Generalized nonholonomic systems and control

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ABSTRACT

In this work we present a class of mechanical systems, which we have called **generalized nonholonomic systems (GNHS)**, given by Lagrangian systems with constraints whose constraint forces do not satisfy D'Alembert principle. They provide an apropriate scenario for the study of mechanical systems with general constraints. We discuss an application of them to the control of servomechanisms, considering as an example the stabilization of the inertia wheel pendulum. Before that, we motivate the definition of a GNHS by analyzing simple mechanical systems with constraints.

1. CONSTRAINED SYSTEMS OF N PARTICLES

Let us call $\mathbf{r}_i \in \mathbb{R}^3$ the position of the i-th particle of the system. Of course, if we apply on it a *known* force $\mathbf{f}_i \in \mathbb{R}^3$, then (if $m_i = 1$)

$$\ddot{\mathbf{r}}_i (t) = \mathbf{f}_i (\mathbf{r}_1 (t), ..., \mathbf{r}_N (t), \dot{\mathbf{r}}_1 (t), ..., \dot{\mathbf{r}}_N (t)),$$

i.e. curves $\mathbf{r}_i (t)$'s satisfy **Newton equations**. Writing

$$\mathbf{R} = (\mathbf{r}_1, ..., \mathbf{r}_N) \in \mathbb{R}^{3N} \qquad and \qquad \mathbf{F} = (\mathbf{f}_1, ..., \mathbf{f}_N) \in \mathbb{R}^{3N}$$

such equations reduce to

$$\ddot{\mathbf{R}} (t) = \mathbf{F} \left(\mathbf{R} (t), \dot{\mathbf{R}} (t)\right).$$

- $3N$ unknowns $\mathbf{R} (t)$, $3N$ normal ODE $\Longrightarrow$ existence and uniqueness of solutions.

Let us consider for this system a set of **constraints**

$$\omega_k\left(\mathbf{R},\dot{\mathbf{R}}\right)=0,\qquad k=1,...,K.$$

Note: We say such constraints are **holonomic** if there exist functions ϕ_k such that

$$\omega_k\left(\mathbf{R},\dot{\mathbf{R}}\right)=\frac{\partial\phi_k}{\partial\mathbf{R}}\left(\mathbf{R}\right)\cdot\dot{\mathbf{R}}\qquad\left(i.e.\;\;\omega_k=\frac{d\phi_k}{dt}\right).$$

In this case last conditions can be derived from equations

$$\phi_k\left(\mathbf{R}\right)=cte,\qquad k=1,...,K.$$

Otherwise, we say the constraints are **nonholonomic**. □

If we ask that $\mathbf{R}(t)$ satisfies the constraints, i.e.

$$\omega_k\left(\mathbf{R}(t),\dot{\mathbf{R}}(t)\right)=0,\qquad k=1,...,K,$$

we must add a force on the system, **the constraint force**, which we shall denote $\mathbf{F}_C$. Then, a trajectory $\mathbf{R}(t)$ of the system must satisfy

$$\ddot{\mathbf{R}}(t)=\mathbf{F}\left(\mathbf{R}(t),\dot{\mathbf{R}}(t)\right)+\mathbf{F}_C(t)\quad and\quad \omega_k\left(\mathbf{R}(t),\dot{\mathbf{R}}(t)\right)=0,$$

for some force $\mathbf{F}_C(t)$.

- $3N+3N=6N$ unknowns $\mathbf{R}(t)$ and $\mathbf{F}_C(t)$; $3N+K$ equations. It is not a good model for the physical system. We need more information; e.g. information about constraint forces.

2. D'ALEMBERT PRINCIPLE

Assume that constraints are linear in velocities, i.e.

$$\omega_k\left(\mathbf{R},\dot{\mathbf{R}}\right)=\boldsymbol{\omega}_k\left(\mathbf{R}\right)\cdot\dot{\mathbf{R}}=0.$$

Defining the distribution $\mathcal{C}_{\mathbf{R}}=\left\{\boldsymbol{\omega}_1\left(\mathbf{R}\right),...,\boldsymbol{\omega}_K\left(\mathbf{R}\right)\right\}^{\perp}$, $\mathbf{R}\in\mathbb{R}^{3N}$, **D'Alembert principle** says:

$$\mathbf{F}_C\big|_{\mathbf{R}}\in\mathcal{C}_{\mathbf{R}}^{\perp}.$$

Equations of motions result

$$\begin{cases}\ddot{\mathbf{R}}\left(t\right)=\mathbf{F}\left(\mathbf{R}\left(t\right),\dot{\mathbf{R}}\left(t\right)\right)+\mathbf{F}_C\left(t\right),\\ \dot{\mathbf{R}}\left(t\right)\in\mathcal{C}_{\mathbf{R}(t)},\quad\mathbf{F}_C\left(t\right)\in\mathcal{C}_{\mathbf{R}(t)}^{\perp}.\end{cases}$$

- $6N$ unknowns and $3N+K+(3N-K)=6N$ equations.

In Lagrangian terms, we have a so-called **nonholonomic system**: a trajectory $\gamma : I \to Q$ of the system is given by

$$\mathcal{EL}\left(L\right)\left(\gamma^{(2)}\left(t\right)\right) = f\left(t\right), \quad \gamma'\left(t\right) \in \mathcal{C}_{\gamma(t)}, \qquad f\left(t\right) \in \mathcal{C}^{o}_{\gamma(t)},$$

where $L : TQ \to \mathbb{R}$, $\mathcal{EL}\left(L\right) : T^{(2)}Q \to T^{*}Q$ and $\mathcal{C} \subset TQ$ is a distributio on Q. If L is **simple** we have existence and uniqueness of solutions.

Some questions:

- Is this principle always valid?
  Systems with constraints implemented by the contact of punctual masses and rigid bodies usually satisfy the principle. But it is easy to build up systems which do not: for instance, using servomechanisms.

- What about more general (nonlinear and/or higher order) constraints?
  Chetaev’s principle is the natural generalization [Chetaev (1934); Valcovici (1958); Pironneau (1983)]. Unfortunately, there do not exist interesting mechanical systems fulfilling this principle.
- Why insist on deriving, by a universal procedure, the space of constraint forces from constraint itself?
  This is, probably, a consequence of a misunderstanding of the concept of virtual displacement; mainly in relation with variational principles.

3. GENERALIZED NONHOLONOMIC SYSTEMS

Idea: Consider the constraints and the space where constraint forces take their values as independent data; and do not attempt to derive one from another by a universal procedure [Dazord (1994); Marle (1996); Cendra et al (2004)].

Generalized nonholonomic systems (GNHS):

1. Data: $(L, \mathcal{C}, \mathcal{F})$, $\mathcal{C} \subset TQ$, $\mathcal{F} \subset T^*Q$ a codistribution.
2. Equations of motion:

$$\mathcal{EL}\left(L\right)\left(\gamma^{(2)}\left(t\right)\right) = f\left(t\right), \quad \gamma'\left(t\right) \in \mathcal{C}_{\gamma(t)}, \quad f\left(t\right) \in \mathcal{F}_{\gamma(t)}.$$

D'Alembert principle: $\mathcal{F} = \mathcal{C}^o$.

In general [Cendra et al (2004); Cendra & Grillo (2007)]:

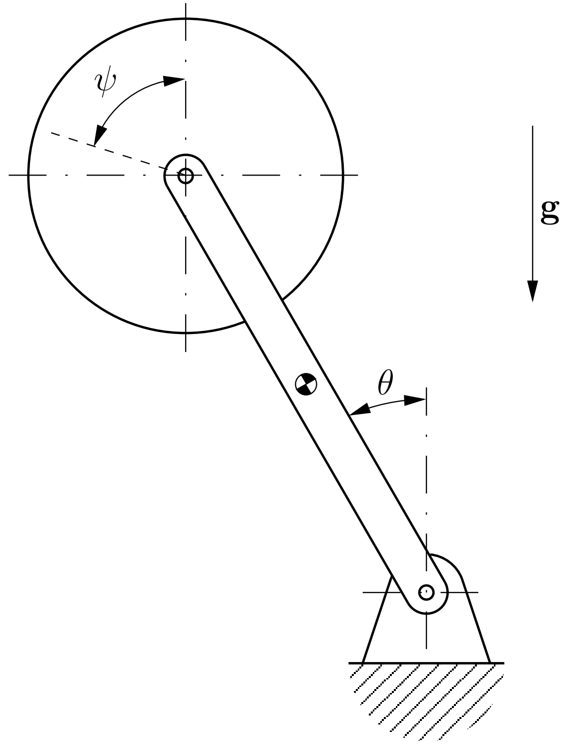
1. Data: $(L, \mathcal{C}, \mathcal{F})$, $\mathcal{C} \subset T^{(k)}Q$, $\mathcal{F} \subset T^{(l)}Q \times_Q T^*Q$.
2. Equations of motion:

$$\mathcal{EL}\,(L)\left(\gamma^{(2)}\,(t)\right) = f\,(t)\,, \quad \gamma^{(k)}\,(t) \in \mathcal{C}_{\gamma(t)}, \quad \left(\gamma^{(l)}\,(t)\,, f\,(t)\right) \in \mathcal{F}_{\gamma(t)}$$

Examples: Elastic rolling bodies (like pneumatic tires), systems with friction forces, underactuated systems...

4. CONTROL OF SERVOMECHANISMS

Inertia wheel pendulum: $Q = S^1 \times S^1$



- Lagrangian: $L = \frac{1}{2}\, I\, \dot{\theta}^2 + \frac{1}{2}\, J\, \left(\dot{\theta} + \dot{\psi}\right)^2 - M\, g\, (1 + \cos\theta)$.
- Space of actuators: $\mathcal{F}_{(\theta,\psi)} = \{(f_\theta, f_\psi) : f_\theta = 0\} \subset T^*_{(\theta,\psi)} Q$.

(Almost) Every triple $(L, \mathcal{C}, \mathcal{F})$ with second (or lower) order constraints defines a Lagrangian system with external forces (L, u), with $u : TQ \to T^*Q$, the related constraint force, such that $\mathrm{Im}\,(u) \subset \mathcal{F}$. The pair (L, u) can be interpreted as a **closed-loop mechanical system (CLMS)**, being u the *control law*. Then, given an underactuated system $(L, \mathcal{F})$ and a set of constraints $\mathcal{C}$, we have in this way a CLMS

$$(L, \mathcal{F}) \oplus \mathcal{C} \rightsquigarrow (L, \mathcal{C}, \mathcal{F}) \rightsquigarrow (L, u)\,, \quad \mathrm{Im}\,(u) \subset \mathcal{F}.$$

Idea: In order to stabilize a given underactuated system $(L, \mathcal{F})$, fix a set of constraint $\mathcal{C}$ that make stabilization possible, and derive control law u as the related constraint force [Marle (1996); Shiriaev, Perram, Canudas-de-Wit (2005)].

Example (inertia wheel pendulum): We can consider the kinematic constraints

$$\mathcal{C}_{(\theta,\psi)} = \left\{ \left(\dot{\theta}, \dot{\psi}\right) : \dot{\theta} + a\,\dot{\psi} = b\,\sin\theta \right\},$$

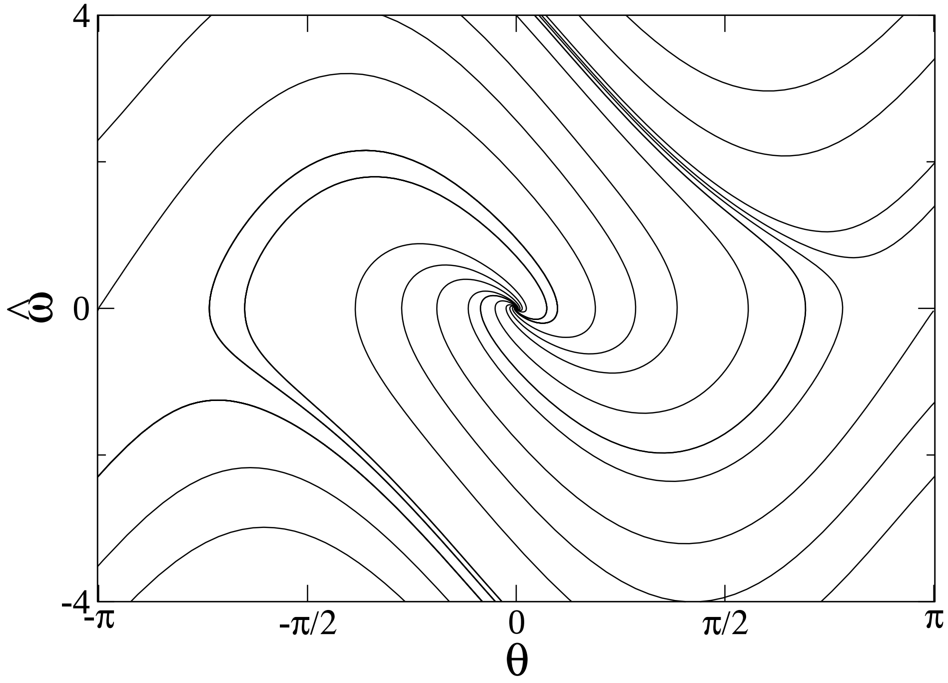
and implement it by using the actuators (i.e. a torque on the disc), so

$$\mathcal{F}_{(\theta,\psi)} = \{(f_\theta, f_\psi) : f_\theta = 0\}\,.$$

This gives a CLMS with $u = (u_\theta, u_\psi) = (0, u_\psi)$, where

$$u_\psi\left(\theta, \psi, \dot{\theta}, \dot{\psi}\right) = \frac{(I\,J\,b/a)\ \dot{\theta}\ \cos\theta + (1 - 1/a)\ M\,g\,J\ \sin\theta}{I + J - J/a}\,.$$

For certain values of a and b we have quasi-global asymptotic stability:



Here, $\widehat{\omega}$ is a variable proportional to $\dot{\theta}$ [Pérez (2006)].

Final comments

- Is this method systematic?

It is, by now, just an alternative.

- Any CLMS can be constructed from a GNHS; i.e. any control law can be seen as the constraint force related to a given set of constraints?

It can be shown that every CLMS is equivalent to a second order GNHS. So, there is a deep connection between CLMS and constrained mechanical systems [Grillo, Maciel & Pérez (2008)].

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