where** $A$**,** $B$**, and** $C$ **are arbitrary constants and** $\omega = \sqrt{k(M+m)/Mm}$**.**

**Without loss of generality, we set** $A = 0$ **and consider periodic solutions**

$$r(t) = \dot{r}_0/\omega \sin\omega t, \qquad p_1 = 0, \qquad p_2 = p_2^0, \tag{0.66}$$

**which correspond to the initial conditions**

$$r(0) = 0, \quad \dot{r}(0) = \dot{r}_0, \quad p_1(0) = 0, \quad p_2(0) = p_2^0.$$

**We now perturb solutions** (0.66) **by setting**

$$r(0) = 0, \quad \dot{r}(0) = \dot{r}_0, \quad p_1(0) = p_1^0, \quad p_2(0) = p_2^0. \tag{0.67}$$

**Assuming that $p_1^0$ is small and using a continuity argument, there exists $\tau = \tau_{p,\dot{r}_0} > 0$ such that**

$$r(\tau_{p,\dot{r}_0}) = 0$$

**for solutions subject to initial conditions** $(0.67)$**. That is, the** $r$**-component is** $2\tau$**-periodic if** $p_1^0$ **is sufficiently small.**

**Using equation** $(0.61)$ **and periodicity of** $r(t)$**, we conclude that** $p_1$ **is** $2\tau$**-periodic as well. Equation** $(0.62)$ **then implies that** $p_2(t)$ **is also** $2\tau$**-periodic.**

**Thus,** *the reduced dynamics is integrable in an open subset of the reduced phase space***. The invariant tori are one-dimensional, and the reduced flow is periodic. A generic periodic trajectory in the direct product of the shape and momentum spaces is shown in Figure 0.7.**

**Using the quasi-periodic reconstruction theorem we obtain the following theorem:**

**Theorem 0.1** ***Generic trajectories of the coupled sleigh-oscillator***

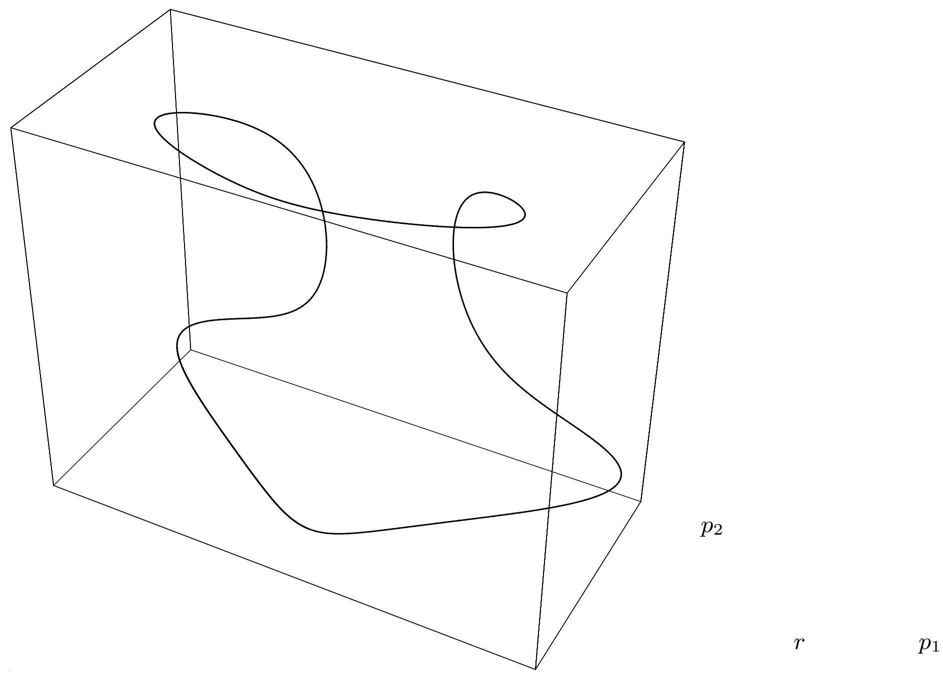


Figure 0.7: A reduced trajectory of the sleigh-mass system.

***system in the full phase space are quasi-periodic motions on two-dimensional invariant tori.***

**Typical trajectories of the contact point of the sleigh with the plane are shown in Figure 0.8.**

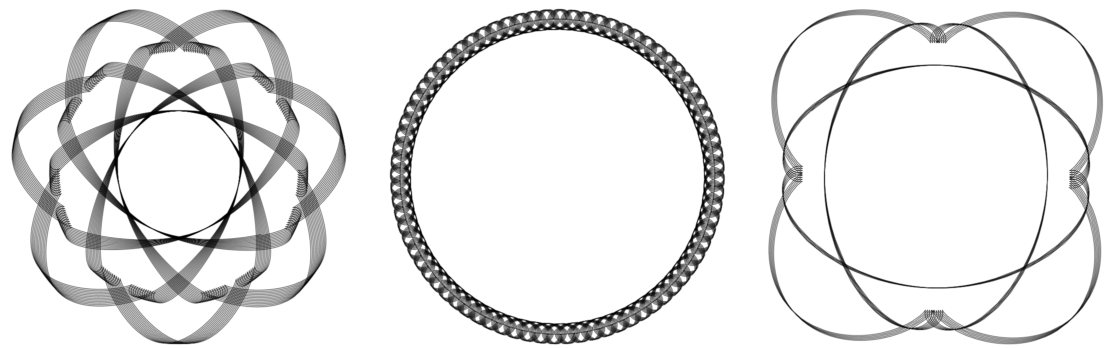

Figure 0.8: Trajectories of the contact point of the blade for various initial states.

**Some key refs:**

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**and T. Mestdag).**
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