Review Example 2, Chapter 8.

(a) *Let* $\mathbf{F} = (x^2 + y - 4)\mathbf{i} + 3xy\mathbf{j} + (2xz + z^2)\mathbf{k}$. *Calculate the divergence and curl of* $\mathbf{F}$.

(b) *Find the flux of the curl of* $\mathbf{F}$ *across the surface* $x^2 + y^2 + z^2 = 16$, $z \geq 0$.

(c) *Find the flux of* $\mathbf{F}$ *across the surface of the unit cube* $[0, 1] \times [0, 1] \times [0, 1]$.

Solution.

(a) By a direct computation,

$$\nabla \cdot \mathbf{F} = 7x + 2z, \qquad \nabla \times \mathbf{F} = -2z\mathbf{j} + (3y - 1)\mathbf{k}.$$

(b) From Gauss' theorem and the identity $\nabla \cdot (\nabla \times \mathbf{F}) = 0$, we can conclude that the answer is 0.

(c) Applying Gauss' theorem once again, we get

$$\iint_S \mathbf{F} \cdot \mathbf{n}\, dS = \int_0^1 \int_0^1 \int_0^1 7x + 2z\, dx\, dy\, dz = \frac{9}{2}.$$