Template-based stabilization of relative equilibria in systems with symmetry

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Goals

• Stabilization of unstable *relative equilibria* in systems with a *continuous* symmetry, using feedback control; examples include traveling waves, co-ordinated motions of vehicles
•The closed loop system should retain the symmetry of the original system

Idea

• A *template-based* procedure to factor out the *group dynamics*, and obtain *shape dynamics* on the quotient space
• Relative equilibria are steady states of template-dynamics, governed by differential-algebraic equations (DAEs)
• Use linear control techniques to stabilize steady states of these DAEs

Template-based symmetry reduction

• Consider a vector field X on a n-dimensional manifold M, equivariant under the action of a d-dimensional group G

$$\dot{z} = X(z,u) \qquad \text{(dynamics)}$$
$$X(\Phi_g(z), \Psi_g(u)) = T\Phi_g(X(z,u)) \qquad \text{(equivariance)}$$

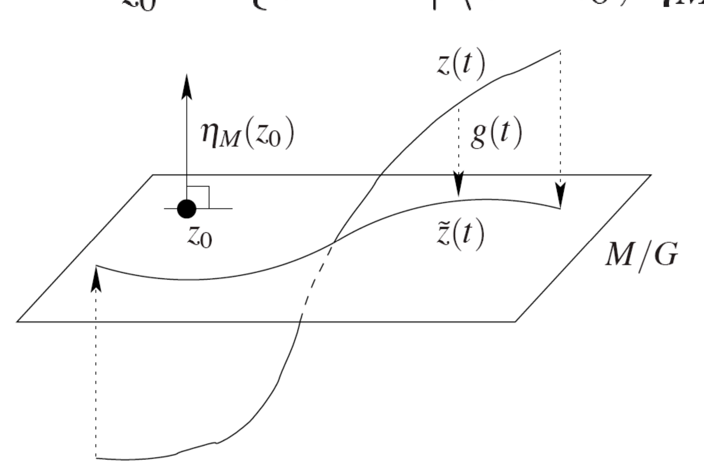
State $z \in M$, control $u \in U$
$g \in G$, symmetry group acting on M, U via actions Φ_g, Ψ_g

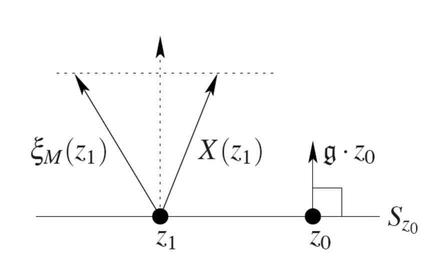
• Decompose the state *z(t)* and the input *u(t)* as

$$\left.\begin{aligned} z(t) &= g(t)\cdot\tilde{z}(t) \\ u(t) &= g(t)\cdot\tilde{u}(t)\end{aligned}\right\} \quad \dot{\tilde{z}} = X(\tilde{z},\tilde{u}) - \xi_M(\tilde{z})$$

• Define a *template* $z_0 \in M$, and choose *g(t)* such that $\tilde{z}(t)$ remains "aligned" with z_0 for all time. Equivalent to restricting *z(t)* to an affine *slice* S_{z_0} defined by

$$S_{z_0} = \{\tilde{z} \in M \,|\, \langle \tilde{z} - z_0, \eta_M(z_0)\rangle = 0, \forall \eta \in \mathfrak{g}\}$$





• Gives algebraic equations for ξ, that restrict the dynamics to the slice

$$\langle X(\tilde{z},\tilde{u}) - \xi_M(\tilde{z}), \eta_M(z_0)\rangle = 0, \quad \forall \eta \in \mathfrak{g}.$$

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Control Design

• Slice dynamics form a set of index-1 differential-algebraic equations (DAEs)

$$\dot{\tilde{z}} = X(\tilde{z},\tilde{u}) - \xi_M(\tilde{z}) \qquad (n \text{ equations})$$
$$\langle X(\tilde{z},\tilde{u}) - \xi_M(\tilde{z}), \eta_M(z_0)\rangle = 0, \quad \forall \eta \in \mathfrak{g}. \qquad (d \text{ equations})$$

• Relative equilibria are steady states of slice dynamics
• Eliminate the algebraic equations to obtain an equivalent set of *n-d* differential equations
• Linearize about the unstable relative equilibrium and use linear control theory to find stabilizing feedback laws
• **Special case:** if the relative equilibrium z_s is also a steady state of the original equations, and also is the template chosen to define the slice, then the following diagram commutes:

$$\begin{array}{ccc} \dot{z} = X(z,u) & \xrightarrow{\text{lin. about } \tilde{z}_s} & \dot{w} = Aw + Bv \\ \downarrow \text{slice dynamics} & & \downarrow \mathbb{P}_{S_{z_s}} \\ \dot{\tilde{z}} = X(\tilde{z},\tilde{u}) - \xi_M(\tilde{z}) & \xrightarrow{\text{lin. about } \tilde{z}_s} & \dot{w} = \mathbb{P}_{S_{z_s}}(Aw+Bv) \end{array}$$

Equivariant actuation

• Assume the dynamics to be affine in control, and actuation *H(z)* to be equivariant:

$$X(z,u) = X(z) + H(z)u. \qquad H(\Phi_g(z))\circ\Psi_g = T_z\Phi_g \circ H(z)$$

• Define a quadratic cost, also invariant under the action of G

$$J[\tilde{z},\tilde{u}] = \int_0^\infty \left(\langle \tilde{z}-\tilde{z}_s, Q(\tilde{z}-\tilde{z}_s)\rangle + \langle \tilde{u}, R\tilde{u}\rangle \right) dt \qquad \begin{aligned} Q &= \Phi_{h^{-1}} \circ Q \circ \Phi_h \\ R &= \Psi_{h^{-1}} \circ R \circ \Psi_h \end{aligned}$$

• Use LQR to determine the optimal feedback gain K that minimizes the cost
• The form of control input in the original equations is

$$u = \Psi_g \circ K(\Phi_{g^{-1}}(z) - \tilde{z}_s)$$

Amplitude and phase actuation

• We consider m general actuators, which are not equivariant, but we have a freedom to translate the actuators in the group direction

$$\dot{z} = X(z) + \sum_{i=1}^{m} T\Phi_{h_i} \circ b_i u_i.$$

Control inputs: amplitude u_i and phase h_i

• The slice dynamics of these equations are

$$\dot{\tilde{z}} = X(\tilde{z}) + \sum_{i=1}^{m} T\Phi_{g^{-1}h_i} \circ b_i u_i - \xi_M(\tilde{z})$$

• Choose the phase $h_i = g \cdot c_i$, for constant c_i. Choose c_i such that the resulting control term has no action in the group direction; that is, h_i does not contribute to ξ
• Linearize about the relative equilibrium and use LQR to determine the amplitude u_i

Example 1: Kuramoto-Sivashinsky equation

• A PDE in one spatial dimension, governing flame dynamics in combustion
• Symmetry in the form of translational invariance; vector-field equivariant to the action of $G = (\mathbb{R}, +)$

$$z_t = -zz_x - z_{xx} - \nu z_{xxxx}, \qquad x \in [0, 2\pi)$$

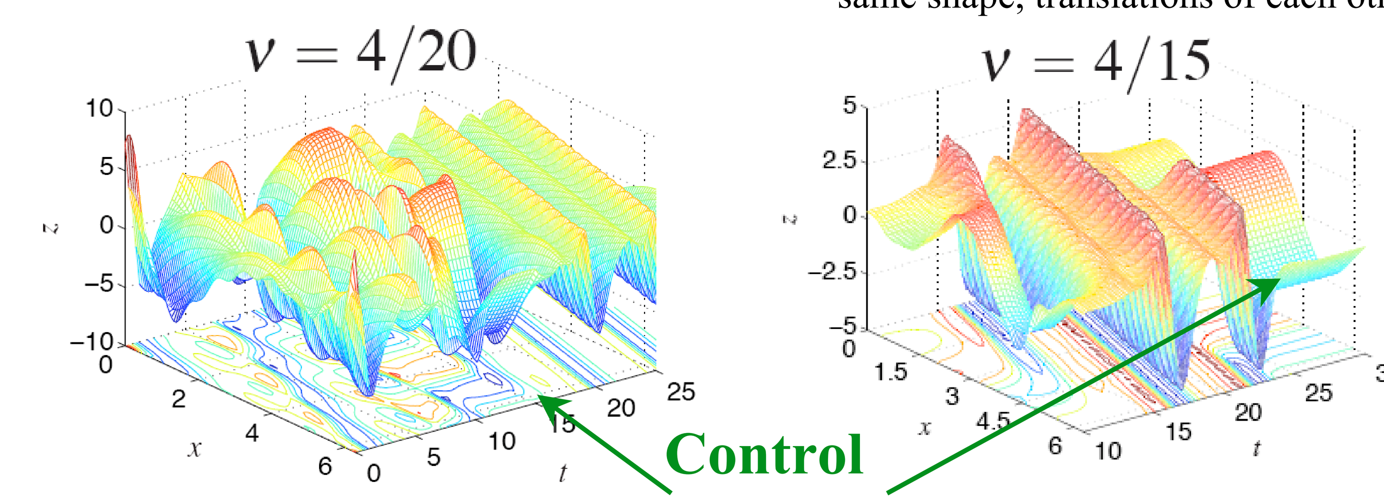
Periodic boundary conditions ⇒ translational invariance

• Shape dynamics are obtained by subtracting $\xi_M(z)$, where

$$\xi_M(z) = \xi z_x \qquad \xi = \text{speed of translation}$$

• **Unstable relative equilibria:**

• 1-parameter family of traveling waves
• 1-parameter family of steady states; same shape, translations of each other



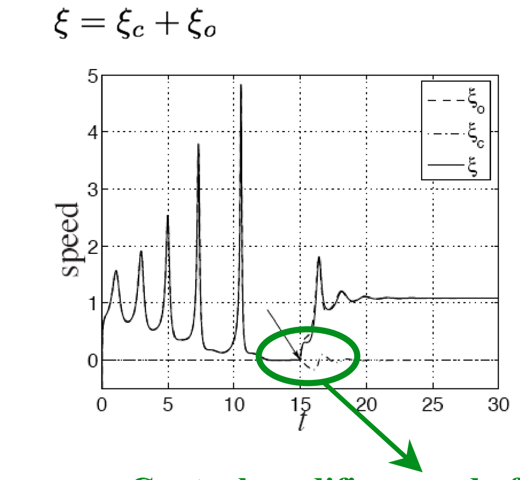
Control ON

Equivariant actuation

• Actuator modeled as a sinusoid

$$\dot{z} = X(z) + \sum_{k=1}^{m} u_k(t)\exp(ikx)$$

• ξ_c: contribution of control term to ξ
$\xi = \xi_c + \xi_o$

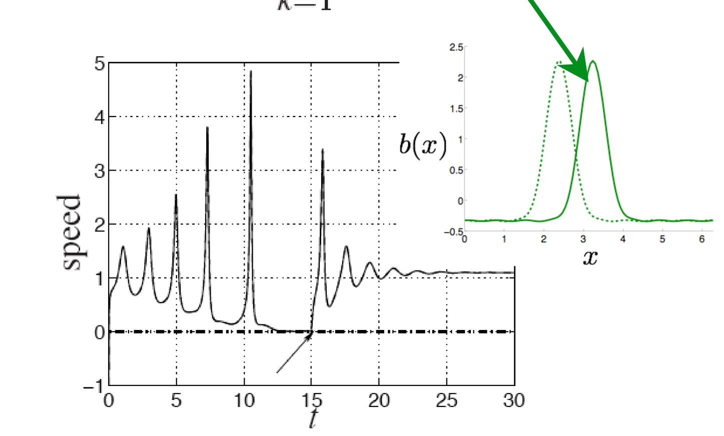


Control modifies speed of translation (action in the group direction)

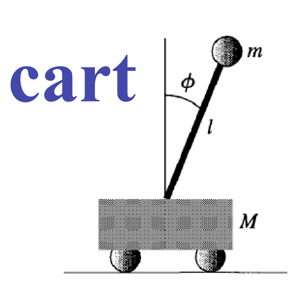
Amplitude and phase actuation

• Actuator modeled as a localized Gaussian, with the phase given by the location of the Gaussian peak

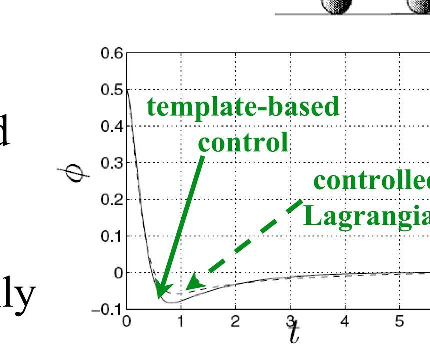
$$\dot{z} = X(z) + \sum_{k=1}^{m} u_k(t) b_k(x + h_k)$$



Example 2: Inverted pendulum on a cart



• Four equations of motion, for the cart position, pendulum angle and their velocities
• Symmetry group is (*G*,+); equations are invariant under translations of the cart
• Template dynamics are equivalent to those obtained by eliminating the equation for the cart position
• Template-based controller compares with that obtained using controlled Lagrangian, but ensures only local stability



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