OBERWOLFACH WORKSHOP

APPLIED DYNAMICS AND GEOMETRIC MECHANICS

Affine Euler-Poincaré formulation of reduction for principal bundle field theories

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1. THE PROBLEM SETTING

Covariant and dynamical reduction for principal bundle field theories

Reduction for field theories with symmetry can be done either covariantly – that is, on spacetime – or dynamically – that is, after spacetime is split into space and time, see [5]. In [1] it was shown that these two reduction procedures are, in an appropriate sense, equivalent for a class of field theories whose fields take values in a principal bundle.

The purpose of this note is to give a different approach to the dynamical reduction for principal bundle field theories, by using the new process of *affine Euler-Poincaré reduction*. We then show that this approach is equivalent to the covariant Euler-Poincaré reduction theorem for principal bundle field theories, see [2].

It is interesting to note that the affine Euler-Poincaré reduction has been initially developed in the context of *complex fluids*, such as liquid crystals, superfluids, microfluids, and Yang-Mills magnetofluids (see [4]).

2. COVARIANT REDUCTION

2.1 Covariant Euler-Poincaré reduction

We recall here from [2] the covariant Euler-Poincaré reduction as it applies to principal bundle field theories.

Let $\pi : P \to X$ be a *right* principal G-fiber bundle over a manifold X with volume form μ and let $\mathcal{L} = \bar{\mathcal{L}}\mu : J^1P \to \Lambda^{n+1}X$ be a G invariant Lagrangian density. Let $\ell = \bar{\ell}\mu : J^1P/G \to \Lambda^{n+1}X$ be the reduced Lagrangian density associated to $\mathcal{L}$.

For a local section $s : U \subset X \to P$, let $\sigma := [j^1s] : U \to J^1P/G$ be the reduced first jet extension of s and let $\sigma^{\mathcal{A}} : U \to L(TU, \operatorname{Ad} P)$ be the vector bundle section associated to σ and to a principal connection $\mathcal{A}$ on the principal bundle $P_U := P|_U \to U$.

Then the following are equivalent :

(i) The variational principle $\delta \int_U \mathcal{L}(j^1s) = 0$ holds, for vertical variations along s with compact support.

(ii) The local section s satisfies the **covariant Euler-Lagrange equations** for $\mathcal{L}$.

(iii) The **variational principle** $\delta \int_U \ell(\sigma) = 0$ holds, using variations of the form

$$\delta\sigma = \nabla^{\mathcal{A}}\eta - [\sigma^{\mathcal{A}}, \eta],$$

where $\eta : U \subset X \to \operatorname{Ad} P_U$ is a section with compact support, and $\nabla^{\mathcal{A}} : \Gamma(\operatorname{Ad} P_U) \to \Omega^1(U, \operatorname{Ad} P_U)$ denotes the affine connection induced by the principal connection $\mathcal{A}$ on P_U.

(iv) The **covariant Euler-Poincaré equations**

$$\operatorname{div}^{\mathcal{A}} \frac{\delta \bar{\ell}}{\delta \sigma} = -\operatorname{ad}^*_{\sigma^{\mathcal{A}}} \frac{\delta \bar{\ell}}{\delta \sigma}$$

hold, where $\frac{\delta \bar{\ell}}{\delta \sigma}$ is the fiber derivative of $\bar{\ell}$ and

$$\operatorname{div}^{\mathcal{A}} : \mathfrak{X}(U, \operatorname{Ad} P_U^*) \to \Gamma(\operatorname{Ad} P_U^*)$$

is the covariant divergence associated to the $\mathcal{A}$.

Reconstruction : The covariant Euler-Poincaré equations are not sufficient for reconstructing the solution of the problem. One must impose that the curvature of the section σ (as a connection on P) vanishes.

2.2 The case of a trivial bundle

Consider a trivial principal G-bundle $\pi : P = X \times G \to X$, $\pi(p) = x$, where $p = (x, g)$. An element γ_p in the first jet bundle reads $\gamma_p = id_{T_xX} \times b_{(x,g)}$, where $b_{(x,g)} \in L(T_xX, T_gG)$. Any principal connection $\mathcal{A}$ on P writes

$$\mathcal{A}(x,g)(u_x, \xi_g) = \operatorname{Ad}_{g^{-1}}(\bar{\mathcal{A}}(x)(u_x) + TR_{g^{-1}}\xi_g),$$

where $\bar{\mathcal{A}} \in \Omega^1(X, \mathfrak{g})$. A local section $s : U \subset X \to P$ reads $s(x) = (x, \bar{s}(x))$ where $\bar{s} \in \mathcal{F}(U, G)$. The reduced first jet extensions $\sigma := [j^1s]$ and $\sigma^{\mathcal{A}}$ can be identified with $\bar{\sigma}, \bar{\sigma}^{\mathcal{A}} \in \Omega^1(U, \mathfrak{g})$ given by $\bar{\sigma}(x) = TR_{\bar{s}(x)^{-1}}T_x\bar{s}$ and $\bar{\sigma}^{\mathcal{A}}(x) = \bar{\mathcal{A}}(\bar{s}(x)) + \bar{\sigma}(x)$. As a consequence, in the trivial case, the covariant Euler-Poincaré equations reads

$$\operatorname{div} \frac{\delta \bar{\ell}}{\delta \bar{\sigma}} = -\operatorname{ad}^*_{\bar{\sigma}} \frac{\delta \bar{\ell}}{\delta \bar{\sigma}} \quad \text{or} \quad \operatorname{div}^{-\bar{\sigma}} \frac{\delta \bar{\ell}}{\delta(-\bar{\sigma})} = 0.$$

and the constrained variational principle becomes

$$\delta\bar{\sigma} = \mathbf{d}\eta - [\bar{\sigma}, \eta].$$

Note that, as expected in the trivial case, the connection $\mathcal{A}$ can be eliminated from the covariant Euler-Poincaré equations.

3. DYNAMICAL REDUCTION

The affine Euler-Poincaré reduction

We now recall from [4] the process of affine Euler-Poincaré reduction as it applies to the group $\mathcal{F}(M, G)$ of G-valued maps on a manifold M, acting on the space $\Omega^1(M, \mathfrak{g})$. Here $\mathcal{F}(M, G)$ is interpreted as the group of *gauge transformations* and $\Omega^1(M, \mathfrak{g})$ is interpreted as the space of *connections*, on the trivial bundle $M \times G \to M$.

- Assume that we have a (*possibly time-dependent*) Lagrangian $L : T\mathcal{F}(M,G) \times \Omega^1(M, \mathfrak{g}) \to \mathbb{R}$ which is *right* invariant under the *affine action* of $\Lambda \in \mathcal{F}(M, G)$ given by

$$(\chi, \dot{\chi}, \gamma) \mapsto (\chi\Lambda, \dot{\chi}\Lambda, \theta_\Lambda(\gamma)), \quad \theta_\Lambda(\gamma) := \Lambda^{-1}\gamma\Lambda + \Lambda^{-1}\mathbf{d}\Lambda. \qquad (1)$$

- Fix $\gamma_0 \in \Omega^1(M, \mathfrak{g})$ and define the Lagrangian $L_{\gamma_0} : T\mathcal{F}(M,G) \to \mathbb{R}$ by $L_{\gamma_0}(\chi, \dot{\chi}) := L(\chi, \dot{\chi}, \gamma_0)$. Then L_{γ_0} is right invariant under the lift to $T\mathcal{F}(M,G)$ of the right action of $\mathcal{F}(M,G)_{\gamma_0}$ on $\mathcal{F}(M,G)$, where $\mathcal{F}(M,G)_{\gamma_0}$ denotes the isotropy group of γ_0 with respect to the affine action θ.
- Right G-invariance of L permits us to define the reduced Lagrangian $l = l(\nu, \gamma) : \mathcal{F}(M, \mathfrak{g}) \times \Omega^1(M, \mathfrak{g}) \to \mathbb{R}$.
- For a curve $\chi(t) \in \mathcal{F}(M,G), \chi(0) = e$, let $\nu(t) := \dot{\chi}(t)\chi(t)^{-1} \in \mathcal{F}(M, \mathfrak{g})$ and define the curve $\gamma(t)$ as the unique solution of the equation

$$\dot{\gamma} + \mathbf{d}^\gamma\nu = 0, \quad \gamma(0) = \gamma_0.$$

The solution can be written as $\gamma(t) = \chi(t)\gamma_0\chi(t)^{-1} + \chi(t)\mathbf{d}\chi(t)^{-1}$.

With the preceding notations, the following are equivalent :

(i) With γ_0 fixed, the **variational principle** $\delta \int_{t_1}^{t_2} L_{\gamma_0}(\chi, \dot{\chi})dt = 0$, holds, for variations $\delta\chi(t)$ of $\chi(t)$ vanishing at the endpoints.

(ii) $\chi(t)$ satisfies the **Euler-Lagrange equations** for L_{γ_0} on $\mathcal{F}(M, G)$.

(iii) The **constrained variational principle** $\delta \int_{t_1}^{t_2} l(\nu, \gamma)dt = 0$, holds on $\mathcal{F}(M, \mathfrak{g}) \times \Omega^1(M, \mathfrak{g})$, upon using variations of the form

$$\delta\nu = \dot{\zeta} - [\nu, \zeta], \quad \delta\gamma = -\mathbf{d}^\gamma\zeta,$$

where $\zeta(t) \in \mathcal{F}(M, \mathfrak{g})$ vanishes at the endpoints.

(iv) The **affine Euler-Poincaré equations**

$$\frac{\partial}{\partial t}\frac{\delta l}{\delta \nu} = -\operatorname{ad}^*_\nu \frac{\delta l}{\delta \nu} + \operatorname{div}^\gamma \frac{\delta l}{\delta \gamma} \qquad (2)$$

hold on $\mathcal{F}(M, \mathfrak{g}) \times \Omega^1(M, \mathfrak{g})$.

4. FROM COVARIANT TO DYNAMICAL REDUCTION

4.1 Slicing of the covariant Euler-Poincaré equations

We now consider the specific case in which the principal bundle $P \to X$ is sliced. For simplicity, we restrict to the case where the bundle and the slicing are trivial, that is, we have $P = X \times G \to X$ and $X = \mathbb{R} \times M$. We assume that M has a volume form μ_M and we endow X with the volume form $dt \wedge \mu_M$.

Using the notations of §2.2, any local section $s : U = I \times V \subset X \to P$ reads $s(x) = (x, \bar{s}_t(m))$ where $x = (t, m)$ and $\bar{s}_t \in \mathcal{F}(V, G), t \in I$. The reduced first jet extension $\sigma = [j^1s]$ can be identified with the time dependent quantities $\bar{\sigma}^1_t \in \mathcal{F}(V, \mathfrak{g})$ and $\bar{\sigma}^2_t \in \Omega^1(V, \mathfrak{g})$ given by $\bar{\sigma}^1_t(m) = TR_{\bar{s}(x)^{-1}}\dot{\bar{s}}(x)$ and $\bar{\sigma}^2_t(m) = TR_{\bar{s}(x)^{-1}}\mathbf{d}\bar{s}(x)$, where $x = (t, m)$. Here $\dot{\bar{s}}$ and $\mathbf{d}\bar{s}$ denote the tangent maps with respect to I and M.

The Lagrangian densities $\mathcal{L}$ and ℓ can be written $\mathcal{L} = \mathcal{L}(t, \dot{\bar{s}}, \mathbf{d}\bar{s})$ and $\ell = \ell(t, \bar{\sigma}^1, \bar{\sigma}^2)$ and the covariant Euler-Poincaré equations are

$$\frac{\partial}{\partial t}\frac{\delta \bar{\ell}}{\delta \bar{\sigma}^1} = -\operatorname{ad}^*_{\bar{\sigma}^1} \frac{\delta \bar{\ell}}{\delta \bar{\sigma}^1} + \operatorname{div}^{(-\bar{\sigma}_2)} \frac{\delta \bar{\ell}}{\delta(-\bar{\sigma}^2)}.$$

Remarkably, these equations are formally identical to the affine Euler-Poincaré equations (2). We will explain how this fact can be understood from a reduction point of view.

4.2 Definition and invariance of the instantaneous Lagrangian

Using the hypotheses and notations of the previous box and given a G-invariant Lagrangian density $\mathcal{L} : J^1P \to \Lambda^{n+1}X$, we define the time dependent Lagrangian $L^{\mathcal{L}} : I \times T\mathcal{F}(V, G) \times \Omega^1(V, \mathfrak{g}) \to \mathbb{R}$ by

$$L^{\mathcal{L}}(t, \chi, \dot{\chi}, \gamma) = L^{\mathcal{L}}_\gamma(t, \chi, \dot{\chi}) := \int_V \bar{\mathcal{L}}(t, \dot{\chi}(m), \mathbf{d}\chi(m) - \chi(m)\gamma(m))\mu_M.$$

This Lagrangian has the remarkable property to be invariant under the affine action (1). Thus, we can apply the affine Euler-Poincaré reduction for any initial value γ_0 of γ. We will choose $\gamma_0 = 0$.

4.3 The main result

THEOREM. Consider a local section $\bar{s} = \bar{s}(x) = \bar{s}(t, m) : U = I \times V \to G$ of the trivial principal bundle $X \times G \to X = \mathbb{R} \times M$. The reduced first jet extension can be written $\bar{\sigma} = (\bar{\sigma}^1, \bar{\sigma}^2)$.

Given a local section $s : I \times V \subset X \to P$, we can define the curve $\chi(t) \in \mathcal{F}(V, G)$ by $\chi(t)(m) := \bar{s}(x)$. Given a curve $\chi(t) \in \mathcal{F}(V, G)$, we define the curves $\nu(t) = \dot{\chi}(t)\chi(t)^{-1}$ and $\gamma(t) = -\mathbf{d}\chi(t)\chi(t)^{-1}$. Note that we have $\nu(t)(m) = \bar{\sigma}^1(x)$ and $\gamma(t)(m) = -\bar{\sigma}^2(x)$, where $x = (t, m)$.

Consider a G invariant Lagrangian density $\mathcal{L} : J^1P \to \Lambda^{n+1}X$ and define the corresponding time dependent and affine invariant Lagrangian $L^{\mathcal{L}} : I \times T\mathcal{F}(V, G) \times \Omega^1(V, \mathfrak{g}) \to \mathbb{R}$.

Then the corresponding reduced Lagrangians verify the relation

$$l(\nu, \gamma) = \int_V \ell(\nu, -\gamma)\mu_V$$

and the following eight conditions are equivalent :

(i) The **variational principle** $\delta \int_{t_1}^{t_2} L^{\mathcal{L}}_0(t, \chi(t), \dot{\chi}(t))dt = 0$, holds for variations $\delta\chi(t)$ of $\chi(t)$ vanishing at the endpoints.

(ii) $\chi(t)$ satisfies the **Euler-Lagrange equations** for L_0 on $\mathcal{F}(V, G)$.

(iii) The **constrained variational principle** $\delta \int_{t_1}^{t_2} l(\nu(t), \gamma(t))dt = 0$, holds on $\mathcal{F}(V, \mathfrak{g}) \times \Omega^1(V, \mathfrak{g})$, upon using variations of the form

$$\delta\nu = \dot{\zeta} - [\nu, \zeta], \quad \delta\gamma = -\mathbf{d}^\gamma\zeta,$$

where $\zeta(t) \in \mathcal{F}(V, \mathfrak{g})$ vanishes at the endpoints.

(iv) The **affine Euler-Poincaré equations**

$$\frac{\partial}{\partial t}\frac{\delta l}{\delta \nu} = -\operatorname{ad}^*_\nu \frac{\delta l}{\delta \nu} + \operatorname{div}^\gamma \frac{\delta l}{\delta \gamma}$$

hold on $\mathcal{F}(V, \mathfrak{g}) \times \Omega^1(V, \mathfrak{g})$.

(v) The **variational principle** $\delta \int_U \mathcal{L}(j^1s) = 0$ holds, for variations with compact support.

(vi) The section s satisfies the **covariant Euler-Lagrange equations** for $\mathcal{L}$.

(vii) The **variational principle** $\delta \int_U \ell(\bar{\sigma}(x)) = 0$ holds, using variations of the form

$$\delta\bar{\sigma} = \mathbf{d}\eta - [\bar{\sigma}, \eta],$$

where $\eta : U \subset X \to \mathfrak{g}$ has compact support.

(viii) The **covariant Euler-Poincaré equations** hold :

$$\operatorname{div} \frac{\delta \bar{\ell}}{\delta \bar{\sigma}} = -\operatorname{ad}^*_{\bar{\sigma}} \frac{\delta \bar{\ell}}{\delta \bar{\sigma}}.$$

4.5 Example : spin glasses

We now apply the theory developed here to the model of *spin glasses* considered by Dzyaloshinskii in [3], see [4] for the dynamical approach. Consider the trivial principal bundle $P = X \times G \to X$ and the Lagrangian density $\mathcal{L}(j^1s) := \|T\bar{s}\|^2$, where the norm is associated to the right invariant metric $(g\gamma)$ on J^1P, constructed from g and γ. Here g is the spacetime metric $g = dt^2 - g_M$ on X, g_M is a Riemannian metric on M, and γ is an adjoint-invariant inner product on $\mathfrak{g}$. We can write $\mathcal{L}(j^1s) := \|\dot{\bar{s}}\|^2 - \|\mathbf{d}\bar{s}\|^2$. The reduced Lagrangian density reads $\ell(\bar{\sigma}^1, \bar{\sigma}^2) = \|\bar{\sigma}^1\|^2 - \|\bar{\sigma}^2\|^2$.

The corresponding instantaneous Lagrangians $L^{\mathcal{L}}$ and l are

$$L^{\mathcal{L}}(\chi, \dot{\chi}, \gamma) = \int_M \left(\|\dot{\chi}\|^2 - \|\mathbf{d}\chi - \chi\gamma\|^2\right)\mu_M, \quad l(\nu, \gamma) = \int_M \left(\|\nu\|^2 - \|\gamma\|^2\right)\mu_M.$$

Thus we have recovered the spin glasses Lagrangian l considered in [3]. The motion equations can be obtained from $L^{\mathcal{L}}$ by dynamical reduction (see [4]), or from $\mathcal{L}$ by covariant reduction.

4.6 Future directions

(1) Explore the Hamiltonian side of the theory.
(2) Treat the case of a general slicing of the spacetime X.

Références

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