1. Do each of the following calculations.

(a) If a particle follows the curve
\[ c(t) = e^{t-1}i - (t - 1)j + \sin(\pi t)k \]
and flies off on a tangent at \( t = 1 \), where is it at \( t = 2 \)?

**Solution.** First note that \( c(1) = i \). We compute that \( c'(t) = e^{t-1}i - j + \pi \cos(\pi t)k \).
Therefore \( c'(1) = i - j - \pi k \). Hence the position of the particle at time \( t = 2 \) is
\[ i + (2 - 1)(i - j - \pi k) = 2i - j - \pi k. \]

(b) Find the equation of the tangent plane to the surface \( x^2 - e^{xy} + z^2 = 1 \) at the point \( (1, 0, 1) \).

**Solution.** The tangent plane consists of vectors based at \( (1, 0, 1) \) that are perpendicular to the gradient of \( f(x, y, z) = x^2 - e^{xy} + z^2 \). We compute that
\[ \nabla f(x, y, z) = (2x - ye^{xy}, -xe^{xy}, 2z). \]
Evaluating at \( (1, 0, 1) \), we obtain
\[ \nabla f(1, 0, 1) = (2, -1, 2). \]
Therefore, the tangent plane at \( (1, 0, 1) \) is defined by
\[ (2, -1, 2) \cdot (x - 1, y, z - 1) = 0, \]
namely by
\[ z = -x + \frac{y}{2} + 2. \]

(c) Let \( f(x, y, z) = 5 + xy - zx \) be the concentration of chemical X. Find the direction at \( (1, 1, 1) \) in which X is decreasing the fastest. In which directions is it decreasing at 30% of its maximum rate? Give your answer in terms of the angle made with the direction of fastest decrease.

**Solution.** The concentration of X is increasing the fastest in the direction of the gradient of \( f \), and hence decreasing the fastest in the opposite of this direction. We compute that
\[ \nabla f(x, y, z) = (y - z, x, -x), \]
and hence that
\[ \nabla f(1, 1, 1) = (0, 1, -1). \]
Therefore the concentration of $X$ is decreasing fastest in the direction $-\nabla f(1,1,1) = (0,-1,1)$.

Let $\mathbf{n}$ be a unit vector, and let $\theta$ be the angle between $\mathbf{n}$ and $-\nabla f(1,1,1)$. Then the directional derivative of $f$ in the direction $\mathbf{n}$ is given by $\nabla f(1,1,1) \cdot \mathbf{n} = \|\nabla f(1,1,1)\| \|\mathbf{n}\| \cos(\pi - \theta)$. This is 30% of its most negative value when $\cos(\pi - \theta) = -0.3$, which is equivalent to $\cos(\theta) = 0.3$. This means that $\theta = \cos^{-1}(0.3)$. 

2. Answer each of the following questions.

(a) Let $f(r,s)$ be a (smooth) function of $r$ and $s$ and, let $r = x + 2y$ and $s = x - 2y$. Define the function $h$ by $h(x,y) = f(x + 2y, x - 2y)$. Calculate

$$\frac{\partial^2 h}{\partial x \partial y}$$

in terms of the partial derivatives of $f$.

Solution. Let $r(x,y) = x + 2y$ and $s(x,y) = x - 2y$. Then we have that

$$\frac{\partial r}{\partial x} = 1, \quad \frac{\partial r}{\partial y} = 2, \quad \frac{\partial s}{\partial x} = 1, \quad \frac{\partial s}{\partial y} = -2.$$

Applying the chain rule to $h(x,y) = f(r(x,y), s(x,y))$, we find that

$$\frac{\partial^2 h}{\partial x \partial y} = \frac{\partial}{\partial x} \left( \frac{\partial h}{\partial y} \right) = \frac{\partial}{\partial x} \left( \frac{\partial f}{\partial r} \frac{\partial r}{\partial y} + \frac{\partial f}{\partial s} \frac{\partial s}{\partial y} \right) = \frac{\partial}{\partial x} \left( 2 \frac{\partial f}{\partial r} - \frac{\partial f}{\partial s} \right) = 2 \frac{\partial}{\partial x} \left( \frac{\partial f}{\partial r} \right) - 2 \frac{\partial}{\partial x} \left( \frac{\partial f}{\partial s} \right) = 2 \left( \frac{\partial^2 f}{\partial r^2} \frac{\partial r}{\partial x} + \frac{\partial^2 f}{\partial r \partial s} \frac{\partial s}{\partial x} \right) - 2 \left( \frac{\partial^2 f}{\partial r \partial s} \frac{\partial r}{\partial x} + \frac{\partial^2 f}{\partial s^2} \frac{\partial s}{\partial x} \right) = 2 \left( \frac{\partial^2 f}{\partial r^2} + \frac{\partial^2 f}{\partial s \partial r} \right) - 2 \left( \frac{\partial^2 f}{\partial r \partial s} + \frac{\partial^2 f}{\partial s^2} \right).$$

Because $f(r,s)$ is a smooth function, $\partial^2 f / \partial s \partial r = \partial^2 f / \partial r \partial s$. Thus

$$\frac{\partial^2 h}{\partial x \partial y} = 2 \frac{\partial^2 f}{\partial r^2} - 2 \frac{\partial^2 f}{\partial s^2},$$

or, more precisely,

$$\frac{\partial^2 h}{\partial x \partial y}(x,y) = 2 \frac{\partial^2 f}{\partial r^2}(x + 2y, x - 2y) - 2 \frac{\partial^2 f}{\partial s^2}(x + 2y, x - 2y).$$
(b) Let \( f(u, v, w) \) be a given (differentiable) real valued function of three variables and let
\[
g(x, y) = f(x^2 + y^2, 2xy, x^2 - y^2)
\]
Writing gradients as column vectors, write the gradient of \( g \) as \( \nabla g = \mathbf{M} \cdot \nabla f \) where \( \mathbf{M} \) is a matrix function of \( x \) and \( y \) and the dot is matrix multiplication. Explicitly determine this matrix \( \mathbf{M} \).

**Solution.** Let \( u = x^2 + y^2, v = 2xy, \) and \( w = x^2 - y^2 \). Then, as \( g(x, y) = f(u, v, w) \), the chain rule gives that
\[
\nabla g = \nabla f \cdot \begin{bmatrix}
\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\
\frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \\
\frac{\partial w}{\partial x} & \frac{\partial w}{\partial y}
\end{bmatrix},
\]
where \( \nabla g \) and \( \nabla f \) are treated as row vectors. Transposing both sides of the equality, we get \( \nabla g = \mathbf{M} \cdot \nabla f \), where \( \nabla g \) and \( \nabla f \) are treated as column vectors, with
\[
\mathbf{M} = \begin{bmatrix}
\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\
\frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \\
\frac{\partial w}{\partial x} & \frac{\partial w}{\partial y}
\end{bmatrix} = \begin{bmatrix}
2x & 2y & 2x \\
2y & 2x & -2y
\end{bmatrix}.
\]

3. Answer each of the following questions.

(a) Let a curve in space \( \mathbf{c}(t) \) satisfy \( \mathbf{c}'(t) = \nabla T(\mathbf{c}(t)) \) where \( T(x, y, z) = x^3 - zy^3 \) and \( \mathbf{c}(0) = (1, 1, 0) \). Show that \( \mathbf{c}'(0) \) is perpendicular to the surface \( x^3 - zy^3 = 1 \).

**Solution.** At any given point \((x_0, y_0, z_0)\), the gradient \( \nabla T \) is perpendicular to the level set of \( T \) containing \( T(x_0, y_0, z_0) \). Taking \( \mathbf{c}(0) = (1, 1, 0) \) for \((x_0, y_0, z_0)\), we get \( T(1, 1, 0) = 1 \), so \( \nabla T(\mathbf{c}(0)) = \mathbf{c}'(0) \) is perpendicular to the level set \( T(x, y, z) = x^3 - zy^3 = 1 \).

(b) In the preceding question, even though we might not know \( \mathbf{c}(1) \) explicitly, must \( T(\mathbf{c}(0)) \leq T(\mathbf{c}(1)) \)?

**Solution.** Yes. By the Fundamental Theorem of Calculus, the difference \( T(\mathbf{c}(1)) - T(\mathbf{c}(0)) \) is equal to
\[
\int_0^1 \frac{d}{dt} T(\mathbf{c}(t)) dt.
\]
Applying the chain rule, we obtain \( \frac{d}{dt} T(\mathbf{c}(t)) = \nabla T(\mathbf{c}(t)) \cdot \mathbf{c}'(t) \), which by the assumption in (a) is equal to \( \mathbf{c}'(t) \cdot \mathbf{c}'(t) = \| \mathbf{c}'(t) \|^2 \) and so is nonnegative everywhere. Thus the integral expression for \( T(\mathbf{c}(1)) - T(\mathbf{c}(0)) \) is nonnegative, so \( T(\mathbf{c}(0)) \leq T(\mathbf{c}(1)) \).

(c) Let the curve in space \( \mathbf{c}(t) \) satisfy \( \mathbf{c}'(t) = \nabla f(\mathbf{c}(t)) \) where \( f(x, y, z) = x^2 - xy + z^2 \) and satisfy \( \mathbf{c}(0) = (1, 1, 1) \). Calculate the *acceleration* of the curve \( \mathbf{c}(t) \) at \( t = 0 \).
Solution. The acceleration of \( c(t) \) at \( t = 0 \) is the second derivative
\[
\frac{d}{dt}(c'(t))|_{t=0}
\]
By the hypothesis, we have
\[
c'(t) = (\nabla f)(c(t))
\]
and thus by the chain rule,
\[
c''(t) = \frac{d}{dt}((\nabla f)(c(t)))
\]
\[
= (D\nabla f)(c(t))c'(t),
\]
so that
\[
c''(0) = (D\nabla f)(c(0))c'(0)
\]
\[
= (D\nabla f)(1, 1, 1)(\nabla f)(c(0))
\]
\[
= (D\nabla f)(1, 1, 1)(\nabla f)(1, 1, 1).
\]
Now, the gradient of \( f \) can be computed as
\[
\nabla f(x, y, z) = \begin{pmatrix}
\frac{\partial f}{\partial x} \\
\frac{\partial f}{\partial y} \\
\frac{\partial f}{\partial z}
\end{pmatrix}
= \begin{pmatrix}
2x - y \\
-x \\
2z
\end{pmatrix};
\]
in particular, \( D\nabla f \) is the matrix whose columns are the partial derivatives of this column vector:
\[
(D\nabla f)(x, y, z) = \begin{pmatrix}
2 & -1 & 0 \\
-1 & 0 & 0 \\
0 & 0 & 2
\end{pmatrix}.
\]
Evaluating at \((1, 1, 1)\) and multiplying gives
\[
c''(0) = \begin{pmatrix}
2 & -1 & 0 \\
-1 & 0 & 0 \\
0 & 0 & 2
\end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}
= \begin{pmatrix}
3 \\
-1 \\
4
\end{pmatrix}.
\]
(In Math 2a, you’ll learn how to solve such equations; one finds that
\[
c(t) = \left( \frac{e^{(1+\sqrt{2})t} + e^{(1-\sqrt{2})t}}{2}, \frac{(1 - \sqrt{2})e^{(1+\sqrt{2})t} + (1 + \sqrt{2})e^{(1-\sqrt{2})t}}{2}, e^{2t} \right),
\]
from which one can read off the acceleration by direct differentiation.)
4. (a) Consider the function

\[ f(x, y) = 2x^2y - x^2 - y^2. \]

Find and classify the critical points of \( f \) as local maxima, minima, or saddles.

**Solution.** First observe that \( f \), as a polynomial, is not only defined and continuous on all of \( \mathbb{R}^3 \), but also \( C^3 \) on all of \( \mathbb{R}^3 \). Thus, we can apply Theorem 3.5 to classify the critical points of \( f \) on \( \mathbb{R}^3 \).

We compute that

\[
\nabla f(x, y) = \left( \frac{\partial f}{\partial x}(x, y), \frac{\partial f}{\partial y}(x, y) \right) \\
= (4xy - 2x, 2x^2 - 2y) \\
= 2(2x(y - \frac{1}{2}), x^2 - y),
\]

so that \( \nabla f(x, y) = 0 \) if and only if \( 2x(y - \frac{1}{2}) = 0 \) and \( x^2 - y = 0 \). Now, \( 2x(y - \frac{1}{2}) = 0 \) if and only if \( x = 0 \) or \( y = \frac{1}{2} \); in the case that \( x = 0 \), \( x^2 - y = 0 \) if and only if \( y = 0 \), and in the case that \( y = \frac{1}{2} \), \( x^2 - y = 0 \) if and only if \( x = \pm \sqrt{\frac{1}{2}} \). Thus, \( \nabla f(x, y) = 0 \) if and only if \( (x, y) = (0, 0) \) or \( (\pm \sqrt{\frac{1}{2}}, \frac{1}{2}) \); since \( f \) is differentiable on all of \( \mathbb{R}^3 \), these are all the critical points of \( f \).

Now, the Hessian matrix of \( f \) is given by

\[
Hf(x, y) = \begin{pmatrix}
\frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\
\frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2}
\end{pmatrix} = \begin{pmatrix}
4y - 2 & 4x \\
4x & -2
\end{pmatrix},
\]

so that

\[
Hf(0, 0) = \begin{pmatrix}
-2 & 0 \\
0 & -2
\end{pmatrix}
\]

and

\[
Hf(\pm \sqrt{\frac{1}{2}}, \frac{1}{2}) = \begin{pmatrix}
0 & \pm 2\sqrt{\frac{1}{2}} \\
\pm 2\sqrt{\frac{1}{2}} & 0
\end{pmatrix}.
\]

On the one hand, \( Hf(0, 0) \) is manifestly negative definite, so that by Theorem 3.5, \( f \) has a local maximum at \( (0, 0) \). On the other hand, \( Hf(\pm \sqrt{\frac{1}{2}}, \frac{1}{2}) \) is symmetric, and thus is diagonalizable with two (not necessarily distinct) real eigenvalues \( \lambda_1 \) and \( \lambda_2 \). Hence,

\[
\lambda_1 \lambda_2 = \text{det} \left( Hf(\pm \sqrt{\frac{1}{2}}, \frac{1}{2}) \right) = \text{det} \left( \begin{pmatrix}
0 & \pm 2\sqrt{\frac{1}{2}} \\
\pm 2\sqrt{\frac{1}{2}} & 0
\end{pmatrix} \right) = -8 < 0,
\]

so that \( Hf(\pm \sqrt{\frac{1}{2}}, \frac{1}{2}) \) has one positive eigenvalue and one negative eigenvalue, i.e. \( f \) has a saddle point at \( (\pm \sqrt{\frac{1}{2}}, \frac{1}{2}) \).

Thus, the critical points of \( f \) are the local maximum \( (0, 0) \) and the saddle points \((\sqrt{\frac{1}{2}}, \frac{1}{2})\) and \((-\sqrt{\frac{1}{2}}, \frac{1}{2})\). \(\square\)
(b) Use the method of Lagrange multipliers to determine if the maximum of the function $f$ in part (a) on the region $x^2 + y^2 \leq 1$ is on the boundary circle $x^2 + y^2 = 1$.

**Solution.** Since $f$ is continuous on $\mathbb{R}^3$, it is, in particular, continuous on $\{(x, y) \in \mathbb{R}^3 \mid x^2 + y^2 \leq 1\} = D \cup \partial D$, for $D = \{(x, y) \in \mathbb{R}^3 \mid x^2 + y^2 < 1\}$. Since $D \cup \partial D$ is closed and bounded, it follows from Theorem 3.7 that $f$ restricted to $D \cup \partial D$ attains a global maximum and a global minimum.

On the one hand, by (a), since $(0, 0), \left(\frac{1}{\sqrt{2}}, \frac{1}{2}\right)$ and $\left(-\frac{1}{\sqrt{2}}, \frac{1}{2}\right)$ all lie in $D$, the critical points of $f$ restricted to $D$ are precisely $(0, 0), \left(\frac{1}{\sqrt{2}}, \frac{1}{2}\right)$ and $\left(-\frac{1}{\sqrt{2}}, \frac{1}{2}\right)$.

On the other hand, $\partial D = \{(x, y) \in \mathbb{R}^3 \mid x^2 + y^2 = 1\} = \{(x, y) \in \mathbb{R}^3 \mid g(x, y) = 0\}$, where $g(x, y) = x^2 + y^2 - 1$, so that since $\nabla g(x, y) = 2(x, y) \neq 0$ on $\partial D$, we can apply the method of Lagrange multipliers (i.e. Theorem 3.8) to find the possible local maxima and minima of $f$ on $\partial D$. Now, let $\lambda \in \mathbb{R}$. Then $\nabla f(x, y) = \lambda \nabla g(x, y)$ if and only if $2(2x - y - \frac{1}{2}), x^2 - y = 2\lambda(x, y)$, if and only if $x(2y - \lambda - 1) = 0$ and $x^2 = (1 + \lambda)y$. Thus, we must solve the following system of equations:

\[
\begin{align*}
x(2y - \lambda - 1) &= 0 \\
x^2 &= (1 + \lambda)y \\
x^2 + y^2 &= 1.
\end{align*}
\]

Now, by the first equation, there are two possible cases, namely, $x = 0$ or $y = \frac{1 + \lambda}{2}$.

First, suppose that $x = 0$. Then by the third equation, $y^2 = 1$, so that $y = \pm 1$, and hence, by the second equation, $\lambda = -1$. Hence, we obtain the candidate points $(0, \pm 1)$. Now, suppose that $y = \frac{1 + \lambda}{2}$. Then, by the second equation, $x^2 = \frac{(1 + \lambda)^2}{2}$, so that $x = \pm \frac{|1 + \lambda|}{\sqrt{2}}$, and hence, by the third equation,

\[
1 = \frac{(1 + \lambda)^2}{2} + \frac{(1 + \lambda)^2}{4} = \frac{3}{4}(1 + \lambda)^2,
\]

so that $1 + \lambda = \pm \frac{2}{\sqrt{3}}$. Hence, we obtain the candidate points $\left(\frac{\sqrt{2}}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right), \left(\frac{-\sqrt{2}}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right), \left(-\frac{\sqrt{2}}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right), \text{and} \left(-\frac{\sqrt{2}}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right)$.

Thus, the possible global maxima and minima of $f$ on $D \cup \partial D$ are $(0, 0), \left(\frac{1}{\sqrt{2}}, \frac{1}{2}\right), (0, \pm 1), \left(\frac{\sqrt{2}}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right), \left(\frac{-\sqrt{2}}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right), \left(-\frac{\sqrt{2}}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right), \text{and} \left(-\frac{\sqrt{2}}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right)$. Since

\[
\begin{align*}
f(0, 0) &= 0 \\
f\left(\frac{1}{\sqrt{2}}, \frac{1}{2}\right) &= -\frac{1}{2} \\
f(0, \pm 1) &= -1 \\
f\left(\frac{\sqrt{2}}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right) &= \frac{4}{3\sqrt{3}} - 1 \\
f\left(\frac{\sqrt{2}}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right) &= -\frac{4}{3\sqrt{3}} - 1 \\
f\left(-\frac{\sqrt{2}}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right) &= \frac{4}{3\sqrt{3}} - 1 \\
f\left(-\frac{\sqrt{2}}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right) &= -\frac{4}{3\sqrt{3}} - 1,
\end{align*}
\]

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it follows that \( f \) attains a global maximum at \((0,0)\), which is not on the boundary circle \( \partial D \).

(c) Consider the function \( f(x,y) = ax^2 + 2bxy + cy^2 \) defined on the whole \( xy \)-plane. Suppose that the eigenvalues of the matrix

\[
\begin{bmatrix}
a & b \\
b & c \\
\end{bmatrix}
\]

are both positive. Is the origin a local minimum of \( f \)? Must it be a global minimum of \( f \)?

**Solution.** Yes and Yes.

We calculate the partial derivatives

\[
\frac{\partial f}{\partial x} = 2ax + 2by \\
\frac{\partial f}{\partial y} = 2bx + 2cy.
\]

These both vanish at the origin so the origin is indeed a critical point. Further,

\[
\frac{\partial^2 f}{\partial x^2} = 2a \\
\frac{\partial^2 f}{\partial y^2} = 2c \\
\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x} = 2b,
\]

so the Hessian is twice the matrix given in the problem. If the eigenvalues are both positive then the origin is a local minimum.

While the fact that the origin is a local minimum does not imply that it is a global minimum, it turns out that in this case the origin is also a global minimum. If both eigenvalues are positive then the diagonal derivatives must be positive. That is, \( a > 0 \) and \( ac - b^2 > 0 \) implying as well that \( c > 0 \). Now for fixed \((x,y)\), we have either

\[
ax^2 + 2bxy + cy^2 > ax^2 + 2 \sqrt{ac}xy + cy^2 = (\sqrt{ac})^2
\]

or

\[
ax^2 + 2bxy + cy^2 > ax^2 - 2 \sqrt{ac}xy + cy^2 = (\sqrt{ac})^2
\]

depending on whether the middle term is positive or negative. Either way we have expressed \( f(x,y) \) as a square so it is nonnegative, and we know that \( f(0,0) = 0 \).

5. (a) Let \( \mathbf{F}(x,y) = f(x^2 + y^2)[-yi + xj] \) for a given function \( f \) of one variable. Find an equation that \( g(t) \) should satisfy so that

\[
\mathbf{c}(t) = [\cos g(t)]i + [\sin g(t)]j
\]

will be a flow line for \( \mathbf{F} \).
Solution. By definition, \( c \) is a flow line of \( F \) if \( F(c(t)) = c'(t) \). We calculate that

\[
F(c(t)) = f(\cos^2(g(t)) + \sin^2(g(t)))(-\sin(g(t))i + \cos(g(t))j)
\]

= \( f(1)(-\sin(g(t))i + \cos(g(t))j) \)

and

\[
c'(t) = \frac{d}{dt} \cos(g(t))i + \frac{d}{dt} \sin(g(t))j
\]

= \( g'(t)(-\sin(g(t))i + \cos(g(t))j) \).

Thus the condition that \( c(t) \) is a flow line of \( F \) means that

\[
f(1)(-\sin(g(t))i + \cos(g(t))j) = g'(t)(-\sin(g(t))i + \cos(g(t))j).
\]

Since \(-\sin(g(t))i + \cos(g(t))j\) is always a nonzero vector, we may divide through by it to obtain \( f(1) = g'(t) \). Therefore we conclude that \( g(t) = f(1)t + C \) for some constant \( C \).

(b) Let \( F = (x^2 + y - 4)i + 3xyj + (2xz + z^2)k \). Calculate the divergence and curl of \( F \).

**Solution.** We compute that

\[
\text{div } F = \frac{\partial}{\partial x}(x^2 + y - 4) + \frac{\partial}{\partial y}3xy + \frac{\partial}{\partial z}(2xz + z^2)
\]

= \( (2x) + (3x) + (2x + 2z) \)

= \( 7x + 2z \).

and that

\[
\text{curl } F = (\partial_y(2xz + z^2) - \partial_z 3xy)i + (\partial_z(x^2 + y - 4) - \partial_x(2xz + z^2))j
\]

\[
+ (\partial_x 3xy - \partial_y(x^2 + y - 4))k
\]

= \( (0 - 0)i + (0 - 2z)j + (3y - 1)k \)

= \(-2zj + (3y - 1)k \).