Example 1 Evaluate the surface integral of the vector field \( \mathbf{F} = 3x^2\mathbf{i} - 2yx\mathbf{j} + 8\mathbf{k} \) over the surface \( S \) that is the graph of \( z = 2x - y \) over the rectangle \([0, 2] \times [0, 2]\).

Solution. Use the formula for a surface integral over a graph \( z = g(x,y) \):

\[
\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_D \left[ \mathbf{F} \cdot \left( -\frac{\partial g}{\partial x} \mathbf{i} - \frac{\partial g}{\partial y} \mathbf{j} + \mathbf{k} \right) \right] dxdy.
\]

In our case we get

\[
\int_0^2 \int_0^2 (3x^2, -2yx, 8) \cdot (-2, 1, 1) dxdy = \int_0^2 \int_0^2 (-6x^2 - 2yx + 8) dxdy
\]

\[
= \int_0^2 -2x^3 - yx^2 + 8x \bigg|_{x=0}^2 dy
\]

\[
= \int_0^2 -4y dy = -2y^2 \bigg|_0^2 = -8.
\]
Example 2 Let $S$ be the triangle with vertices $(1,0,0), (0,2,0),$ and $(0,1,1),$ and let $F = xyz(i+j).$ Calculate the surface integral

$$\int \int_S F \cdot dS,$$

if the triangle is oriented by the “downward” normal.

Solution. Since $S$ lies in a plane (see the right hand part of the Figure), it is part of the graph of a linear function $z = ax + by + c.$

Substituting the vertices of the triangle for $(x, y, z)$, we get the equation

$$0 = a + c, \quad 0 = 2b + c, \quad 1 = b + c,$$

which we can solve to find $b = -1, c = 2, a = -2,$ i.e., $z = -2x - y + 2.$ We may take $x$ and $y$ as parameters; i.e.,

$$x = u, \quad y = v, \quad z = -2u - v + 2,$$

or $\Phi(u, v) = (u, v, -2u - v + 2).$ The domain $D$ of the parametrization is the triangle with vertices at $(1, 0), (0, 2),$ and $(0, 1)$ in the $(u, v)$ plane. For this parametrization,

$$T_u \times T_v = (1, 0, -2) \times (0, 1, -1) = (2, 1, 1).$$

Since the third component of this vector is positive, the orientation determined by $\Phi$ is “upward,” so we will have to multiply our find answer by $-1$ to get the surface integral with the downward orientation.

Now we have (with the minus sign reminding us that the orientation is wrong),

$$-\int \int_D F \cdot dS = \int \int_D xyz(i+j) \cdot (2i + j + k)du \ dv$$

$$= \int \int_D 3xyz \ du \ dv = \int \int_D 3uv(-2u - v + 2)du \ dv.$$

To compute the double integral, we draw the integration domain $D$ in the $uv$-plane, in the left hand part of the Figure.

By reduction to iterated integrals,

$$\int \int_D 3uv(-2u - v + 2) \ du \ dv = \int_0^1 \int_{1-u}^{2-2u} (-6u^2v - 3uv^2 + 6uv) dv \ du$$
Carrying out the \( v \)-integration, we get

\[
\int_0^1 \left[ -3u^2v^2 - uv^3 + 3uv^2 \right] \frac{2-2u}{1-u} du
\]

\[
= \int_0^1 uv^2[-3u - v + 3] \frac{2(1-u)}{1-u} du
\]

\[
= \int_0^1 [4u(1-u)^2(1-u) - u(1-u)^2 2(1-u)] du
\]

Multiplying out and simplifying, this integral becomes

\[
2 \int_0^1 u(1-u)^3 du
\]

\[
= 2 \int_0^1 (u - 3u^2 + 3u^3 - u^4) du
\]

\[
= 2 \left( \frac{1}{2} - \frac{3}{3} + \frac{3}{4} - \frac{1}{5} \right) = \frac{1}{10},
\]

and so

\[
\int\int_S \mathbf{F} \cdot d\mathbf{S} = -\frac{1}{10}.
\]

**Note.** This example is related to the scalar integral example in homework set 7 (exercise 2 in section 7.5), which asks you to evaluate

\[
\int\int_S xyz dS
\]

where \( S \) is the same triangle with vertices \((1,0,0),(0,2,0)\) and \((0,1,1)\). Perhaps you can make use of some of the calculations from that exercise and the formula

\[
dS = \frac{dx\;dy}{\mathbf{n} \cdot \mathbf{k}} = \sqrt{6} dx\;dy
\]

obtained in the solution to that problem.
Example 3. The equations

\[ z = 12, \quad x^2 + y^2 \leq 25 \]

describe a disk of radius 5 lying in the plane \( z = 12 \). Suppose that \( \mathbf{r} \) is the position vector field

\[ \mathbf{r}(x, y, z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}. \]

Compute

\[ \iint_S \mathbf{r} \cdot dS. \]

Solution. Since the disk is parallel to the \( xy \) plane, the outward unit normal is \( \mathbf{k} \). Hence \( \mathbf{n}(x, y, z) = \mathbf{k} \) and so \( \mathbf{r} \cdot \mathbf{n} = z \). Thus,

\[
\iint_S \mathbf{r} \cdot dS = \iint_S \mathbf{r} \cdot \mathbf{n} dS = \iint_S z \, dS = \iint_D 12 \, dx \, dy = 300\pi.
\]

Alternatively we may solve this problem by using the formula for surface integrals over graphs:

\[
\iint_S \mathbf{F} \cdot dS = \iint_D \mathbf{F} \cdot \left( -\frac{\partial g}{\partial x} \mathbf{i} - \frac{\partial g}{\partial y} \mathbf{j} + \mathbf{k} \right) \, dx \, dy.
\]

With \( g(x, y) = 12 \) and \( D \) the disk \( x^2 + y^2 \leq 25 \), we get

\[
\iint_S \mathbf{r} \cdot dS = \iint_D (x \cdot 0 + y \cdot 0 + 12) \, dx \, dy = 12(\text{area of } D) = 300\pi.
\]
Example 4. Let $S$ be the closed surface that consists of the hemisphere $x^2 + y^2 + z^2 = 1, z \geq 0$, and its base $x^2 + y^2 \leq 1, z = 0$. Let $\mathbf{E}$ be the electric field defined by $\mathbf{E}(x, y, z) = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}$. Find the electric flux across $S$.

Solution. Write $S = H \cup D$ where $H$ is the upper hemisphere and $D$ is the disk. Hence
\[
\iint_{S} \mathbf{E} \cdot d\mathbf{S} = \iint_{H} \mathbf{E} \cdot d\mathbf{S} + \iint_{D} \mathbf{E} \cdot d\mathbf{S}.
\]

(i) Let $x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ be the unit normal $\mathbf{n}$ pointing outward from $H$. Then
\[
\iint_{H} \mathbf{E} \cdot d\mathbf{S} = \iint_{H} \mathbf{E} \cdot \mathbf{n} dS = \iint_{H} (2x, 2y, 2z) \cdot (x, y, z)dS = 2 \iint_{H} (x^2 + y^2 + z^2)dS = 2 \iint_{H} dS = 4\pi.
\]

(ii) The unit normal is $-\mathbf{k}$ and $z = 0$ on $D$. Hence,
\[
\iint_{D} \mathbf{E} \cdot d\mathbf{S} = \iint_{D} \mathbf{E} \cdot \mathbf{n} dS = \iint_{D} (2x, 2y, 2z) \cdot (0, 0, -1)dS = 0.
\]

Therefore,
\[
\iint_{S} \mathbf{E} \cdot d\mathbf{S} = 4\pi.
\]