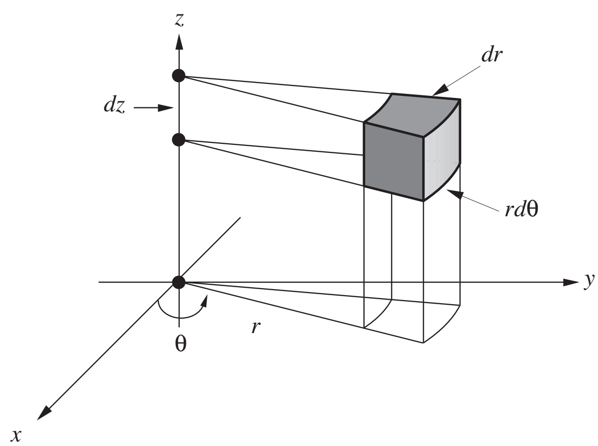
Example 1. Find a formula for the divergence of a vector field $\mathbf{F}$ in cylindrical coordinates.

Solution. We proceed along the same lines as the discussion in the text at the end of §8.4. Consider the situation of the Figure.



The divergence is the flux per unit volume. If $\mathbf{F} = F_r\mathbf{e}_r + F_\theta\mathbf{e}_\theta + F_z\mathbf{k}$, then the flux out of this cube is, approximately (using the linear approximation),

$$\begin{aligned}\text{Flux} &\approx [(r+dr)F_r(r+dr,\theta,z) - rF_r(r,\theta,z)]d\theta\, dz\\ &+ [F_\theta(r,\theta+d\theta,z) - F_\theta(r,\theta,z)]dr\, dz\\ &+ [F_z(r,\theta,z+dz) - F_z(r,\theta,z)]dr\cdot r d\theta\\ &\approx \frac{\partial(rF_r)}{\partial r}dr\, d\theta\, dz + \frac{\partial F_\theta}{\partial\theta}dr\, d\theta\, dz + \frac{\partial F_z}{\partial z}r dr\, d\theta\, dz.\end{aligned}$$

Thus, the flux per unit volume is

$$\text{div}\ \mathbf{F} = \frac{1}{r}\frac{\partial}{\partial r}(rF_r) + \frac{1}{r}\frac{\partial F}{\partial\theta} + \frac{\partial F_z}{\partial z}. \quad \Diamond$$

Example 2.

(a) Use Gauss' theorem to show that

$$\iint_{S_1}(\nabla\times\mathbf{F})\cdot\mathbf{n}\, dS = \iint_{S_2}(\nabla\times\mathbf{F})\cdot\mathbf{n}\, dS,$$

where S_1 and S_2 are two surfaces having a common boundary.

(b) Prove the same assertion using Stokes' theorem.

Solution.

(a) Since S_1 and S_2 have a common boundary, we can apply Gauss' theorem to the region U enclosed by their union:

$$\iint_{S_1 \cup S_2} (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dS = \iiint_U \nabla \cdot (\nabla \times \mathbf{F}) dV = 0,$$

since $\nabla \cdot (\nabla \times \mathbf{F}) = 0$ for *any* vector field $\mathbf{F}$ by the equality of mixed partials. Assuming that S_1 and S_2 are initially oriented so that the induced orientations of their common boundary are the same, we then get:

$$0 = \iint_{S_1 \cup S_2} (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dS = \iint_{S_1} (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dS - \iint_{S_2} (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dS.$$

(b) By an application of Stokes' theorem, we find that

$$\iint_{S_1} (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dS = \int_{\partial S_1} \mathbf{F} \cdot d\mathbf{S},$$

and

$$\iint_{S_2} (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dS = \int_{\partial S_2} \mathbf{F} \cdot d\mathbf{S},$$

where S_1 and S_2 are assumed to have the same orientations relative to one another as in (a) above. Since $\partial S_1 = \partial S_2$, that does it. $\diamondsuit$

Example 3. Let

$$\mathbf{F}(\mathbf{x}) = \frac{1}{4\pi} \sum_{i=1}^{8} 10^i \frac{\mathbf{x} - \mathbf{x}_i}{\|\mathbf{x} - \mathbf{x}_i\|^3},$$

where $\mathbf{x}_1, \ldots, \mathbf{x}_8$ are eight different points in $\mathbb{R}^3$. If S is a closed surface such that $\iint_S \mathbf{F} \cdot d\mathbf{S} = 11010$, which of the eight points lie inside S?

Solution. By Gauss' law, we find that

$$\iint_S \mathbf{F} \cdot \mathbf{n}\, dS = \frac{1}{4\pi} \sum_{i=1}^{8} 10^i \iint_S \frac{\mathbf{x} - \mathbf{x}_i}{\|\mathbf{x} - \mathbf{x}_i\|^3} = \sum_{i=1}^{8} 10^i \cdot \begin{cases} 1 & \text{if } \mathbf{x}_i \text{ lies inside } S \\ 0 & \text{otherwise} \end{cases}$$

From the equation, it is apparent that the only way for the integral to equal 11010 is for S to contain the points $\mathbf{x}_1, \mathbf{x}_3, \mathbf{x}_4$.