Problem setting

Dynamical system
- Deterministic: \( S : \Omega \rightarrow \Omega \)
- Stochastic: \( p : \Omega \times B \rightarrow [0, 1] \), the transition function
  Treated as a random pert. of the deterministic case

Transfer operator
\[
P \mu = \mu \circ S^{-1} \quad \text{or} \quad P \mu(A) = \int_{\Omega} p(x, A) \mu(dx)
\]
for \( \mu \in \mathcal{M} \).

Or for densities
\[
\int_A Pf = \int_{S^{-1}(A)} f \quad \text{or} \quad Pf(y) = \int_{\Omega} k(x, y) f(x)m(dx)
\]

Spectral properties

Let \( Pf = \lambda f \). Then:
\[
\lambda = 1
\]
In the case of ergodicity, the eigenfunction characterizes long term behaviour:
\[
\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{k=1}^{N} \phi(S^k x) = \int \phi f
\]

\[\lambda < 1\]
The state space \( \Omega \) can be splitted into two almost invariant sets, \( \text{supp}(f^+) \) and \( \text{supp}(f^-) \). The eigenvalue \( \lambda \) is a measure for the almost invariance of these sets.

Long term dynamical behaviour

\[\Downarrow\]
Eigenvalue problem in \( L^2 \)

Numerical computation

Central issues
- Basis functions: Gaussians with \( n \)-dependent variance.
- Norm computation:
\[
\| Pf \|_2^2 = m(\Omega) \sum_{k=1}^{N} (s_k)^2
\]
Needs pointwise evaluation of \( Pf \). Leads to
\[
\min_{\| w \|_2 = 1} \| (A - \lambda B) w \|_2
\]
- Computing the eigenvalue: Find local minima of

\[\lambda \rightarrow \| (A - \lambda B) w \|_2\]

Use the reduced QR factorization of \( B = QR \) to obtain
\[
w^T (A^T Q - \lambda R^T) R w = 0
\]
Solve generalized eigenproblem
\[
Q^T Aw = \lambda Rw
\]

Examples

The four legs map

The standard map

Using 15 \times 15 basis functions: