Review Example 1, Chapter 8.
Let W be the region in the octant $x \geq 0, y \geq 0, z \geq 0$, bounded by the three planes $y = 0, z = 0, x = y$, and by the sphere $x^2 + y^2 + z^2 = 1$.

(a) *Find the volume of W.*

(b) *Set up a triple integral giving the integral of a function $f(x, y, z)$ over this region using spherical coordinates.*

(c) *Calculate the surface integral*

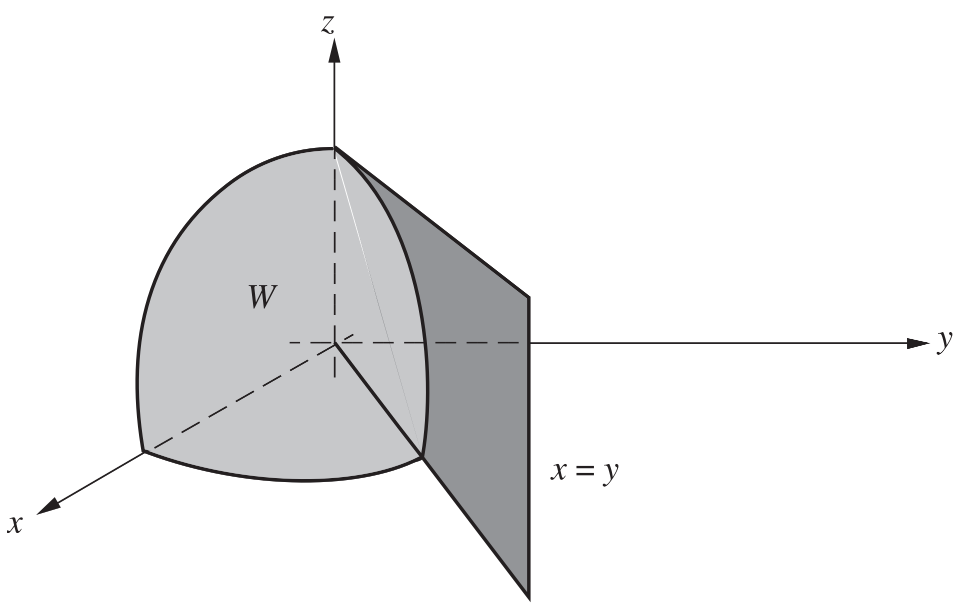
$$\iint_S \mathbf{F} \cdot d\mathbf{S}$$

where

$$F(x, y, z) = (3x - z^4)\mathbf{i} - (x^2 - y)\mathbf{j} + (xy^2)\mathbf{k}$$

and S is the boundary of the set W.

Solution. First we draw the following figure.



(a) The volume of the region W is 1/16th that of a unit sphere, so it is $[(4/3)\pi]/16 = \pi/12$.

(b) The required integral is

$$\int_{\phi=0}^{\pi/2} \int_{\theta=0}^{\pi/4} \int_{\rho=0}^{1} f(\rho \sin\phi \cos\theta, \rho \sin\phi \sin\theta, \rho \cos\phi)\rho^2 \sin\phi \, d\rho \, d\theta \, d\phi$$

by the change of variables formula.

(c) Notice that div $\mathbf{F} = 4$, so that by the divergence theorem,

$$\begin{aligned}\iint_S \mathbf{F} \cdot d\mathbf{S} &= \iiint_W \operatorname{div} \mathbf{F} \, dx \, dy \, dz \\ &= \iiint_W 4 \, dx \, dy \, dz = 4 \times \operatorname{vol}(W) = \pi/3\end{aligned}$$

using part (a).