1. THE PROBLEM SETTING

Covariant and dynamical reduction for principal bundle field theories

Reduction for field theories with symmetry can be done either covariantly—that is, on space-time—or dynamically—that is, after spacetime is split into space and time, see [5]. In [1] it was shown that these techniques, in an appropriate sense, are equivalent for a class of field theories whose fields take values in a principal bundle.

The purpose of this note is to give a different approach to the dynamical reduction for principal bundle field theories, by using the new process of affine Euler-Poincaré reduction. We then show that this approach is equivalent to the covariant Euler-Poincaré reduction framework for principal bundle field theories, see [2].

It is interesting to note that the affine Euler-Poincaré reduction has been initially developed in the context of complex fluids, such as liquid crystals, superfluids, microtubuli, and Yang Mills magnetotubuli (see [4]).

3. DYNAMICAL REDUCTION

The affine Euler-Poincaré reduction

We now recall from [4] the process of affine Euler-Poincaré reduction as it applies to principal bundle field theories. Let \( E \rightarrow \mathcal{M} \) be a right principal \( G \)-fiber bundle over a manifold \( \mathcal{M} \) with volume form \( \omega \) and let \( \mathcal{L} : E \times \mathcal{M} \rightarrow \mathbb{R} \) be the \( G \)-invariant Lagrangian density. Let \( \mathcal{L} = \mathcal{L}(x, \dot{x}, \frac{\partial \mathcal{L}}{\partial x}, \frac{\partial \mathcal{L}}{\partial \dot{x}}) \) be the reduced Lagrangian density on \( \mathcal{M} \).

For a local section \( s: U \subset \mathcal{M} \rightarrow \mathcal{M} \) of the right principal \( G \)-fiber bundle \( E \rightarrow \mathcal{M} \) and a connection \( A \) on the principal bundle \( P = E/G \), the following are equivalent:

(i) The variational principle \( F \) of \( \mathcal{L} \) holds, for vertical variations along \( s \) with compact support.

(ii) The variational principle \( s (\mathcal{L}) \) holds, using variations of the form

\[ \delta \mathcal{L} |_{s(x)} = \mathcal{L}_{x} (\delta_{A} x, \dot{x}, \frac{\partial \mathcal{L}}{\partial x}, \frac{\partial \mathcal{L}}{\partial \dot{x}}) \]

where \( \frac{\partial \mathcal{L}}{\partial \dot{x}} \) is the fiber derivative of \( \mathcal{L} \) and the connection \( A \) is interpreted as a connection induced by the principal connection \( A \) on \( P = E/G \).

(iii) The covariant Euler-Lagrange equations hold, which means

\[ \frac{\partial \mathcal{L}}{\partial \dot{x}} - \frac{\partial}{\partial x} \left( \frac{\partial \mathcal{L}}{\partial \dot{x}} \right) + \mathcal{L}_{x} = 0 \]

where \( \mathcal{L}_{x} \) is the Lagrangian density on \( \mathcal{M} \) and \( \mathcal{L}_{x} \) is interpreted as the Lagrangian density on \( \mathcal{M} \).

Applying these results to the case of the massless Klein-Gordon equation on \( \mathbb{R}^n \) and using the Killing vectors of \( \mathbb{R}^n \), we obtain the affine Euler-Poincaré reduction for this equation, see [4].

4. FROM DYNAMICAL TO REDUCTIVE

4.1 SLIDING OF THE COVARIANT EULER-Poincaré EQUATIONS

We now consider the specific case in which the principal bundle \( E \rightarrow \mathcal{M} \) is solved. For simplicity, we restrict to the case where the bundle and the sliding are trivial, that is, we have \( \mathcal{L} = \mathcal{L}(x, \dot{x}) \) and \( x = x(t) \in \mathcal{M} \). Assume that \( \mathcal{M} \) has a volume form \( \omega \) and we endow \( \mathcal{M} \) with the volume form \( \partial \omega \).

Using the notations of \( \mathcal{L} \), any local section \( s: U \subset \mathcal{M} \rightarrow \mathcal{M} \) reads \( \mathcal{L}(x, \dot{x}) = \mathcal{L}(x, \dot{y}) \). The reduced first jet extension \( \mathcal{L}^{(1)} \) can be identified with the time-dependent quantities \( s_{1}(x) \) given by \( s_{1}(x) = \mathcal{L}_{x} \in \mathcal{M} \). If the Lagrangian density \( \mathcal{L}^{(1)} \) on \( \mathcal{M} \) is interpreted as the Lagrangian density

\[ \mathcal{L}(x, \dot{x}) = \mathcal{L}_{x}(x, \dot{x}) + \mathcal{L}_{1}(x, \dot{x}) \]

The Lagrangian densities \( \mathcal{L} \) and \( \mathcal{L}^{(1)} \) can be written \( \mathcal{L} = \mathcal{L}(x, \dot{x}) + \mathcal{L}_{1}(x, \dot{x}) \). The Lagrangian densities \( \mathcal{L} \) and \( \mathcal{L}^{(1)} \) are interpreted as the Lagrangian densities on \( \mathcal{M} \).

5.4 Example : spin glasses

We now apply the theory developed here to the model of spin glasses considered by Dzyaloshinskii in [3], see [4] for the dynamical approach. Consider the trivial principal bundle \( E = \mathcal{M} = \mathcal{S} \times \mathcal{M} \) and the Lagrangian density \( \mathcal{L}(x, \dot{x}) = \mathcal{L}(x, \dot{y}) \), where the norm is associated to the right invariant metric \( g(x) \) on \( \mathcal{M} \), constructed from \( \mathcal{S} \) and \( \mathcal{M} \) is the space-time metric \( g = -\sqrt{-1} g_{\mathcal{M}} \). We can write \( \mathcal{L}(x, \dot{x}) = \mathcal{L}(x, \dot{y}) = \mathcal{L}(x, \dot{y}) \). The reduced Lagrangian density reads

\[ \mathcal{L}(x, \dot{x}) = \mathcal{L}(x, \dot{y}) \]

Thus we have recovered the spin glasses Lagrangian \( \mathcal{L} \) considered in [3]. The motion equations can be obtained from \( \mathcal{L} \) by reduction (see [4]), or from \( \mathcal{L} \) by covariant reduction.

6. Future directions

(1) Explore the Hamiltonian side of the theory.

(2) Treat the case of a general slicing of the space-time \( \mathcal{X} \).

Références


