1. If true, justify and if false, give a counterexample, or explain why.

   (a) Let $f(x, y, z) = y - x$. Then the line integral of ∇f around the unit circle $x^2 + y^2 = 1$ in the xy plane is π, the area of the circle.

   Solution. This is false. The line integral of any gradient around a closed curve is zero.

   (b) Let $\mathbf{F}$ be a smooth vector field in space and suppose that the circulation of $\mathbf{F}$ around the circle of radius 1 centered at $(0, 0, 0)$ and lying in the xy-plane, is zero. Then $(\nabla \times \mathbf{F})(0, 0, 0) = 0$.

   Solution. This is false. There are two reasons this is wrong. First, one has to have zero circulation for circles of arbitrarily small radius. Second, one has to have the circles in planes with arbitrary normal vectors. An explicit counterexample is $\mathbf{F} = y\mathbf{k}$, which has $\nabla \times F = \mathbf{i}$. The circulation around any circle in the xy-plane is zero, but the curl at $(0, 0, 0)$ is not zero.

   (c) The center of mass of the region between the spheres $x^2 + y^2 + z^2 = 4$ and $x^2 + y^2 + z^2 = 9$ having mass density

   $$\delta(x, y, z) = \sin[\pi(7 - x^2 - y^2 + 5z)]$$

   lies somewhere on the z-axis between $z = -3$ and $z = 3$.

   Solution. This is true. The mass density is symmetric about the z-axis (this is because the function δ depends on x and y only in the combination $r^2 = x^2 + y^2$) and so the center of mass lies on this axis of symmetry. On the other hand, the center of mass should lie between the greatest and least values of z, namely between $z = -3$ and $z = 3$.

   (d) If f is a smooth function of (x, y, z), there is a point (x_0, y_0, z_0) on the sphere $x^2 + y^2 + z^2 = 1$ such that

   $$\nabla f(x_0, y_0, z_0) = k(x_0\mathbf{i} + y_0\mathbf{j} + z_0\mathbf{k})$$

   for some constant k.

   Solution. This is true. If we let the point (x_0, y_0, z_0) be a maximum point for f, and let $g(x, y, z) = x^2 + y^2 + z^2$, then by the Lagrange multiplier theorem, there is a constant k such that $\nabla f(x_0, y_0, z_0) = k\nabla g(x_0, y_0, z_0)$, which is the result desired.

2. Find the center of mass of the solid region that consists of all points (x, y, z) that lie inside the hemisphere

   $$x^2 + y^2 + z^2 = 1,$$

   lie above the xy plane (*i.e.*, $z \geq 0$), and lie in the cone $z^2 \geq x^2 + y^2$ if the mass density (mass per unit volume at (x, y, z)) is $\delta(x, y, z) = 1 - z$.

Solution. After drawing a figure, we see that the region in question is described in spherical coordinates by $0 \le \rho \le 1, 0 \le \theta \le 2\pi, 0 \le \phi \le \pi/4$. By symmetry, the center of mass will lie on the z-axis, so we need only compute $\bar{z}$. First of all, the mass is given by integrating the mass density $\delta(x, y, z) = 1 - z = 1 - \rho\cos\phi$ over the region in question:

$$\begin{aligned}
m &= \int_0^{2\pi}\int_0^{\pi/4}\int_0^1 (1 - \rho\cos\phi)\rho^2 \sin\phi \, d\rho d\phi d\theta \\
&= \int_0^{2\pi}\int_0^{\pi/4}\int_0^1 (\rho^2 \sin\phi - \rho^3 \sin\phi\cos\phi) \, d\rho d\phi d\theta \\
&= \frac{1}{3}(-\cos\phi)|_0^{\pi/4} \cdot 2\pi - \frac{1}{4}\frac{1}{2}(\sin^2\phi)|_0^{\pi/4} \cdot 2\pi \\
&= 2\pi\left(\frac{1}{3}\left(1 - \frac{\sqrt{2}}{2}\right) - \frac{1}{4}\cdot\frac{1}{4}\right) \\
&= \frac{13}{24}\pi - \frac{\sqrt{2}}{3}\pi
\end{aligned}$$

The numerator in the formula for the z-component of the center of mass is given by

$$n = \int_0^{2\pi}\int_0^{\pi/4}\int_0^1 \rho\cos\phi(1 - \rho\cos\phi)\rho^2 \sin\phi \, d\rho d\phi d\theta.$$

This is evaluated as with the mass to give

$$n = \frac{\pi}{8} - \frac{2\pi}{15}\left(1 - \frac{\sqrt{2}}{4}\right).$$

Thus, the location of the center of mass is given by $(0, 0, n/m)$.

3. (a) Let S be the surface $x^2 + 2y^2 + 2z^2 = 1$. Find a parametrization of S and use it to find the tangent plane to S at $(\frac{1}{\sqrt{2}}, \frac{1}{2}, 0)$.

   Solution. This parametrization is given by modifying the spherical coordinate parametrization of the sphere:

$$\mathbf{\Phi} = \left(\cos\theta\sin\phi, \frac{1}{\sqrt{2}}\sin\theta\sin\phi, \frac{1}{\sqrt{2}}\cos\phi\right)$$

   The normal vector to the surface is given by

$$\begin{aligned}
\mathbf{\Phi}_\phi \times \mathbf{\Phi}_\theta &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos\theta\cos\phi & \frac{1}{\sqrt{2}}\sin\theta\cos\phi & -\frac{1}{\sqrt{2}}\sin\phi \\ -\sin\theta\sin\phi & \frac{1}{\sqrt{2}}\cos\theta\sin\phi & 0 \end{vmatrix} \\
&= \sin\phi\left(\frac{1}{2}\sin\phi\cos\theta, \frac{1}{\sqrt{2}}\sin\phi\sin\theta, \frac{1}{\sqrt{2}}\cos\phi\right)
\end{aligned}$$

At the point $\left(\frac{1}{\sqrt{2}}, \frac{1}{2}, 0\right)$, where $\phi = \pi/2$ and $\theta = \pi/4$, this becomes

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{2} & 0 \end{vmatrix} = \left(\frac{1}{2\sqrt{2}}, \frac{1}{2}, 0\right)$$

Thus, the tangent plane is given by

$$\frac{1}{2\sqrt{2}}\left(x - \frac{1}{\sqrt{2}}\right) + \frac{1}{2}\left(y - \frac{1}{2}\right) = 0,$$

that is, $x + \sqrt{2}y = \sqrt{2}$.

(b) Verify that the curve $\mathbf{c}(t) = (\cos t)\mathbf{i} + \frac{1}{\sqrt{2}}(\sin t)\mathbf{j}$ where $0 \leq t \leq 2\pi$ lies in the surface S in part (a) and that $\mathbf{c}'(\frac{\pi}{4})$ lies in the tangent plane you found in (a).

Solution. Since $x = \cos t, y = \frac{1}{\sqrt{2}} \sin t, z = 0$ satisfies $x^2 + 2y^2 + 2z^2 = 1$, the curve $\mathbf{c}(t)$ lies on the surface. Its tangent vector at the point $t = \pi/4$ is

$$\mathbf{c}'(t) = \left(-\sin\frac{\pi}{4}, \frac{1}{\sqrt{2}}\cos\frac{\pi}{4}, 0\right) = \left(\frac{-1}{\sqrt{2}}, \frac{1}{2}, 0\right)$$

This lies in the tangent plane found in (a) after translation to the point $(\frac{1}{\sqrt{2}}, \frac{1}{2}, 0)$. That is, the point

$$(x, y, z) = \left(\frac{1}{\sqrt{2}}, \frac{1}{2}, 0\right) + \mathbf{c}'(\pi/4) = (0, 1, 0)$$

satisfies the equation $x + \sqrt{2}y = \sqrt{2}$, which is clearly true.

4. Let C be the unit circle $x^2 + z^2 = 1, y = 0$ oriented counterclockwise when viewed from along the positive y axis. Let S_1 be the surface $x^2 + z^2 \leq 1, y = 0$ and let S_2 be the surface $x^2 + y^2 + z^2 = 1, \quad y \geq 0$.

(a) Draw a figure showing possible orientations for S_1 and S_2.

Solution. See the following figure.

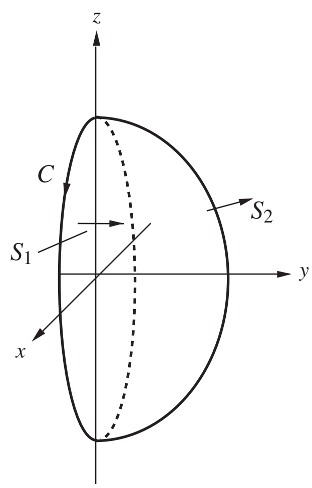

Figure for part (a)

(b) For $\mathbf{F} = y\mathbf{i} - z\mathbf{j} + yz^2\mathbf{k}$, show that

$$\iint_{S_1} (\nabla \times \mathbf{F}) \cdot d\mathbf{S} = \iint_{S_2} (\nabla \times \mathbf{F}) \cdot d\mathbf{S}.$$

Solution. Each side equals the line integral $\int_C \mathbf{F} \cdot d\mathbf{s}$ by Stokes' theorem and so they are equal to each other.

(c) Evaluate

$$\int_C \mathbf{F} \cdot d\mathbf{s}.$$

Solution. Write

$$\int_C \mathbf{F} \cdot d\mathbf{s} = \iint_{S_1} (\nabla \times \mathbf{F}) \cdot d\mathbf{S}$$

The curl of $\mathbf{F}$ is given by

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ y & -z & yz^2 \end{vmatrix} = (z^2 + 1)\mathbf{i} - \mathbf{k},$$

and the area element on S_1 is given by $d\mathbf{S} = \mathbf{j}\, dx\, dz$. Thus, $(\nabla \times \mathbf{F}) \cdot d\mathbf{S} = ((z^2+1)\mathbf{i} - \mathbf{k}) \cdot \mathbf{j}\, dx\, dz = 0$, so the integral is zero.