Spiral Waves and Spiral Solitons in Lattice Differential Equations

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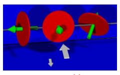
I The Isotropic XY Chain

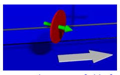
Local Hamilton function:

$$H(n) = -\sum_{x=-n}^{n-1}\left[s_1(x)s_1(x+1) + s_2(x)s_2(x+1)\right] - 2\lambda \sum_{x=-n}^{n} s_3(x) \qquad (1.1)$$

$s_i(x,t) \in \mathbb{R}$; $i = 1,2,3$: Spin components at the site $x \in \mathbb{Z}$ and time $t \in \mathbb{R}$.



Nearest neighbour interaction.



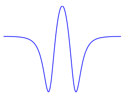
External magnetic field of strength $\lambda \in \mathbb{R}$.

Using $\dot{s}_i(x,t) = \lim_{n\to\infty}\left[s_i(x,t), H(n)\right]$, we obtain the following equations of motion:

$$\begin{aligned}
\dot{s}_1(x,t) &= -s_3(x,t)\left[s_2(x+1,t) + s_2(x-1,t)\right] + 2\lambda s_2(x,t)\\
\dot{s}_2(x,t) &= s_3(x,t)\left[s_1(x+1,t) + s_1(x-1,t)\right] - 2\lambda s_1(x,t)\\
\dot{s}_3(x,t) &= -s_2(x,t)\left[s_1(x+1,t) + s_1(x-1,t)\right]\\
&\quad + s_1(x,t)\left[s_2(x+1,t) + s_2(x-1,t)\right]
\end{aligned} \qquad (1.2)$$

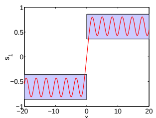
We restrict ourselves to spins of the absolute value $\sqrt{s_1^2(x,t) + s_2^2(x,t) + s_3^2(x,t)} = 1$.

Discrete Symmetry Group



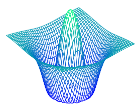
Coupled bistable units.

For coupled bistable units, travelling wave *kink solutions* can be observed. Their tails are trapped by energetic arguments to the neighbourhood of two degenerate groundstates.



Kink solution.

Continuous Symmetry Group



Coupled SO(2) symmetric units.

In the case of a continuous symmetry group, the tails of generalized kink solutions are no longer trapped. They might drift along orbits of the symmetry group.

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II «Spiral Like» Solutions

2.1 Spiral Waves

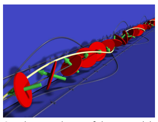
Spiral waves are solutions of the form

$$s(x,t) = \left(a\cos(\varphi_0 + \Delta x + \omega t), a\sin(\varphi_0 + \Delta x + \omega t), b\right)^T \qquad (2.3)$$

with $a, b \in \mathbb{R}$, the angular shift $\Delta \in \mathbb{R}$ and the angular velocity $\omega \in \mathbb{R}$.

If we plug this ansatz into the equations of motion (1.2), we obtain

$$\omega = 2\sqrt{1-|a|^2}\cos\Delta\,. \qquad (2.4)$$



Spiral wave solution of the XY model.

2.2 Spiral Solitons

Definition 2.1 *A L^∞ rotating travelling wave solution*

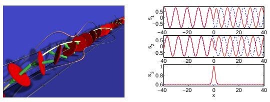
$$s(x,t) = \left(\mathrm{Re}(e^{i\omega t}\eta(x-ct)), \mathrm{Im}(e^{i\omega t}\eta(x-ct)), \sqrt{1-|\eta(x-ct)|^2}\right) \qquad (2.5)$$

of equation (1.2) with $\eta: \mathbb{R} \to \{z \in \mathbb{C} \mid |z| \le 1\}$, the angular velocity $\omega \in \mathbb{R}$ and the velocity $c \in \mathbb{R}$ is called a spiral soliton solution *if there exist an angular shift $\Delta \in \mathbb{R}$ and two constants $\eta_\pm \in \mathbb{C}$ with*

$$\lim_{x\to\pm\infty} e^{-i\Delta x}\eta(x) = \eta_\pm \qquad (2.6)$$

and $\eta_- \neq e^{-i\Delta x}\eta(x)$ for at least one $x \in \mathbb{R}$. The latter condition holds automatically if $\eta_- \neq \eta_+$. By $\Delta_\infty = \arg\eta_+ - \arg\eta_-$, we denote the asymptotic phase shift *of the spiral soliton solution. In many cases, η_- and η_+ have the same absolute value. Then, we denote by $r_\infty = |\eta_\pm|$ the* asymptotic amplitude.

Spiral solitons interpolate spatially between a pair of spiral wave solutions. They can be understood as an (essentially) localized defect (i.e., a phase shift) which travels with the velocity c through the lattice.



Spiral soliton solution of the XY model with velocity $c = 1$.

Theorem 2.2 *Let $c \in]-2,2[$, $\xi = x - ct$, $a \in]-1,0[\cup]0,1[$ and $\varepsilon > 0$ sufficiently small. Then, equation (1.2) has spiral soliton solutions of velocity c and asymptotic amplitude $r_\infty = \varepsilon$. These solutions have the form*

$$\eta(\xi) = \varepsilon e^{i(\varphi_0 + \Delta\xi + \omega t)} g(\varepsilon)(\varepsilon\xi) \qquad (2.7)$$

with $\varphi_0 \in \mathbb{R}$, $\Delta = \varepsilon\frac{a}{\sqrt{2}} + \arcsin\frac{c}{2}$, $\omega = 2\sqrt{(1-\varepsilon^2)}\cos\Delta$ and $g(\varepsilon) \in L^\infty(\mathbb{R},\mathbb{C})$. The function $g(\varepsilon)$ depends continuously on ε in the L^∞ norm and behaves as $\varepsilon \to 0$ asymptotically like

$$g(0)(x) = a + i\sqrt{1-a^2}\tanh\left(\sqrt{\frac{1-a^2}{2}}\,x\right). \qquad (2.8)$$

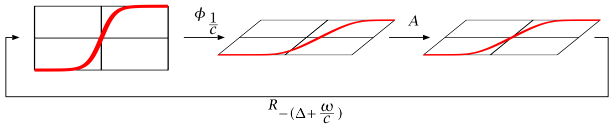
The asymptotic phase shift Δ_∞ depends continuously on ε and tends to $2\arccos a$ as $\varepsilon \to 0$.

In the case of $c = 0$, the function $g(\varepsilon)$ is evaluated only at the discrete set of points $x \in \varepsilon\mathbb{Z}$. Therefore, every function $g(\varepsilon)$ leads by the shift $x \to x + x_0$ with $x_0 \in \mathbb{R}$ to a one parameter family of spiral soliton solutions.

III Spectral Stability

3.1 Strategy

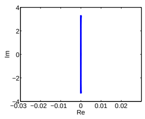
- Transform Spiral wave or Spiral soliton solution into the fixed point of an equivalent dynamical system.
- Linearize the transformed equations of motion around the fixed point.
- Spectral analysis of this linearization with respect to l^2 perturbations.
- The spectrum provides information on the stability properties of the fixed point (and, thus, of the original solution).



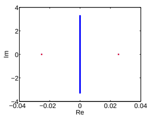
Construction of a fixed point corresponding to a spiral soliton with nonzero velocity $c \neq 0$.

3.2 Stationary Spiral Soliton Solutions

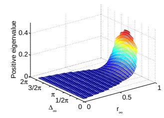
CB solutions turn out to be spectrally stable, while CS solutions are linearly unstable.



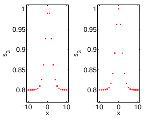
Spectrum of a CB solution for $r_\infty = 0.8$ and $\Delta_\infty = 2.59$.



Spectrum of a CS solution for $r_\infty = 0.8$ and $\Delta_\infty = 2.59$.



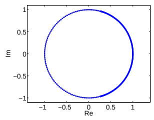
Positive eigenvalue of the CS solution as a function of r_∞ and Δ_∞.



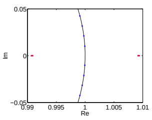
CB and CS solutions.

3.3 The Case of Nonzero Velocity

Single spiral soliton solutions with velocity $c \neq 0$ are always spectrally stable. The concatenation of multiple spiral soliton solutions of the same velocity c becomes linearly unstable.



Spectrum of a spiral soliton solution with $c = 1$, $r_\infty = 0.3$ and $\Delta_\infty = 2.63$. The solution is spectrally stable.

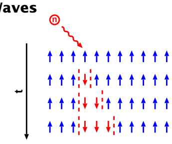


Detail of the spectrum of a double spiral soliton solution with $c = 0.5$, $r_\infty = 0.3$ and $\Delta_\infty = 5.19$. The solution is linearly unstable.

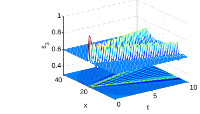
IV Interactions

4.1 Perturbation of Spiral Waves

FRM II, Garching.



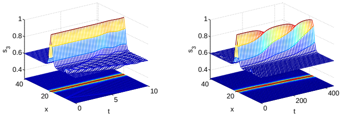
Schematic run of a neutron scattering experiment.



Time development of a local perturbation applied to a spiral wave solution with $a = 0.8$, $\Delta = 0.8$ and $\omega = 1.2$.

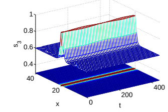
4.2 Perturbation of Spiral Solitons

The effect of a small perturbation on an (unstable) CS soliton. The solution begins to travel through the lattice with some small velocity $c \neq 0$.



Local perturbation of an (unstable) CS solution with $r_\infty = 0.8$ and $\Delta_\infty = 2.59$.

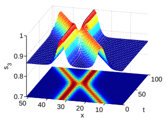
On the other hand, a CB soliton is stable under the same perturbation. It only starts oscillating around its original position. This effect is special to spatially discrete systems (have in mind that in PDEs there is no distinction between CS and CB solitons) and is called *pinning* or *propagation failure.*



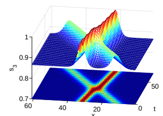
Local perturbation of a (stable) CB solution with $r_\infty = 0.8$ and $\Delta_\infty = 2.59$.

4.3 Collision of Spiral Solitons

Spiral solitons interact by a phase shift. This is the defining property of *solitons* in the classical sense.

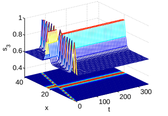


Collision of two spiral solitons with $r_\infty = 0.5$, $\Delta = 0$, $c_1 = 0$ and $c_2 = 0.3$.

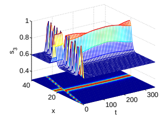


Collision of two spiral solitons with $r_\infty = 0.5$, $\Delta = 0$, $c_1 = -0.1$ and $c_2 = 0.5$.

Of particular interest are collisions where spiral solitons of zero velocity are involved. On the left hand side, the phase shift transforms a CS soliton into a CB soliton. The energetic difference between both solutions is emitted as an additional pair of high-speed spiral solitons.



Collision of two spiral solitons with $r_\infty = 0.8$, $\Delta = 0.2$, $c_1 = 0$ and $c_2 = 0.4422$.



Collision of two spiral solitons with $r_\infty = 0.8$, $\Delta = 0.2$, $c_1 = 0$ and $c_2 = 0.5$.