TECHNISCHE UNIVERSITÄT MÜNCHEN

Meshfree Numerics for Transfer Operators

Wissenschaftliches Rechnen (M3)

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Problem setting

Dynamical system

- Deterministic: $S : \Omega \to \Omega$
- Stochastic: $p : \Omega \times \mathcal{B} \to [0,1]$, the transition function
  Treated as a random pert. of the deterministic case

Transfer operator

$$P\mu = \mu \circ S^{-1} \qquad \text{or} \qquad P\mu(A) = \int_\Omega p(x,A)\mu(\mathrm{d}x)$$

for $\mu \in \mathcal{M}$.
Or for *densities*

$$\int_A Pf = \int_{S^{-1}(A)} f \qquad \text{or} \qquad Pf(y) = \int_\Omega k(x,y)f(x)m(\mathrm{d}x)$$

Spectral properties

Let $Pf = \lambda f$. Then:

$\underline{\lambda = 1}$

In the case of ergodicity, the eigenfunction characterizes long term behaviour:

$$\lim_{N\to\infty} \frac{1}{N}\sum_{k=1}^{N} \psi\left(S^k x\right) = \int \psi f$$

$\underline{\lambda < 1}$

The state space Ω can be splitted into two almost invariant sets, $\mathrm{supp}(f^+)$ and $\mathrm{supp}(f^-)$. The eigenvalue λ is a measure for the almost invariance of these sets.

Long term dynamical behaviour
$\Updownarrow$
Eigenvalue problem in L^2

Idea

Minimize residual

Given approximation space V_n, compute:

$$f_n = \arg \min_{\substack{g \in V_n \\ \|g\|_{L^2}=1}} \|Pg - \lambda_n g\|_{L^2}$$

Theorem. Let $\lambda \in \mathbb{R}$, $P : L^2 \to L^2$ be a bounded operator and $V_n \subset L^2$ a sequence of subspaces with the following properties:

(a) ex. unique eigenvector f,

(b) $0 < C := \inf\left\{ \|(P - \lambda I)g\|_{L^2} \mid g \in f^\perp, \|g\|_{L^2} = 1 \right\}$
„Separation in residual sense“

(c) $\mathrm{dist}(f, V_n) \overset{n\to\infty}{\longrightarrow} 0$.

Then the sequence $\{f_n\}$ with

$$f_n := \arg \min_{\substack{g \in V_n \\ \|g\|_{L^2}=1}} \|Pg - \lambda g\|_{L^2}$$

satisfies

$$\|f_n - f\|_{L^2} = \mathcal{O}\left(\mathrm{dist}\left(f, V_n\right)\right).$$

Compact and selfadjoint operators fulfill assumption (b).

Numerical computation

Central issues

- *Basis functions:* Gaussians with n-dependent variance.
- *Norm computation:*

$$\|f\|_{L^2}^2 \approx \frac{m(\Omega)}{N}\sum_{k=1}^{N} f(s_k)^2$$

Needs pointwise evaluation of Pf. Leads to

$$\min_{\|w\|=1} \|(A - \lambda B)w\|_2$$

- *Computing the eigenvalue:* Find local minima of

$$\lambda \mapsto \|(A - \lambda B)w\|_2$$

Use the reduced QR factorization of $B = QR$ to obtain

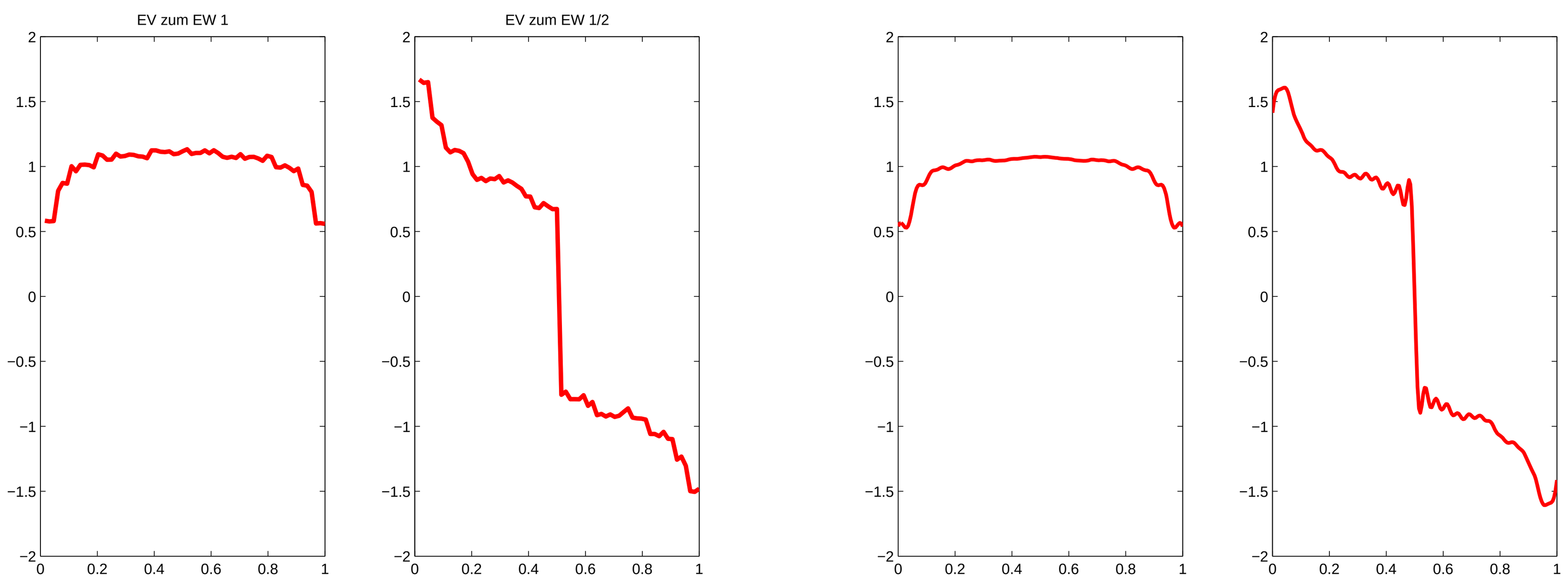
$$w^\top (A^\top Q - \lambda R^\top) R w \overset{!}{=} 0$$

Solve generalized eigenproblem

$$Q^\top A w = \lambda R w$$

Examples

The four legs map



Ulam (2^7) Res. Min. (50)

The standard map

Using 15×15 basis functions: