Review Example 3, Chapter 8.

Example 8.R.4 *For a region W in space with boundary ∂W, unit outward normal $\mathbf{n}$ and functions f and g defined on W and ∂W, prove Green's identities:*

$$\iint_{\partial W} (f\nabla g - g\nabla f) \cdot \mathbf{n}\, dS = \iiint_W (f\nabla^2 g - g\nabla^2 f) dx\, dy\, dz,$$

where

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

is the Laplacian of f.

Solution. To prove this identity, we apply Gauss' theorem to the vector field $f\nabla g - g\nabla f$ to obtain

$$\begin{aligned}
\iint_{\partial W} (f\nabla g - g\nabla f) \cdot \mathbf{n}\, dS &= \iiint_W \nabla \cdot (f\nabla g - g\nabla f)\, dV \\
&= \iiint_W \nabla f \cdot \nabla g + f\nabla^2 g - \nabla g \cdot \nabla f - g\nabla^2 f\, dV \\
&= \iiint_W f\nabla^2 g - g\nabla^2 f\, dV.
\end{aligned}$$