Hamiltonizing Nonholonomic Mechanics

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Nonholonomic Systems

Nonholonomic — Nonholonomic + Invariant Measure — Variational

$$L, \quad a_{ij}\dot{q}^j = 0.$$

Lagrange d'Alembert

$$\int_a^b \left(\delta L + \lambda_i a_{ij} \delta q^j\right)\ dt = 0$$

Physical Equations of Motion

Variational Nonholonomic

$$\delta \int_a^b \left(L + \lambda_i a_{ij} \dot{q}^j\right)\ dt = 0$$

Related to Optimal Control

A link between the two

Nonholonomic

Nonholonomic + Invariant Measure

Conditionally Variational

Variational

Lagrange d'Alembert ⟷ Variational Nonholonomic

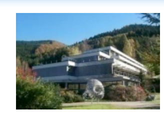
Conditionally Variational

Under the equivalence conditions and certain regularity conditions the Euler-Lagrange equations of the variational Lagrangian L_V defined by

$$L_V = L - \frac{\partial L}{\partial \dot{s}^a}(\dot{s}^a + A^a_\alpha(r)\dot{r}^\alpha)$$

reproduce the nonholonomic mechanics when the initial velocities $\dot{q}_0$ lie in $\mathcal{D}$.

Moreover, for systems with an invariant measure, such conditions depend explicitly on the invariant measure density N.



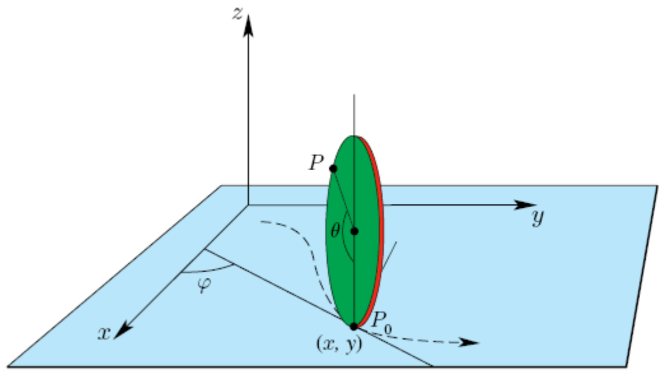
Example: Vertical Rolling Disk

$$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + \frac{1}{2}I\dot{\theta}^2 + \frac{1}{2}J\dot{\varphi}^2, \quad \dot{x} = R\cos\varphi\dot{\theta}, \quad \dot{y} = R\sin\varphi\dot{\theta}$$

Lagrange d'Alembert ← $(L, \mathcal{D})$ | L_V → **Variational Nonholonomic**

Lagrange d'Alembert:

$$\begin{aligned} \ddot{\varphi} &= 0, \\ \ddot{\theta} &= 0, \\ \dot{x} &= R\cos(\varphi)\dot{\varphi}, \\ \dot{y} &= R\sin(\varphi)\dot{\varphi}. \end{aligned}$$



Variational Nonholonomic:

$$\begin{aligned} \ddot{\varphi} &= -\frac{mR}{J}\left[c_1\sin(\varphi) - c_2\cos(\varphi)\right]\dot{\theta}, \\ \ddot{\theta} &= \frac{mR}{(I + mR^2)}\left[c_1\sin(\varphi) - c_2\cos(\varphi)\right]\dot{\varphi}, \\ \dot{x} &= R\cos(\varphi)\dot{\varphi} + c_1, \\ \dot{y} &= R\sin(\varphi)\dot{\varphi} + c_2. \end{aligned}$$

$$L_V = -\frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + \frac{1}{2}I\dot{\theta}^2 + \frac{1}{2}J\dot{\varphi}^2 + mR\dot{\theta}(\dot{x}\cos\varphi + \dot{y}\sin\varphi)$$