Example 1. Let C be the perimeter of the rectangle with sides $x = 1, y = 2, x = 3$, and $y = 3$. Evaluate the integral

$$\int_C (3x^4 + 5)dx + (y^5 + 3y^2 - 1)dy$$

Solution. Use Green's theorem:

$$\int_C (3x^4 + 5)dx + (y^5 + 3y^2 - 1)dy$$
$$= \int_2^3 \int_1^3 \left[\frac{\partial}{\partial x}(y^5 + 3y^2 - 1) - \frac{\partial}{\partial y}(3x^4 + 5) \right] dx\, dy = 0.$$

Example 2. Let

$$\mathbf{F}(x,y) = (2y + e^x)\mathbf{i} + (x + \sin y^2)\mathbf{j}$$

and C be the circle $x^2 + y^2 = 1$. Evaluate

$$\int_C \mathbf{F} \cdot d\mathbf{s}.$$

Solution. Let D be the unit disk bounded by C, then by Green's theorem,

$$\begin{aligned} \int_C \mathbf{F} \cdot d\mathbf{s} &= \iint_D \left[\frac{\partial}{\partial x}(x + \sin y^2) - \frac{\partial}{\partial y}(2y + e^x) \right] dx\, dy \\ &= \iint_D (1 - 2) dx\, dy = -\pi. \end{aligned}$$

Example 3. Let C be the boundary of the rectangle with sides $x = 1, y = 2, x = 3$ and $y = 3$. Evaluate

$$\int_C \left(\frac{2y + \sin x}{1 + x^2} \right) dx + \left(\frac{x + e^y}{1 + y^2} \right) dy.$$

Solution. Let R be the rectangle bounded by C. Then by Green's theorem,

$$\begin{aligned} & \int_C \frac{2y + \sin x}{1 + x^2} dx + \frac{x + e^y}{1 + y^2} dy \\ = & \iint_R \left[\frac{\partial}{\partial x} \left(\frac{x + e^y}{1 + y^2} \right) - \frac{\partial}{\partial y} \left(\frac{2y + \sin x}{1 + x^2} \right) \right] dx\, dy \\ = & \int_2^3 \int_1^3 \left(\frac{1}{1 + y^2} - \frac{2}{1 + x^2} \right) dx\, dy \\ = & (2 \arctan y) \Big|_2^3 - (2 \arctan x) \Big|_1^3 = \frac{\pi}{2} - 2 \arctan 2. \end{aligned}$$