Constrained Discrete Mechanics and Optimal Control - applied to the optimal pitch

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The optimal control of multi-body dynamics is a challenging task. This poster describes an efficient numerical method to find optimal control policies for constrained systems based on variatonal discrete mechanics. As an example of biomotion in sports the optimal pitch of an athlete is investigated where the arm is modelled as kinematic chain.

Optimal control problem

Let Q be a n-dimensional manifold. Find the optimal configuration $q(t) \in Q$ and control force trajectories $f(t)$ that

minimize an objective function $J(q, \dot{q}, f)$

subject to

- the dynamics of a forced mechanical system with Lagrangian $L(q(t), \dot{q}(t))$
- with holonomic constraints that determine the (n-m)-dimensional constraint manifold

$$C = \{q(t) \in Q \mid g(q(t)) = 0\}$$

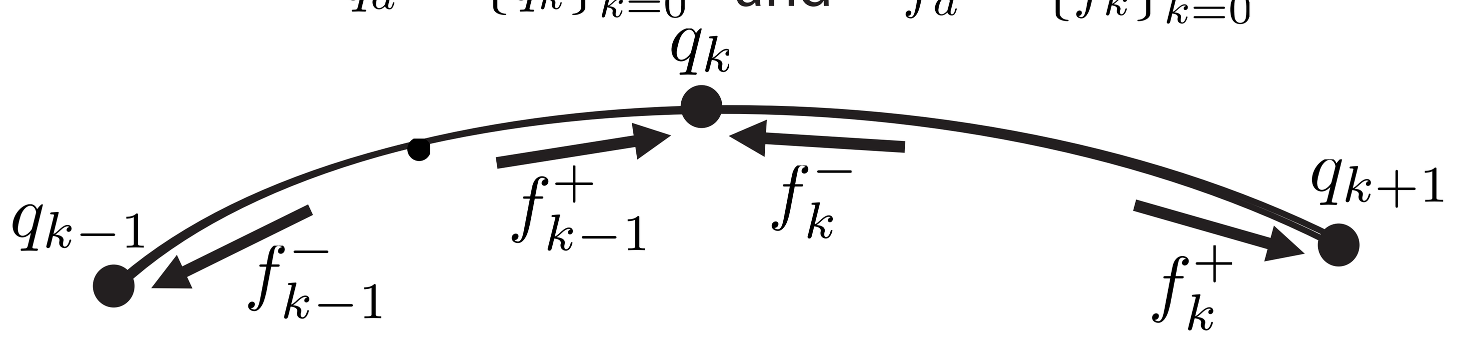
- and boundary conditions

$$r(q(0), \dot{q}(0), q(T), \dot{q}(T))$$

Discretized problem

Consider sequences of discrete configurations and controls

$$q_d = \{q_k\}_{k=0}^N \quad \text{and} \quad f_d = \{f_k\}_{k=0}^N$$



The discrete formulation transforms the optimal control problem into a constrained optimization problem

$$\min_{q_d, f_d} J_d(q_d, f_d)$$

$$\text{s.t. } \delta \sum_{k=0}^{N-1} L_d(q_k, q_{k+1}) - \tfrac{1}{2} g_d^T(q_k) \cdot \lambda_k - \tfrac{1}{2} g_d^T(q_{k+1}) \cdot \lambda_{k+1} + \sum_{k=0}^{N-1} f_k^- \cdot \delta q_k + f_k^+ \cdot \delta q_{k+1} = 0$$

$$r_d(q_0, q_1, q_{N-1}, q_N, f_{N-1}, f_N) = 0$$

with $\lambda_k \in \mathbb{R}^m$ the discrete Langrange multipliers

$\Rightarrow N(n+m)$ discrete variables and Nn discrete controls

Discrete nullspace method

The redundant system can be transformed into a system of minimal dimension of configurations and controls. With the choice of a discrete nodal reparametrization $q_k = F(u_k, q_{k-1})$ such that the constraints are fulfilled, the discrete variational principle leads to

$$\left(\frac{\partial F}{\partial u_k}\right)^T \cdot \left(D_1 L_d(q_k, q_{k+1}) + D_2 L_d(q_{k-1}, q_k) + f_{k-1}^+ + f_k^-\right) = 0$$

Choice of discrete control forces

In practical applications the control forces are usually given as generalized forces. Then, the redundant forces are computed as

$$f_{k-1}^+ = B^T(q_k) \cdot \tau_{k-1}^+ \quad \text{and} \quad f_k^- = B^T(q_k) \cdot \tau_k^+$$

such that $\left(\frac{\partial F}{\partial u_k}\right)^T \cdot (f_{k-1}^+ + f_k^-)$ is the actuation of the generalized degree of freedom effected by the generalized forces.

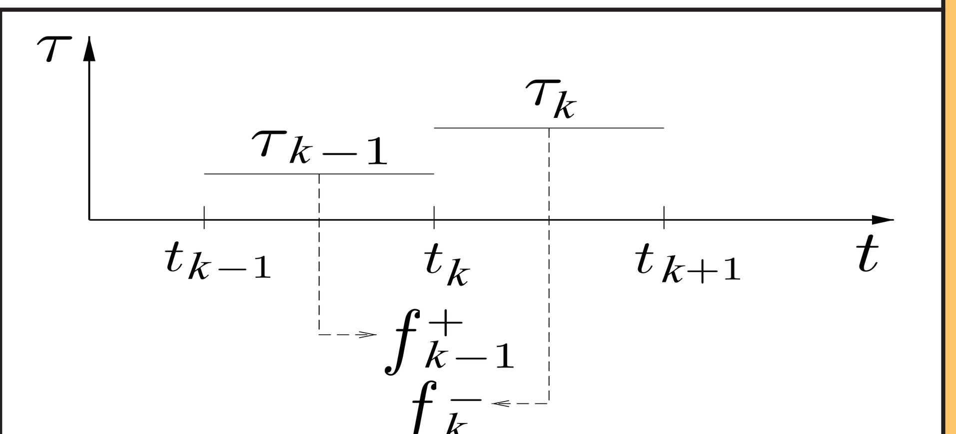


Fig: Relation of redundant forces to discrete generalized forces.

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constrained optimization problem of minimal dimension

$$\min_{u_d, \tau_d} \bar{J}_d(u_d, \tau_d)$$

s.t. reduced discrete dynamical system and boundary conditions

$\Rightarrow N(n-m)$ configurations, $N(n-m)$ controls

Proposition: The definition of the redundant left and right discrete forces guarantees that the change of angular momentum along the solution trajectory is induced only by the effect of the discrete generalized forces. In particular, it is conserved exactly, if the motion of the pair is induced by shape changes only.

Application to the optimal pitch

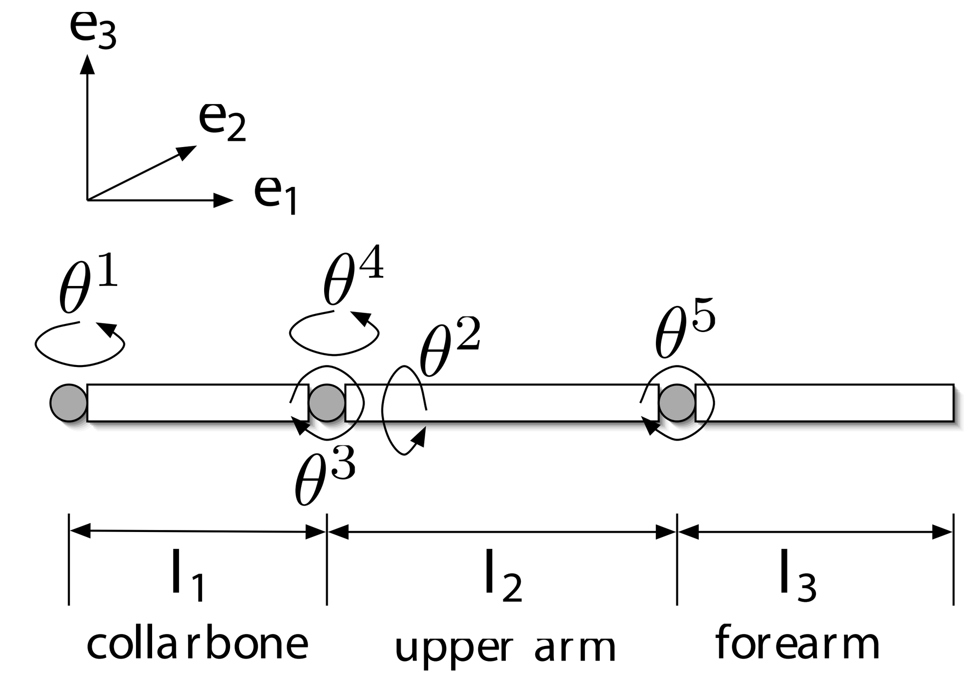
Goal: Maximize the final moment in pitching direction and determine the optimal pitch duration

Model: kinematic chain, 3 rigid bodies

- 5 generalized coordinates

$$\theta \in S^1 \times \mathfrak{so}(3) \times S^1$$

- 5 generalized control torques $\tau \in \mathbb{R}^5$ acting in the joints

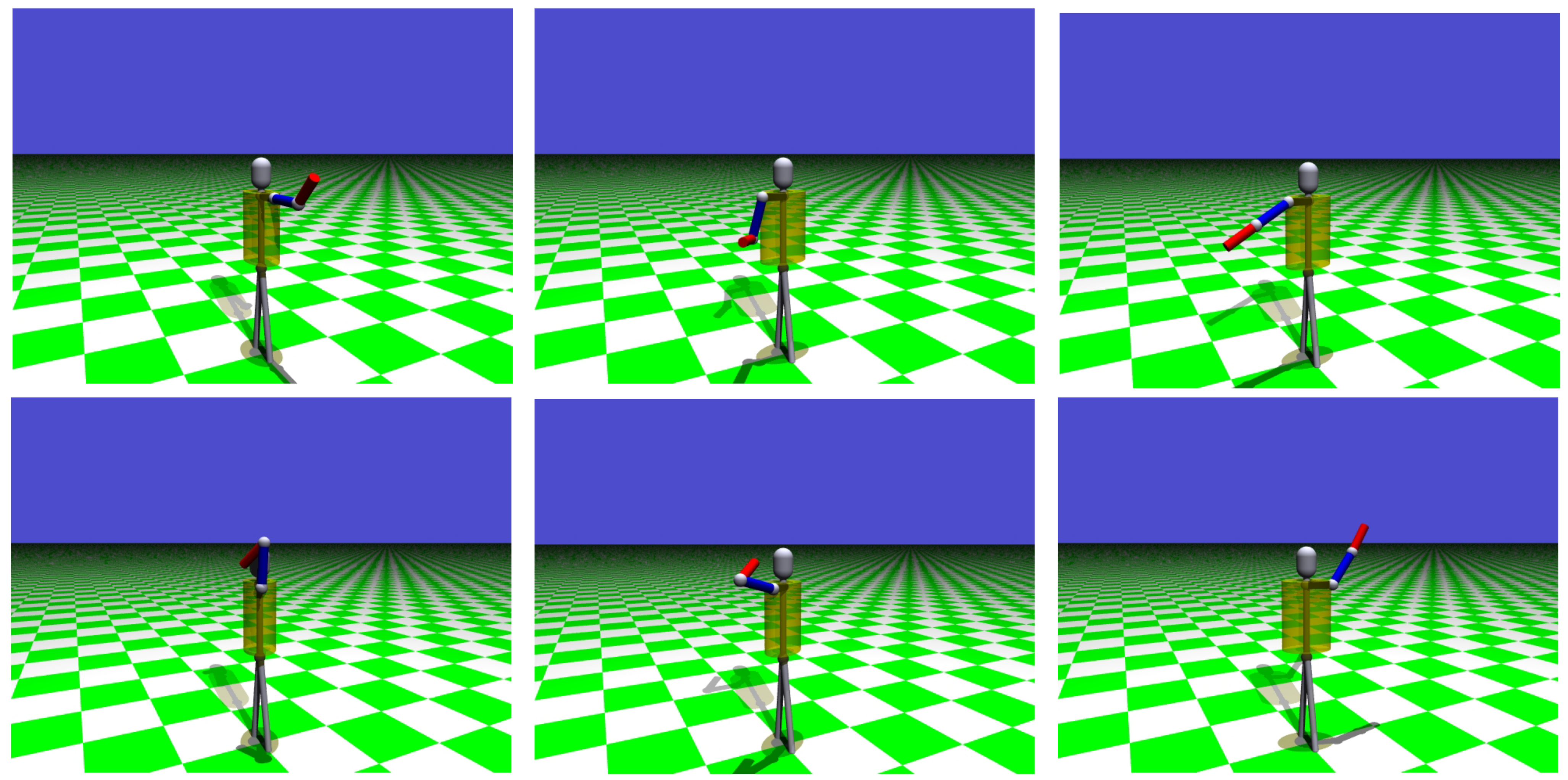


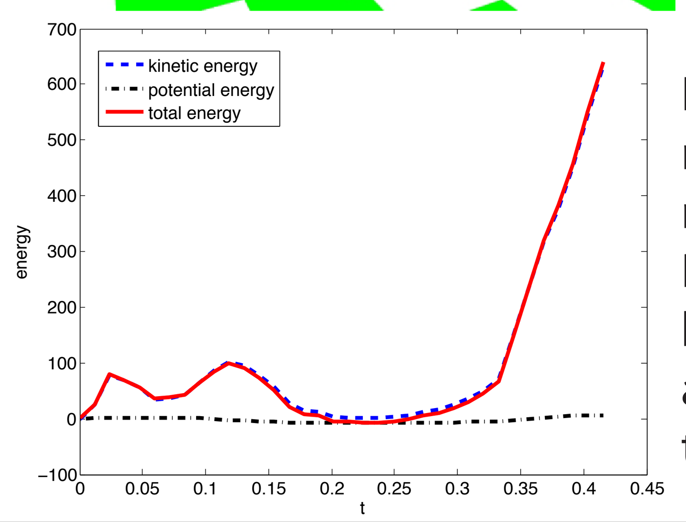
Additional constraints:
bounds on configuration and control torques, fixed initial state, target manifold for the final hand position

Objective function: $\max_{\theta_d, \tau_d, T} \; e_2^T \cdot Q(q_N) \cdot p_N^+$

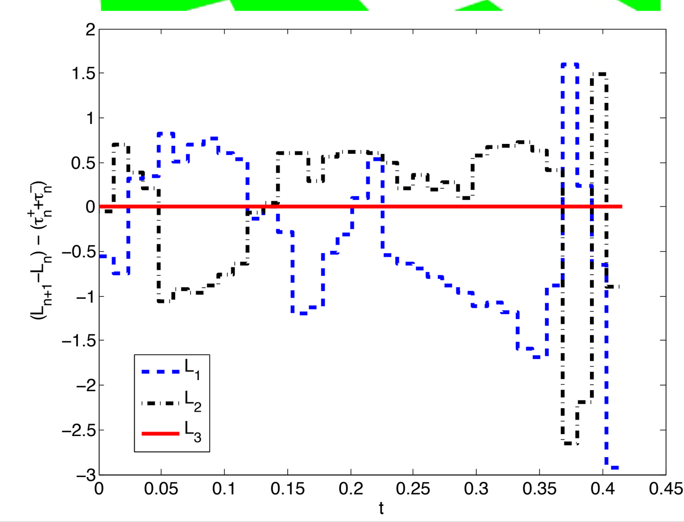
with Q being the projection onto the (n-m)-dimensional submanifold fulfilling the kinematic constraints on momentum level.

Numerical results: The pitcher strikes his arm out, moves it rearwards, pulls it above his head, before he finally moves his arm like a whip.





Left: The kinetic energy increases monotonously in the end of the motion.
Right: The evolution of the angular momentum around the vertical axis is exactly given by the control torque acting at the collarbone.



References:
Ober-Blöbaum, S.: *Discrete Mechancis and Optimal Control*, Dissertation, UPB, 2008
Leyendecker, S., Ober-Blöbaum, S., Marsden, J.E., Ortiz, M.: *Discrete Mechanics and Optimal Control for constrained systems*, Preprint

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