Example 1 Evaluate the surface integral of the vector field $\mathbf{F} = 3x^2\mathbf{i} - 2yx\mathbf{j} + 8\mathbf{k}$ over the surface S that is the graph of $z = 2x - y$ over the rectangle $[0, 2] \times [0, 2]$.

Solution. Use the formula for a surface integral over a graph $z = g(x, y)$:

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_D \left[\mathbf{F} \cdot \left(-\frac{\partial g}{\partial x}\mathbf{i} - \frac{\partial g}{\partial y}\mathbf{j} + \mathbf{k}\right)\right] dx\, dy.$$

In our case we get

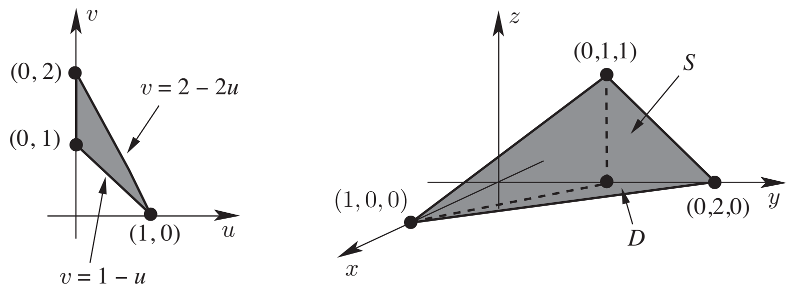
$$\begin{aligned} \int_0^2 \int_0^2 (3x^2, -2yx, 8) \cdot (-2, 1, 1) dx\, dy &= \int_0^2 \int_0^2 (-6x^2 - 2yx + 8) dx\, dy \\ &= \int_0^2 -2x^3 - yx^2 + 8x \Big|_{x=0}^2 dy \\ &= \int_0^2 -4y\, dy = -2y^2|_0^2 = -8. \end{aligned}$$

Example 2 Let S be the triangle with vertices $(1,0,0),(0,2,0)$, and $(0,1,1)$, and let $\mathbf{F} = xyz(\mathbf{i}+\mathbf{j})$. Calculate the surface integral

$$\iint_S \mathbf{F}\cdot d\mathbf{S},$$

if the triangle is oriented by the "downward" normal.

Solution. Since S lies in a plane (see the right hand part of the Figure), it is part of the graph of a linear function $z = ax + by + c$.



Substituting the vertices of the triangle for (x,y,z), we get the equation

$$0 = a + c, \quad 0 = 2b + c, \quad 1 = b + c,$$

which we can solve to find $b = -1, c = 2, a = -2$, *i.e.*, $z = -2x - y + 2$. We may take x and y as parameters; *i.e.,*

$$x = u, \quad y = v, \quad z = -2u - v + 2,$$

or $\mathbf{\Phi}(u,v) = (u, v, -2u-v+2)$. The domain D of the parametrization is the triangle with vertices at $(1,0),(0,2)$, and $(0,1)$ in the (u,v) plane. For this parametrization,

$$\mathbf{T}_u \times \mathbf{T}_v = (1,0,-2)\times(0,1,-1) = (2,1,1).$$

Since the third component of this vector is positive, the orientation determined by $\mathbf{\Phi}$ is "upward", so we will have to multiply our find answer by -1 to get the surface integral with the *downward* orientation.

Now we have (with the minus sign reminding us that the orientation is wrong),

$$-\iint_S \mathbf{F}\cdot d\mathbf{S} = \iint_D xyz(\mathbf{i}+\mathbf{j})\cdot(2\mathbf{i}+\mathbf{j}+\mathbf{k})du\,dv$$
$$= \iint_D 3\,xyz\,du\,dv = \iint_D 3\,uv(-2u-v+2)du\,dv.$$

To compute the double integral, we draw the integration domain D in the uv-plane, in the left hand part of the Figure.

By reduction to iterated integrals,

$$\iint_D 3\,uv(-2u-v+2)\,du\,dv = \int_0^1 \int_{1-u}^{2-2u} (-6u^2v - 3uv^2 + 6uv)dv\,du$$

Carrying out the v-integration, we get

$$\int_0^1 \left[-3u^2v^2 - uv^3 + 3uv^2\right]\Big|_{1-u}^{2-2u} du$$
$$= \int_0^1 uv^2[-3u - v + 3]\Big|_{1-u}^{2(1-u)} du$$
$$= \int_0^1 [4u(1-u)^2(1-u) - u(1-u)^2\, 2(1-u)]du$$

Multiplying out and simplifying, this integral becomes

$$2\int_0^1 u(1-u)^3 du$$
$$= 2\int_0^1 (u - 3u^2 + 3u^3 - u^4)du$$
$$= 2\left(\frac{1}{2} - \frac{3}{3} + \frac{3}{4} - \frac{1}{5}\right) = \frac{1}{10},$$

and so

$$\iint_S \mathbf{F}\cdot d\mathbf{S} = -\frac{1}{10}.$$

Note. This example is related to the scalar integral example in homework set 7 (exercise 2 in section 7.5), which asks you to evaluate

$$\iint_S xyz\, dS$$

where S is the same triangle with vertices $(1,0,0), (0,2,0)$ and $(0,1,1)$.. Perhaps you can make use of some of the calculations from that exercise and the formula $dS = \dfrac{dx\,dy}{\mathbf{n}\cdot\mathbf{k}} = \sqrt{6}\,dx\,dy$ obtained in the solution to that problem.

Example 3. The equations

$$z = 12, \quad x^2 + y^2 \leq 25$$

describe a disk of radius 5 lying in the plane $z = 12$. Suppose that $\mathbf{r}$ is the position vector field

$$\mathbf{r}(x, y, z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}.$$

Compute

$$\iint_S \mathbf{r} \cdot d\mathbf{S}.$$

Solution. Since the disk is parallel to the xy plane, the outward unit normal is $\mathbf{k}$. Hence $\mathbf{n}(x, y, z) = \mathbf{k}$ and so $\mathbf{r} \cdot \mathbf{n} = z$. Thus,

$$\iint_S \mathbf{r} \cdot d\mathbf{S} = \iint_S \mathbf{r} \cdot \mathbf{n}\, dS = \iint_S z\, dS = \iint_D 12\, dx\, dy = 300\pi.$$

Alternatively we may solve this problem by using the formula for surface integrals over graphs:

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_D \mathbf{F} \cdot \left(-\frac{\partial g}{\partial x}\mathbf{i} - \frac{\partial g}{\partial y}\mathbf{j} + \mathbf{k} \right) dx\, dy.$$

With $g(x, y) = 12$ and D the disk $x^2 + y^2 \leq 25$, we get

$$\iint_S \mathbf{r} \cdot d\mathbf{S} = \iint_D (x \cdot 0 + y \cdot 0 + 12) dx\, dy = 12(\text{ area of } D) = 300\pi.$$

Example 4. *Let S be the closed surface that consists of the hemisphere $x^2+y^2+z^2 = 1, z \geq 0$, and its base $x^2+y^2 \leq 1, z = 0$. Let $\mathbf{E}$ be the electric field defined by $\mathbf{E}(x,y,z) = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}$. Find the electric flux across S.*

Solution. Write $S = H \cup D$ where H is the upper hemisphere and D is the disk. Hence

$$\iint_S \mathbf{E} \cdot d\mathbf{S} = \iint_H \mathbf{E} \cdot d\mathbf{S} + \iint_D \mathbf{E} \cdot d\mathbf{S}.$$

(i) Let $x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ be the unit normal $\mathbf{n}$ pointing outward from H. Then

$$\begin{aligned}\iint_H \mathbf{E} \cdot d\mathbf{S} &= \iint_H \mathbf{E} \cdot \mathbf{n}\, dS = \iint_H (2x, 2y, 2z) \cdot (x, y, z) dS \\ &= 2 \iint_H (x^2+y^2+z^2) dS = 2 \iint_H dS = 4\pi.\end{aligned}$$

(ii) The unit normal is $-\mathbf{k}$ and $z = 0$ on D. Hence,

$$\iint_D \mathbf{E} \cdot d\mathbf{S} = \iint_D \mathbf{E} \cdot \mathbf{n}\, dS = \iint_D (2x, 2y, 2z) \cdot (0, 0, -1) dS = 0.$$

Therefore,

$$\iint_S \mathbf{E} \cdot d\mathbf{S} = 4\pi.$$