Template-based stabilization of relative equilibria in systems with symmetry

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Goals
- Stabilization of unstable relative equilibria in systems with a continuous symmetry, using feedback control; examples include traveling waves, co-ordinated motions of vehicles
- The closed loop system should retain the symmetry of the original system

Idea
- A template-based procedure to factor out the group dynamics, and obtain shape dynamics for the quotient space
- Relative equilibria are steady states of template-dynamics, governed by differential-algebraic equations (DAEs)
- Use linear control techniques to stabilize steady states of these DAEs

Template-based symmetry reduction

Consider a vector field \( X \) on an \( n \)-dimensional manifold \( M \), equivariant under the action of a \( d \)-dimensional group \( G \)

\[
\begin{align*}
\dot{z} &= X(z, u) \quad \text{(dynamics)} \\
X(\Phi_g(z), \Psi_g(u)) &= T\Phi_g(X(z, u)) \quad \text{(equivariance)}
\end{align*}
\]

State \( z \in M \), control \( u \in U \)
\( g \in G \), symmetry group acting on \( M \)
- Decompose the state \( z(t) \) and the input \( u(t) \) as

\[
\begin{align*}
z(t) &= g(t) \cdot z(t) \\
u(t) &= g(t) \cdot u(t)
\end{align*}
\]

- Define a template \( \phi_g \in M \), and choose \( g(t) \) such that \( z(t) \) remains “aligned” with \( \phi_g \) for all time. Equivalent to restricting \( z(t) \) to an affine slice \( S_{\phi_g} \) defined by

\[
S_{\phi_g} = \{ z \in M \mid \langle z - \phi_g, \eta_M(\phi_g) \rangle = 0, \forall \eta \in g \} \]

- Gives algebraic equations for \( \xi \), that restrict the dynamics to the slice

\[
\langle X(\tilde{z}, \tilde{u}), \eta_M(\tilde{z}) \rangle = 0, \quad \forall \eta \in g.
\]

Control Design
- Slice dynamics form a set of index-1 differential-algebraic equations (DAEs)

\[
\begin{align*}
\dot{\tilde{z}} &= X(\tilde{z}, \tilde{u}) - \xi M(\tilde{z}) \quad \text{(n equations)} \\
\langle X(\tilde{z}, \tilde{u}) - \xi M(\tilde{z}), \eta_M(\tilde{z}) \rangle &= 0, \quad \forall \eta \in g. \quad \text{(d equations)}
\end{align*}
\]

- Relative equilibria are steady states of slice dynamics
- Eliminate the algebraic equations to obtain an equivalent set of \( n-d \) differential equations
- Linearize about the unstable relative equilibrium and use an equivalent set of \( n-d \) differential equations
- Special case: if the relative equilibrium \( z_i \) is also a steady state of the original equations, and also the template chosen to define the slice, then the following diagram commutes:

\[
\begin{align*}
z &= X(z, u) \quad \text{lin. about } z_i \quad \Rightarrow \quad w = Au + Bu \\
\dot{\tilde{z}} &= X(\tilde{z}, \tilde{u}) - \xi M(\tilde{z}) \quad \text{lin. about } z_i \quad \Rightarrow \quad \dot{w} = \Phi_b(M(\tilde{z})) Au + Bu
\end{align*}
\]

Equivariant actuation
- Assume the dynamics to be affine in control, and actuation \( H(z) \) to be equivariant:

\[
X(z, u) = X(z) \cdot u + H(z) \cdot u = H(\Phi_g(z)) \cdot u = T\Phi_g(H(z)) \cdot u
\]

- Define a quadratic cost, also invariant under the action of \( G \)

\[
j[z, u] = \frac{1}{2} \int (z - z_i, Q(z - z_i)) + (u, R u) \, dt
\]

- Use LQR to determine the optimal feedback gain \( K \)

\[
\dot{u} = \Psi_g \cdot K(\Phi_g^{-1}(z) - z_i)
\]

Amplitude and phase actuation
- We consider \( m \) general actuators, which are not equivariant, but we have a freedom to translate the actuators in the group direction

\[
z = X(z) + \sum_{j=1}^m T\Phi_{h_j} + b_j n_i
\]

- The slice dynamics of these equations

\[
\dot{\tilde{z}} = X(\tilde{z}) + \sum_{j=1}^m T\Phi_{h_j} + b_j u_j - \xi M(\tilde{z})
\]

- Choose the phase \( \phi_j = g \cdot \phi_j \), for constant \( \phi_j \). Choose \( c_j \) such that the resulting control term has no action in the group direction; that is, \( h_j \) does not contribute to \( \xi \)

- Linearize about the relative equilibrium and use LQR to determine the amplitude \( u_j \)

Example 1: Kuramoto-Sivashinsky equation
- A PDE in one spatial dimension, governing flame dynamics in combustion
- Symmetry in the form of translational invariance; vector-field equivariant to the action of \( G \)

\[
\begin{align*}
\xi = -z_{xx} - c_{x} \cdot v_{x,xx} \quad x \in [0, 2\pi] \\
\end{align*}
\]

- Shape dynamics are obtained by subtracting \( \xi_M(z) \), where

\[
\xi_M(z) = \frac{\xi}{\delta z}
\]

- 1-parameter family of traveling waves, same shape, translations of each other

Example 2: Inverted pendulum on a cart
- Four equations of motion, for the cart position, pendulum angle and their velocities
- Symmetry group is \( (G_x) \); equations are invariant under translations of the cart
- Template dynamics are equivalent to those obtained by eliminating the equation for the cart position
- Template-based controller compares with that obtained using controlled Lagrangian, but ensures only local stability

References