A momentum-energy nonholonomic integrator

Sebastián J. Ferraro*, David Iglesias† and David Martín de Diego†

*Universidad Nacional del Sur, Bahía Blanca, Argentina; †ICMAT (CSIC-UAM-UCM-UC3M), Madrid, Spain.
Email addresses: sferraro@UNS.edu.ar – iglesias@imaff.cfmac.csic.es – d.martin@imaff.cfmac.csic.es

1 General setting

Nonholonomic systems where the Lagrangian is of mechanical type $L(v_q) = \frac{1}{2}k(q, v_q) − V(q)$, $v_q ∈ T_qQ$. Here $k$ is a Riemannian metric on $Q$. The discrete Lagrangian $L_d: Q × Q → \mathbb{R}$ represents an approximation of $L: TQ → \mathbb{R}$. Constraints linear in the velocities are given by a distribution $\mathcal{D} ⊂ TQ$. Using the metric $k$, define the complementary projectors

\[ \mathcal{P}: TQ → \mathcal{D}, \quad \mathcal{Q}: TQ → \mathcal{D}^⊥. \]

2 A geometric nonholonomic integrator

Usual discrete Euler–Lagrange equations (for unconstrained systems):

\[ D_1L_d(q_k, q_{k+1}) + D_2L_d(q_{k-1}, q_k) = 0. \]

The proposed discrete nonholonomic equations (14) are

\[ \mathcal{P} \cdot \delta q_k (D_1L_d(q_k, q_{k+1}) + D_2L_d(q_{k-1}, q_k)) = 0 \]
\[ \mathcal{Q} \cdot \delta q_k (D_1L_d(q_k, q_{k+1}) − D_2L_d(q_{k-1}, q_k)) = 0. \]

The first equation is the projection of the discrete Euler–Lagrange equations to the constraint distribution $\mathcal{D}$, while the second one can be interpreted as an elastic impact of the system against $\mathcal{Q}$ (see [5]).

This defines a unique discrete evolution operator if and only if the matrix $(D_1L_d)_k$ is regular, i.e., if the discrete Lagrangian is regular. Define the pre- and post-momenta using the discrete Legendre transformation of $\mathcal{P}$ on $\mathcal{D}$:

\[ p_{k-1,k}^+ = \mathcal{P}^* \cdot \delta q_k (q_k, D_2L_d(q_{k-1}, q_k)) \in T_qQ, \]
\[ p_{k,k+1}^- = \mathcal{Q}^* \cdot \delta q_k (q_k, -D_1L_d(q_{k+1}, q_k)) \in T_qQ. \]

In these terms, equation (1b) can be rewritten as

\[ \mathcal{Q} \cdot \delta q_k \left( \frac{p_{k,k+1}^- + p_{k-1,k}^+}{2} \right) = 0 \]

which means that the average of post- and pre-momenta satisfies the constraints. We can also rewrite the discrete nonholonomic equations as a jump of momenta:

\[ p_{k,k+1}^- = (\mathcal{P}^* - \mathcal{Q}^*) \cdot q_k (p_{k+1,k-1}^+). \]

3 Left-invariant systems on Lie groups

Take $Q = G$ a Lie group, and a left-invariant discrete Lagrangian ([1, 6]). Define the increment $W_k = g_k \cdot g_{k+1}^\leftarrow$. The method becomes

\[ p_{k,k+1}^- = (\mathcal{P}^* - \mathcal{Q}^*) \cdot \left( W_{k+1}^{-1} \cdot p_{k-1,k}^+ \right) \]

where $R^\leftarrow$ is the mapping on $T^\leftarrow G$ induced by right multiplication. If in addition $\mathcal{Q}$ is left-invariant, then

\[ p_k = (\mathcal{P}^* - \mathcal{Q}^*) \cdot \left( Ad_{g_k} \cdot p_{k-1}^- \right), \]

where $p_k$ is the discrete body momentum (3) $p_k = T^\leftarrow g_k p_{k,k+1}^- ∈ \mathfrak{g}^\leftarrow$.

4 Preserving energy on Lie groups

Theorem 1. Let the configuration manifold be a Lie group with a Lagrangian defined by a bi-invariant metric and an arbitrary distribution $\mathcal{D}$, and take a discrete Lagrangian that is left-invariant. Then the proposed discrete nonholonomic method (1) is energy-preserving.

Sketch of proof. The bi-invariant metric $k$ induces a bi-invariant inner product on each fiber of $T^\leftarrow G$. Right translations on $T^\leftarrow G$ and the mapping $(\mathcal{P} - \mathcal{Q})^* : T^\leftarrow G → T^\leftarrow G$ preserve the corresponding norm $\| \cdot \|_k$. Then the method (2) is such that $\| p_{k,k+1}^- \|_k$ (and also $\| p_{k+1,k-1}^+ \|_k$) is preserved. The potential energy must be zero, so the energy is the Hamiltonian $H = \frac{1}{2} \| p \|_k^2$, which is preserved.

5 Discrete nonholonomic momentum map

Suppose that a Lie group $G$ acts on $Q$. Define for each $q ∈ Q$

\[ g^\leftarrow = \{ \xi ∈ g | \xi_q(q) ∈ \mathfrak{g}_q \}. \]

The bundle over $Q$ whose fiber at $q$ is $g^\leftarrow$ is denoted by $g^\leftarrow$. Define the discrete nonholonomic momentum map $J^\leftarrow Q × Q → g^\leftarrow$ as in [2] by

\[ J^\leftarrow (q_{k-1}, q_k) : g_k \rightarrow \mathbb{R}, \quad \xi \rightarrow \left(D_2L_d(q_{k+1}, q_k), \xi_q(q_k)\right). \]

For any smooth section $\xi$ of $g^\leftarrow$ we have a function $(J^\leftarrow)^\xi : Q × Q → \mathbb{R}$, defined as $(J^\leftarrow)^\xi (q_{k-1}, q_k) = J^\leftarrow (q_{k-1}, q_k) (\xi_q(q_k)).$

We state the following result without proof. If $L_d$ is $G$-invariant and $\xi ∈ g$ is a horizontal symmetry (that is, $\xi_q(q) ∈ \mathfrak{g}_q$ for all $q ∈ Q$), then the proposed nonholonomic integrator preserves $(J^\leftarrow)^\xi$.

6 Preservation of the constraint submanifold

Define the average momentum

\[ \bar{p}_k = \frac{1}{2} \left( p_{k-1,k}^+ + p_{k,k+1}^- \right). \]

For a trajectory computed by our method, this momentum satisfies the constraints $\mathcal{P}^* (\bar{p}_k) = 0$, and in addition $\frac{1}{2} \| \bar{p}_k \|^2_k$ (its energy) is also preserved under the hypotheses of Theorem 1. Thus, our method produces a sequence of points lying on the constraint submanifold on the Hamiltonian side, with constant energy.

References


Evolution of momenta (solid arrows) according to Eq. (2).