# Point vortices interacting with a planar rigid body


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## Abstract


*A significant step towards understanding the dynamics of marine animals or underwater vehicles consists of finding out how rigid bodies interact with fluids. Different groups of researchers have studied the latter problem with a view towards applications; here we take a new look at this problem, focussing on the underlying geometry. This is joint work with Jerrold Marsden and Eva Kanso.*


## 1. Overview and problem setting

We consider the dynamics of $N$ point vortices interacting with a planar rigid body. The dynamics of system was first described by Shashikanth et al. [2002] (**SMBK**) and Borisov et al. [2003] (**BMR**), who discovered two different but equivalent formulations of the equations of motion by straightforward calculation. However, many questions remain: most notably, *is there anything intrinsic to be said about these equations, and why are there different formulations of the dynamics*?

Even though the interest in this system is fairly recent, closely related dynamical systems are very well known.

1. **Kirchhoff (1877)** derived the equations of motion for a *rigid body moving in potential flow*:

$$\dot{\mathbf{L}} = \mathbf{A} \times \mathbf{V} \quad \text{and} \quad \dot{\mathbf{A}} = \mathbf{A} \times \mathbf{\Omega}.$$

Here $\mathbf{L} = m\mathbf{V}$ and $\mathbf{A} = \mathbf{I}\mathbf{\Omega}$ are the translational and rotational momentum of the body, respectively.

2. **Lin (1941)** showed that the motion of *point vortices in a bounded domain in* $\mathbf{R}^2$ are Hamiltonian:

$$\Gamma_k \frac{d\mathbf{x}_k}{dt} = J \frac{\partial H}{\partial \mathbf{x}_k},$$

where $H = -W_G(\mathbf{x}_1, \ldots, \mathbf{x}_N)$ and $W_G$ is the *Kirchhoff-Routh function*.

The Kirchhoff equations are Lie-Poisson on $se(3)^*$; this observation was used by Leonard *et al.* who studied the stability of bottom-heavy underwater vehicles. A similar system was used by Kanso et al. [2005] to study bio-locomotion in a perfect fluid. Marsden and Weinstein (1983) showed that the point vortex system arises by symplectic reduction from the Euler equations. These underlying geometric properties partly explain the remarkable accuracy of the Chorin vortex method.

Since both subsystems benefit from reduction theory, it is likely that the combined dynamics of the rigid body interacting with point vortices will also yield to geometric methods. In this poster we gather some evidence for that claim:

- we obtain the Poisson structures of SMBK and BMR and the Poisson map linking them through **reduction by stages**;
- we show how classical results like the **Kutta-Joukowski force** are geometric in nature, arising because of non-zero curvature;
- we make the link with other areas of interest, including **magnetic dynamics** and **coupling to a Lie group**.

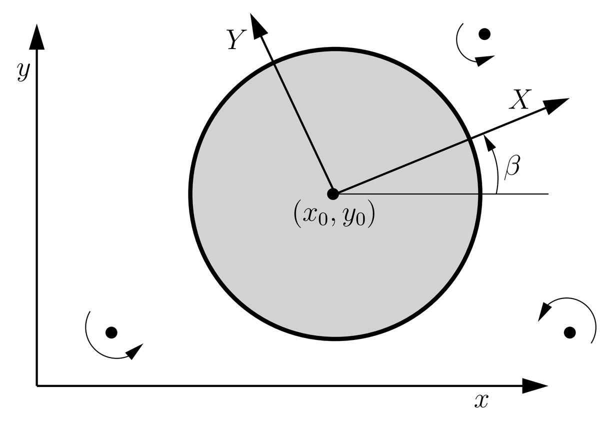



**Figure.** The planar rigid body interacting with point vortices.

## 2. Reduction by stages

We start with a mechanical system on an infinite-dimensional configuration space, keeping track of the individual fluid particles and the rigid body. The configuration space is

$$Q = \mathrm{Emb}_{\mathrm{vol}}(\mathcal{F}_0, \mathbf{R}^2) \times SE(2),$$

of volume-preserving embeddings of the fluid reference space $\mathcal{F}_0$ into $\mathbf{R}^2$, times the Euclidian group in the plane.

On $Q$, two different groups act:

1. The group of **volume-preserving diffeomorphisms** acts on $Q$ from the right, expressing *particle relabelling symmetry*.
2. $SE(2)$, the **Euclidian group** of the plane, acts on $Q$ from the left, since the system is invariant under rotations and translations of both the fluid and the solid.

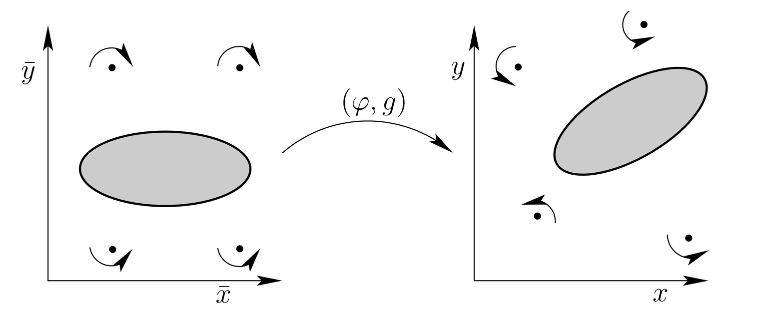



**Figure.** The configuration space of the solid-fluid system, consisting of mappings from the reference configuration into the plane.

## 3. Particle relabelling symmetry

After reduction by the particle relabelling symmetry, we obtain a system on

$$T^*SE(2) \times \mathbf{R}^{2N},$$

where the individual factors describe the rigid body and the point vortices, respectively.

The reduced symplectic form is the sum of the canonical form on $T^*SE(2)$ and a **magnetic term** $\beta_\mu$ on $SE(2) \times \mathbf{R}^{2N}$:

$$\beta_\mu = \sum_{i=1}^{N} \Gamma_i \big( -\mathrm{d}x_i \wedge \mathrm{d}y_i + \mathrm{d}\Psi_A(\mathbf{x}_i) \wedge \mathrm{d}\xi_A \big).$$

The functions $\Psi_A$ are elementary stream functions encoding the response of the fluid to translations and rotations of the rigid body: the magnetic form hence ***encodes the effects of the ambient fluid***. The Helmholtz-Hodge decomposition determines a connection on $Q$, and $\beta_\mu$ is secretly the **curvature** of this connection.

## 4. The $SE(2)$ symmetry

The resulting system is invariant under translations and rotations. To get rid of this symmetry, we use **Poisson reduction**. Physically speaking, this corresponds to rewriting the solid+fluid system in body coordinates.

- By Poisson reducing the magnetic symplectic structure $\Omega - \beta_\mu$, we obtain the non-standard Poisson structure of Borisov, Mamaev and Ramodanov (**BMR**).
- The momentum map of the $SE(2)$ symmetry can be used to construct a **shift map** from $\mathbf{R}^{2N} \times SE(2)$ to itself.
- Pulling back the BMR Poisson structure by the shift map gives the canonical Poisson structure on $\mathbf{R}^{2N} \times se(2)^*$:

$$\{f, g\}_Q = \{f, g\}_{\mathrm{Lie-Poisson}} + \{f, g\}_{\mathrm{vortex}},$$

where the first term is just the Lie-Poisson bracket on $se(2)^*$ and the second term is the vortex bracket on $\mathbf{R}^{2N}$. This is the Poisson structure used by Shashikanth, Marsden, Burdick and Kelly (**SMBK**).

A similar shift map was used by Krishnaprasad and Marsden (1986) in the theory of coupling to a Lie group.

## 5. Illustration: the Kutta-Joukowski force

Classically, the Kutta-Joukowski force on a rigid body moving with velocity $U$ in a fluid with circulation $\Gamma$ and unit density is a force with magnitude $\Gamma U$ at right angles to the velocity.

Having circulation $\Gamma$ is equivalent to placing a vortex of strength $\Gamma$ at the center of the rigid body. Using this ansatz gives the following for the magnetic two-form:

$$\beta_\mu = -\Gamma \mathbf{e}_x^* \wedge \mathbf{e}_y^*,$$

where $\{\mathbf{e}_x, \mathbf{e}_y\}$ is a basis of $\mathbf{R}^2 \subset se(2)$.

The equations of motion can then be written as

$$i_X(\Omega_{\mathrm{can}} - \beta_\mu) = \mathrm{d}H \quad \Leftrightarrow \quad i_X \Omega_{\mathrm{can}} = \mathrm{d}H + i_X \beta_\mu.$$

The set of equations on the right-hand side are Hamilton's equations under the influence of a gyroscopic form $i_X\beta_\mu$. In coordinates, this is precisely the Kutta-Joukowski force.

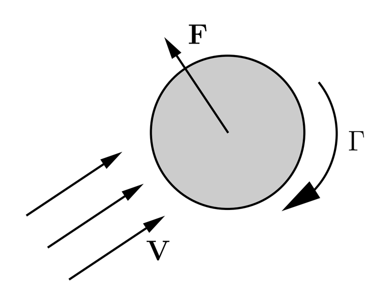
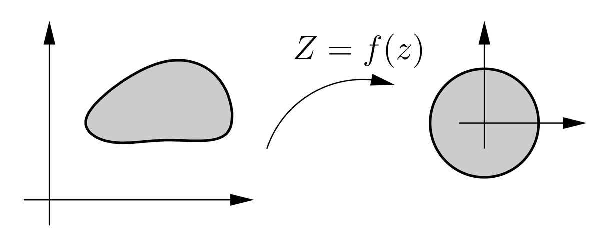



**Figure.** The Kutta-Joukowski force.

## 6. Conclusions and outlook

We have seen how the dynamics of a rigid body interacting with point vortices can be derived in a systematic way by using successive reductions. Even though simplifying assumptions were made along the way, these results continue to hold under less stringent assumptions.

1. **Bodies of arbitrary shape.** By using conformal mapping techniques, an arbitrary closed shape can be mapped onto the unit disc. By pulling back the results obtained for circular bodies, we can therefore extend these methods for bodies of arbitrary shape.




2. **Different distributions of vorticity.** Considering a different vorticity distribution is equivalent to focussing on a different co-adjoint orbit of $\mathrm{Diff}_{\mathrm{vol}}$, but the intrinsic geometry stays the same. For instance, for $N$ vortex rings the co-adjoint orbit would be the space of knots and links with the Arnold-Marsden-Weinstein symplectic structure.

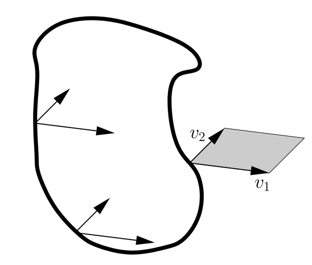


**Figure.** The Arnold-Marsden-Weinstein symplectic structure.

3. **Three-dimensional bodies.** The magnetic form ultimately relies on the Helmholtz-Hodge decomposition, which is independent of dimension. Hence there is also a magnetic form for 3D bodies, interacting with (say) vortex rings.
4. **Stability.** Kanso and Oskouei (2008) study the linear stability of the translational relative equilibrium (Föppl). To prove nonlinear stability, the energy-momentum method or topological methods may be used. For this, a good insight into the geometry of the system is required.
5. **Numerics.** Due to its strong geometric properties, the vortex patch method is remarkably accurate for the simulation of vortical flows. An extension of this method to cover moving boundaries would be interesting.

## References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