Example 1. Let

$$\mathbf{F} = (yze^x + xyze^x)\mathbf{i} + xze^x\mathbf{j} + xye^x\mathbf{k}.$$

Show that the integral of $\mathbf{F}$ around an oriented simple curve C that is the boundary of a surface S is zero.

Solution. By Stokes' theorem,

$$\int_C \mathbf{F} \cdot d\mathbf{s} = \iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dS.$$

We calculate

$$\begin{aligned} \nabla \times \mathbf{F} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ yze^x + xyze^x & xze^x & xye^x \end{vmatrix} \\ &= [xe^x - xe^x]\mathbf{i} - [(y + xy)e^x - (y + xy)e^x]\mathbf{j} \\ &\quad + [(z + xz)e^x - (z + xz)e^x]\mathbf{k} = 0. \end{aligned}$$

Hence the integrals are zero. Alternatively, one can observe that $\mathbf{F} = \nabla(xyze^x)$ and so its integral around any closed curve is zero.

Example 2. Find the integral of

$$\mathbf{F}(x, y, z) = x^2\mathbf{i} + y^2\mathbf{j} - z\mathbf{k}$$

around the triangle with vertices $(0, 0, 0)$, $(0, 2, 0)$ and $(0, 0, 3)$, using Stokes' theorem.

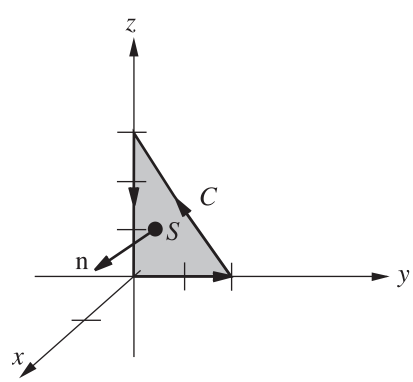
Solution. In the figure, C is the triangle in question and S is a surface it bounds. By Stokes' theorem,

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dS.$$

Now

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x^2 & y^2 & -z \end{vmatrix} = 0\mathbf{i} + 0\mathbf{j} + 0\mathbf{k} = 0.$$

Therefore the integral of $\mathbf{F}$ around C is zero.



Example 3. Let C be the curve $x^2 + y^2 = 1, z = 0$ and let S be the surface S_1 together with S_2, where S_1 is defined by $x^2 + y^2 \leq 1, z = -1$ and S_2 is defined by $x^2 + y^2 = 1, -1 \leq z \leq 0$.

(a) Draw a figure showing an orientation such that Stokes' theorem applies to the surface S and the curve C.

(b) If R is another surface with boundary C, show that

$$\iint_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S} = \iint_R (\nabla \times \mathbf{F}) \cdot d\mathbf{S}$$

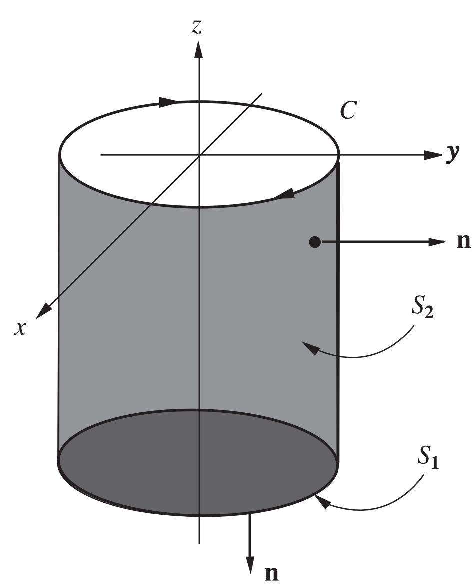
(c) If $\mathbf{F}(x, y, z) = (y^3 + e^{xz})\mathbf{i} - (x^3 + e^{yz})\mathbf{j} + e^{xyz}\mathbf{k}$, calculate

$$\iint_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S}$$

where S is the surface given above.

Solution.

(a) The Figure shows the surface with a chosen orientation:



One could, of course, consistently reverse all the orientations.

(b) Each side equals $\int_C \mathbf{F} \cdot d\mathbf{S}$ by Stokes' theorem, so they are equal to each other.

(c) Note that

$$\begin{aligned}\nabla \times \mathbf{F} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ y^3 + e^{xz} & -x^3 - e^{yz} & e^{xyz} \end{vmatrix} \\ &= (xze^{xyz} + ye^{yz})\mathbf{i} - (yze^{xyz} - xe^{xz})\mathbf{j} + (-3x^2 - 3y^2)\mathbf{k}.\end{aligned}$$

By (b),

$$\iint_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S} = \iint_D (\nabla \times \mathbf{F}) \cdot d\mathbf{S}$$

where D is the unit disk in the xy-plane. For $z = 0$,

$$(\nabla \times \mathbf{F})(x, y, 0) = y\mathbf{i} - x\mathbf{j} - 3(x^2 + y^2)\mathbf{k}$$

and $d\mathbf{S} = -\mathbf{k}\, dx\, dy$ (since it is to be oriented compatible with C). Thus,

$$\begin{aligned}\iint_D (\nabla \times \mathbf{F}) \cdot d\mathbf{S} &= 3 \iint_D (x^2 + y^2) dx\, dy \\ &= 3 \int_0^1 \int_0^{2\pi} r^2 \cdot r\, dr\, d\theta \\ &= 3 \cdot 2\pi \cdot \frac{1}{4} = \frac{3\pi}{2}.\end{aligned}$$