A momentum-energy nonholonomic integrator

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1 General setting

Nonholonomic systems where the Lagrangian is of *mechanical type* $L(v_q) = \frac{1}{2}\kappa(v_q, v_q) - V(q)$, $v_q \in T_qQ$. Here κ is a Riemannian metric on Q. The discrete Lagrangian $L_d\colon Q \times Q \to \mathbb{R}$ represents an approximation of $L\colon TQ \to \mathbb{R}$. Constraints linear in the velocities are given by a distribution $\mathscr{D} \subset TQ$. Using the metric κ, define the complementary projectors

$$\mathscr{P} : TQ \to \mathscr{D}$$
$$\mathscr{Q} : TQ \to \mathscr{D}^{\perp}.$$

2 A geometric nonholonomic integrator

Usual discrete Euler–Lagrange equations (for *unconstrained* systems):

$$D_1L_d(q_k, q_{k+1}) + D_2L_d(q_{k-1}, q_k) = 0.$$

The proposed ***discrete nonholonomic equations*** ([4]) are

$$\mathscr{P}^*_{|q_k}(D_1L_d(q_k, q_{k+1})) + \mathscr{P}^*_{|q_k}(D_2L_d(q_{k-1}, q_k)) = 0 \qquad (1a)$$
$$\mathscr{Q}^*_{|q_k}(D_1L_d(q_k, q_{k+1})) - \mathscr{Q}^*_{|q_k}(D_2L_d(q_{k-1}, q_k)) = 0. \qquad (1b)$$

The first equation is the projection of the discrete Euler–Lagrange equations to the constraint distribution $\mathscr{D}$, while the second one can be interpreted as an elastic impact of the system against $\mathscr{D}$ (see [5]).

This defines a unique discrete evolution operator if and only if the matrix $(D_{12}L_d)$ is regular, i.e., if the discrete Lagrangian is regular.

Define the pre- and post-momenta using the discrete Legendre transformations:

$$p^+_{k-1,k} = \mathbb{F}^+L_d(q_{k-1}, q_k) = (q_k, D_2L_d(q_{k-1}, q_k)) \in T^*_{q_k}Q$$
$$p^-_{k,k+1} = \mathbb{F}^-L_d(q_k, q_{k+1}) = (q_k, -D_1L_d(q_k, q_{k+1})) \in T^*_{q_k}Q.$$

In these terms, equation (1b) can be rewritten as

$$\mathscr{Q}^*_{|q_k}\left(\frac{p^-_{k,k+1} + p^+_{k-1,k}}{2}\right) = 0$$

which means that the average of post- and pre-momenta satisfies the constraints. We can also rewrite the discrete nonholonomic equations as a jump of momenta:

$$p^-_{k,k+1} = (\mathscr{P}^* - \mathscr{Q}^*)\big|_{q_k}(p^+_{k-1,k}).$$

3 Left-invariant systems on Lie groups

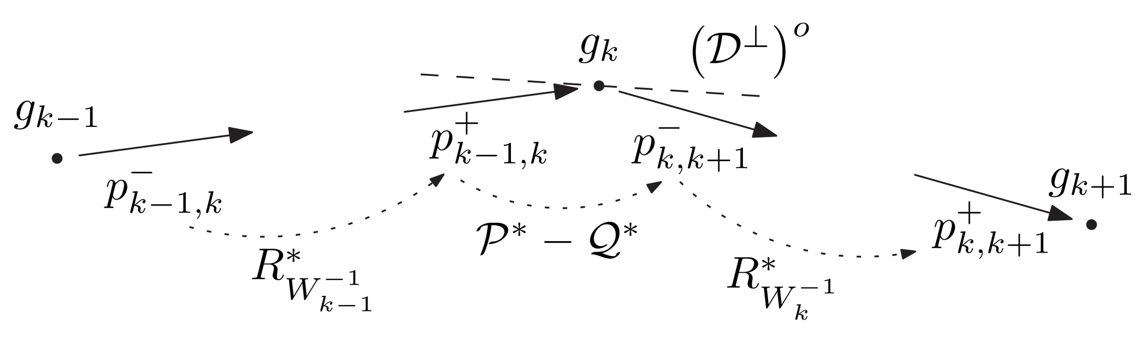
Take $Q = G$ a Lie group, and a left-invariant discrete Lagrangian ([1, 6]). Define the increment $W_k = g_k^{-1}g_{k+1}$. The method becomes

$$p^-_{k,k+1} = (\mathscr{P} - \mathscr{Q})^*\left(R^*_{W^{-1}_{k-1}}p^-_{k-1,k}\right) \qquad (2)$$

where R^* is the mapping on T^*G induced by right multiplication. If in addition $\mathscr{D}$ is left-invariant, then

$$p_k = (\mathscr{P} - \mathscr{Q})^*\left(\mathrm{Ad}^*_{W_{k-1}}p_{k-1}\right),$$

where p_k is the discrete body momentum ([3]) $p_k = L^*_{g_k}p^-_{k,k+1} \in \mathfrak{g}^*$.



Evolution of momenta (solid arrows) according to Eq. (2).

4 Preserving energy on Lie groups

Theorem 1. *Let the configuration manifold be a Lie group with a Lagrangian defined by a bi-invariant metric and with an arbitrary distribution $\mathscr{D}$, and take a discrete Lagrangian that is left-invariant. Then the proposed discrete nonholonomic method* (1) *is energy-preserving.*

Sketch of proof. The bi-invariant metric κ induces a bi-invariant inner product on each fiber of T^*G. Right translations on T^*G and the mapping $(\mathscr{P} - \mathscr{Q})^*\colon T^*G \to T^*G$ preserve the corresponding norm $\|\cdot\|_\kappa$. Then the method (2) is such that $\|p^-_{k,k+1}\|_\kappa$ (and also $\|p^+_{k,k+1}\|_\kappa$) is preserved. The potential energy must be zero, so the energy is the Hamiltonian $H = \frac{1}{2}\|p\|^2_\kappa$, which is preserved. □

5 Discrete nonholonomic momentum map

Suppose that a Lie group G acts on Q. Define for each $q \in Q$

$$\mathfrak{g}^q = \left\{\xi \in \mathfrak{g} \mid \xi_Q(q) \in \mathscr{D}_q\right\}.$$

The bundle over Q whose fiber at q is $\mathfrak{g}^q$ is denoted by $\mathfrak{g}^{\mathscr{D}}$. Define the discrete nonholonomic momentum map $J^{\mathrm{nh}}_d\colon Q \times Q \to (\mathfrak{g}^{\mathscr{D}})^*$ as in [2] by

$$J^{\mathrm{nh}}_d(q_{k-1}, q_k)\colon \mathfrak{g}^{q_k} \to \mathbb{R}$$
$$\xi \mapsto \left\langle D_2L_d(q_{k-1}, q_k), \xi_Q(q_k)\right\rangle.$$

For any smooth section $\widetilde{\xi}$ of $\mathfrak{g}^{\mathscr{D}}$ we have a function $(J^{\mathrm{nh}}_d)_{\widetilde{\xi}}\colon Q \times Q \to \mathbb{R}$, defined as $(J^{\mathrm{nh}}_d)_{\widetilde{\xi}}(q_{k-1}, q_k) = J^{\mathrm{nh}}_d(q_{k-1}, q_k)\left(\widetilde{\xi}(q_k)\right)$.

We state the following result without proof. If L_d is G-invariant and $\xi \in \mathfrak{g}$ is a horizontal symmetry (that is, $\xi_Q(q) \in \mathscr{D}_q$ for all $q \in Q$), then the proposed nonholonomic integrator preserves $(J^{\mathrm{nh}}_d)_\xi$.

6 Preservation of the constraint submanifold

Define the average momentum

$$\widetilde{p}_k = \frac{1}{2}\left(p^+_{k-1,k} + p^-_{k,k+1}\right).$$

For a trajectory computed by our method, this momentum satisfies the constraints $\mathscr{Q}^*(\widetilde{p}_k) = 0$ and in addition $\frac{1}{2}\|\widetilde{p}_k\|^2_\kappa$ (its energy) is also preserved under the hypotheses of Theorem 1. Thus, our method produces a sequence of points lying on the constraint submanifold on the Hamiltonian side, with constant energy.

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