Hamiltonizing Nonholonomic Mechanics

A.M. Bloch, O.E. Fernandez
University of Michigan
Nonholonomic Systems

\[ L, \quad a_{ij} \dot{q}^j = 0. \]

Lagrange d’Alembert:
\[ \int_a^b (\delta L + \lambda_i a_{ij} \delta q^j) \, dt = 0 \]

Physical Equations of Motion

Variational Nonholonomic:
\[ \delta \int_a^b (L + \lambda_i a_{ij} \dot{q}^j) \, dt = 0 \]

Nonholonomic + Invariant Measure

Variational

Related to Optimal Control

Applied Dynamics and Geometric Mechanics
Oberwolfach, 2008
A link between the two

Conditionally Variational

Under the equivalence conditions and certain regularity conditions the Euler-Lagrange equations of the variational Lagrangian $L_V$ defined by

$$L_V = L - \frac{\partial L}{\partial \dot{s}^a}(\dot{s}^a + A^a_\alpha (r) \dot{r}^\alpha)$$

reproduce the nonholonomic mechanics when the initial velocities $\dot{q}_0$ lie in $\mathcal{D}$.

Moreover, for systems with an invariant measure, such conditions depend explicitly on the invariant measure density $N$. 
Example: Vertical Rolling Disk

\[ L = \frac{1}{2} m(\dot{x}^2 + \dot{y}^2) + \frac{1}{2} I \dot{\theta}^2 + \frac{1}{2} J \dot{\varphi}^2, \quad \dot{x} = R \cos \varphi \dot{\theta}, \quad \dot{y} = R \sin \varphi \dot{\theta} \]

Lagrange d’Alembert

\[ (L, D) \]

\[ L_V \]

Variational Nonholonomic

\[ \ddot{\varphi} = -\frac{mR}{J} [c_1 \sin(\varphi) - c_2 \cos(\varphi)] \dot{\theta}, \]
\[ \ddot{\theta} = \frac{mR}{(I + mR^2)} [c_1 \sin(\varphi) - c_2 \cos(\varphi)] \dot{\varphi}, \]
\[ \dot{x} = R \cos(\varphi) \dot{\varphi} + c_1, \]
\[ \dot{y} = R \sin(\varphi) \dot{\varphi} + c_2. \]

\[ L_V = -\frac{1}{2} m(\ddot{x}^2 + \ddot{y}^2) + \frac{1}{2} I \dot{\theta}^2 + \frac{1}{2} J \ddot{\varphi}^2 + mR \dot{\theta} (\dot{x} \cos \varphi + \dot{y} \sin \varphi) \]