Mathematics 1c: Solutions, Midterm Examination
Due: Monday, May 3, at 10am

1. Do each of the following calculations.

   (a) If a particle follows the curve

   $$\mathbf{c}(t) = e^{t-1}\mathbf{i} - (t-1)\mathbf{j} + \sin(\pi t)\mathbf{k}$$

   and flies off on a tangent at $t = 1$, where is it at $t = 2$?

   Solution. First note that $\mathbf{c}(1) = \mathbf{i}$. We compute that $\mathbf{c}'(t) = e^{t-1}\mathbf{i} - \mathbf{j} + \pi\cos(\pi t)\mathbf{k}$. Therefore $\mathbf{c}'(1) = \mathbf{i} - \mathbf{j} - \pi\mathbf{k}$. Hence the position of the particle at time $t = 2$ is

   $$\mathbf{i} + (2-1)(\mathbf{i} - \mathbf{j} - \pi\mathbf{k}) = 2\mathbf{i} - \mathbf{j} - \pi\mathbf{k}. \qquad \square$$

   (b) Find the equation of the tangent plane to the surface $x^2 - e^{xy} + z^2 = 1$ at the point $(1, 0, 1)$.

   Solution. The tangent plane consists of vectors based at $(1, 0, 1)$ that are perpendicular to the gradient of $f(x, y, z) = x^2 - e^{xy} + z^2$. We compute that

   $$\nabla f(x, y, z) = (2x - ye^{xy}, -xe^{xy}, 2z).$$

   Evaluating at $(1, 0, 1)$, we obtain

   $$\nabla f(1, 0, 1) = (2, -1, 2).$$

   Therefore, the tangent plane at $(1, 0, 1)$ is defined by

   $$(2, -1, 2) \cdot (x - 1, y, z - 1) = 0,$$

   namely by

   $$z = -x + \frac{y}{2} + 2. \qquad \square$$

   (c) Let $f(x, y, z) = 5 + xy - zx$ be the concentration of chemical X. Find the direction at $(1, 1, 1)$ in which X is *decreasing* the fastest. In which directions is it decreasing at 30% of its maximum rate? Give your answer in terms of the angle made with the direction of fastest decrease.

   Solution. The concentration of X is increasing the fastest in the direction of the gradient of f, and hence decreasing the fastest in the opposite of this direction. We compute that

   $$\nabla f(x, y, z) = (y - z, x, -x),$$

   and hence that

   $$\nabla f(1, 1, 1) = (0, 1, -1).$$

Therefore the concentration of X is decreasing fastest in the direction

$$-\nabla f(1,1,1) = (0,-1,1).$$

Let $\mathbf{n}$ be a unit vector, and let θ be the angle between $\mathbf{n}$ and $-\nabla f(1,1,1)$. Then the directional derivative of f in the direction $\mathbf{n}$ is given by $\nabla f(1,1,1)\cdot\mathbf{n} = \|\nabla f(1,1,1)\|\|\mathbf{n}\|\cos(\pi-\theta)$. This is 30% of its most negative value when $\cos(\pi-\theta) = -0.3$, which is equivalent to $\cos(\theta) = 0.3$. This means that $\theta = \cos^{-1}(0.3)$. $\square$

2. Answer each of the following questions.

(a) Let $f(r,s)$ be a (smooth) function of r and s and, let $r = x+2y$ and $s = x-2y$. Define the function h by $h(x,y) = f(x+2y, x-2y)$. Calculate

$$\frac{\partial^2 h}{\partial x \partial y}$$

in terms of the partial derivatives of f.

Solution. Let $r(x,y) = x+2y$ and $s(x,y) = x-2y$. Then we have that

$$\frac{\partial r}{\partial x} = 1, \quad \frac{\partial r}{\partial y} = 2, \quad \frac{\partial s}{\partial x} = 1, \quad \frac{\partial s}{\partial y} = -2.$$

Applying the chain rule to $h(x,y) = f(r(x,y), s(x,y))$, we find that

$$\begin{aligned}
\frac{\partial^2 h}{\partial x \partial y} &= \frac{\partial}{\partial x}\left(\frac{\partial h}{\partial y}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial f}{\partial s}\frac{\partial s}{\partial y}\right) \\
&= \frac{\partial}{\partial x}\left(2\frac{\partial f}{\partial r} - 2\frac{\partial f}{\partial s}\right) \\
&= 2\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial r}\right) - 2\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial s}\right) \\
&= 2\left(\frac{\partial^2 f}{\partial r^2}\frac{\partial r}{\partial x} + \frac{\partial^2 f}{\partial s \partial r}\frac{\partial s}{\partial x}\right) - 2\left(\frac{\partial^2 f}{\partial r \partial s}\frac{\partial r}{\partial x} + \frac{\partial^2 f}{\partial s^2}\frac{\partial s}{\partial x}\right) \\
&= 2\left(\frac{\partial^2 f}{\partial r^2} + \frac{\partial^2 f}{\partial s \partial r}\right) - 2\left(\frac{\partial^2 f}{\partial r \partial s} + \frac{\partial^2 f}{\partial s^2}\right).
\end{aligned}$$

Because $f(r,s)$ is a smooth function, $\partial^2 f/\partial s \partial r = \partial^2 f/\partial r \partial s$. Thus

$$\frac{\partial^2 h}{\partial x \partial y} = 2\frac{\partial^2 f}{\partial r^2} - 2\frac{\partial^2 f}{\partial s^2},$$

or, more precisely,

$$\frac{\partial^2 h}{\partial x \partial y}(x,y) = 2\frac{\partial^2 f}{\partial r^2}(x+2y, x-2y) - 2\frac{\partial^2 f}{\partial s^2}(x+2y, x-2y).$$ $\square$

(b) Let $f(u,v,w)$ be a given (differentiable) real valued function of three variables and let

$$g(x,y) = f(x^2+y^2, 2xy, x^2-y^2)$$

Writing gradients as column vectors, write the gradient of g as $\nabla g = M \cdot \nabla f$ where M is a matrix function of x and y and the dot is matrix multiplication. Explicitly determine this matrix M.

Solution. Let $u = x^2+y^2$, $v = 2xy$, and $w = x^2-y^2$. Then, as $g(x,y) = f(u,v,w)$, the chain rule gives that

$$\nabla g = \nabla f \cdot \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} \end{bmatrix},$$

where ∇g and ∇f are treated as row vectors. Transposing both sides of the equality, we get $\nabla g = M \cdot \nabla f$, where ∇g and ∇f are treated as column vectors, with

$$M = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} & \frac{\partial w}{\partial x} \\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} & \frac{\partial w}{\partial y} \end{bmatrix} = \begin{bmatrix} 2x & 2y & 2x \\ 2y & 2x & -2y \end{bmatrix}. \qquad \square$$

3. Answer each of the following questions.

(a) Let a curve in space $\mathbf{c}(t)$ satisfy $\mathbf{c}'(t) = \nabla T(\mathbf{c}(t))$ where T is the function $T(x,y,z) = x^3 - zy^3$ and $\mathbf{c}(0) = (1,1,0)$. Show that $\mathbf{c}'(0)$ is perpendicular to the surface $x^3 - zy^3 = 1$.

Solution. At any given point (x_0, y_0, z_0), the gradient ∇T is perpendicular to the level set of T containing $T(x_0,y_0,z_0)$. Taking $\mathbf{c}(0) = (1,1,0)$ for (x_0,y_0,z_0), we get $T(1,1,0) = 1$, so $\nabla T(\mathbf{c}(0)) = \mathbf{c}'(0)$ is perpendicular to the level set $T(x,y,z) = x^3 - zy^3 = 1$. $\square$

(b) In the preceding question, even though we might not know $\mathbf{c}(1)$ explicitly, must $T(c(0)) \leq T(c(1))$?

Solution. Yes. By the Fundamental Theorem of Calculus, the difference $T(\mathbf{c}(1)) - T(\mathbf{c}(0))$ is equal to

$$\int_0^1 \frac{d}{dt} T(c(t)) dt.$$

Applying the chain rule, we obtain $\frac{d}{dt}T(\mathbf{c}(t)) = \nabla T(\mathbf{c}(t) \cdot \mathbf{c}'(t)$, which by the assumption in (a) is equal to $\mathbf{c}'(t) \cdot \mathbf{c}'(t) = \|\mathbf{c}'(t)\|^2$ and so is nonnegative everywhere. Thus the integral expression for $T(\mathbf{c}(1)) - T(\mathbf{c}(0))$ is nonnegative, so $T(\mathbf{c}(0)) \leq T(c(1))$. $\square$

(c) Let the curve in space $\mathbf{c}(t)$ satisfy $\mathbf{c}'(t) = \nabla f(\mathbf{c}(t))$ where $f(x,y,z) = x^2 - xy + z^2$ and satisfy $\mathbf{c}(0) = (1,1,1)$. Calculate the *acceleration* of the curve $\mathbf{c}(t)$ at $t = 0$.

Solution. The acceleration of $\mathbf{c}(t)$ at $t = 0$ is the second derivative

$$\frac{d}{dt}(\mathbf{c}'(t))|_{t=0}$$

By the hypothesis, we have

$$\mathbf{c}'(t) = (\nabla f)(\mathbf{c}(t))$$

and thus by the chain rule,

$$\begin{aligned}\mathbf{c}''(t) &= \frac{d}{dt}((\nabla f)(\mathbf{c}(t))) \\ &= (D\nabla f)(\mathbf{c}(t))\mathbf{c}'(t),\end{aligned}$$

so that

$$\begin{aligned}\mathbf{c}''(0) &= (D\nabla f)(\mathbf{c}(0))\mathbf{c}'(0) \\ &= (D\nabla f)(1,1,1)(\nabla f)(\mathbf{c}(0)) \\ &= (D\nabla f)(1,1,1)(\nabla f)(1,1,1).\end{aligned}$$

Now, the gradient of f can be computed as

$$\begin{aligned}\nabla f(x,y,z) &= \begin{pmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \\ \frac{\partial f}{\partial z} \end{pmatrix} \\ &= \begin{pmatrix} 2x - y \\ -x \\ 2z \end{pmatrix};\end{aligned}$$

in particular, $D\nabla f$ is the matrix whose columns are the partial derivatives of this column vector:

$$(D\nabla f)(x,y,z) = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

Evaluating at $(1,1,1)$ and multiplying gives

$$\begin{aligned}\mathbf{c}''(0) &= \begin{pmatrix} 2 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} \\ &= \begin{pmatrix} 3 \\ -1 \\ 4 \end{pmatrix}.\end{aligned}$$

(In Math 2a, you'll learn how to solve such equations; one finds that

$$\mathbf{c}(t) = \left(\frac{e^{(1+\sqrt{2})t} + e^{(1-\sqrt{2})t}}{2}, \frac{(1-\sqrt{2})e^{(1+\sqrt{2})t} + (1+\sqrt{2})e^{(1-\sqrt{2})t}}{2}, e^{2t} \right),$$

from which one can read off the acceleration by direct differentiation.) $\square$

4. (a) Consider the function

$$f(x,y) = 2x^2y - x^2 - y^2.$$

Find and classify the critical points of f as local maxima, minima, or saddles.

Solution. First observe that f, as a polynomial, is not only defined and continuous on all of $\mathbb{R}^3$, but also C^3 on all of $\mathbb{R}^3$. Thus, we can apply Theorem 3.5 to classify the critical points of f on $\mathbb{R}^3$.

We compute that

$$\begin{aligned}\nabla f(x,y) &= \left(\frac{\partial f}{\partial x}(x,y), \frac{\partial f}{\partial y}(x,y)\right) \\ &= (4xy - 2x, 2x^2 - 2y) \\ &= 2(2x(y - \tfrac{1}{2}), x^2 - y),\end{aligned}$$

so that $\nabla f(x,y) = 0$ if and only if $2x(y - \frac{1}{2}) = 0$ and $x^2 - y = 0$. Now, $2x(y - \frac{1}{2}) = 0$ if and only if $x = 0$ or $y = \frac{1}{2}$; in the case that $x = 0$, $x^2 - y = 0$ if and only if $y = 0$, and in the case that $y = \frac{1}{2}$, $x^2 - y = 0$ if and only if $x = \pm\frac{1}{\sqrt{2}}$. Thus, $\nabla f(x,y) = 0$ if and only if $(x,y) = (0,0)$ or $(\pm\frac{1}{\sqrt{2}}, \frac{1}{2})$; since f is differentiable on all of $\mathbb{R}^3$, these are all the critical points of f.

Now, the Hessian matrix of f is given by

$$Hf(x,y) = \begin{pmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} \end{pmatrix} = \begin{pmatrix} 4y - 2 & 4x \\ 4x & -2 \end{pmatrix},$$

so that

$$Hf(0,0) = \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix}$$

and

$$Hf(\pm\tfrac{1}{\sqrt{2}}, \tfrac{1}{2}) = \begin{pmatrix} 0 & \pm 2\sqrt{2} \\ \pm 2\sqrt{2} & 0 \end{pmatrix}.$$

On the one hand, $Hf(0,0)$ is manifestly negative definite, so that by Theorem 3.5, f has a local maximum at $(0,0)$. On the other hand, $Hf(\pm\frac{1}{\sqrt{2}}, \frac{1}{2})$ is symmetric, and thus is diagonalizable with two (not necessarily distinct) real eigenvalues λ_1 and λ_2. Hence,

$$\lambda_1 \lambda_2 = \det\left(Hf(\pm\tfrac{1}{\sqrt{2}}, \tfrac{1}{2})\right) = \det\begin{pmatrix} 0 & \pm 2\sqrt{2} \\ \pm 2\sqrt{2} & 0 \end{pmatrix} = -8 < 0,$$

so that $Hf(\pm\frac{1}{\sqrt{2}}, \frac{1}{2})$ has one positive eigenvalue and one negative eigenvalue, i.e. f has a saddle point at $(\pm\frac{1}{\sqrt{2}}, \frac{1}{2})$.

Thus, the critical points of f are the local maximum $(0,0)$ and the saddle points $(\frac{1}{\sqrt{2}}, \frac{1}{2})$ and $(-\frac{1}{\sqrt{2}}, \frac{1}{2})$. □

(b) Use the method of Lagrange multipliers to determine if the *maximum* of the function f in part (a) on the region $x^2 + y^2 \leq 1$ is on the boundary circle $x^2 + y^2 = 1$.

Solution. Since f is continuous on $\mathbb{R}^3$, it is, in particular, continuous on $\{(x, y) \in \mathbb{R}^3 \mid x^2 + y^2 \leq 1\} = D \cup \partial D$, for $D = \{(x, y) \in \mathbb{R}^3 \mid x^2 + y^2 < 1\}$. Since $D \cup \partial D$ is closed and bounded, it follows from Theorem 3.7 that f restricted to $D \cup \partial D$ attains a global maximum and a global minimum.

On the one hand, by (a), since $(0, 0)$, $(\frac{1}{\sqrt{2}}, \frac{1}{2})$ and $(-\frac{1}{\sqrt{2}}, \frac{1}{2})$ all lie in D, the critical points of f restricted to D are precisely $(0, 0)$, $(\frac{1}{\sqrt{2}}, \frac{1}{2})$ and $(-\frac{1}{\sqrt{2}}, \frac{1}{2})$.

On the other hand, $\partial D = \{(x, y) \in \mathbb{R}^3 \mid x^2 + y^2 = 1\} = \{(x, y) \in \mathbb{R}^3 \mid g(x, y) = 0\}$, where $g(x, y) = x^2 + y^2 - 1$, so that since $\nabla g(x, y) = 2(x, y) \neq 0$ on ∂D, we can apply the method of Lagrange multipliers (i.e. Theorem 3.8) to find the possible local maxima and minima of f on ∂D. Now, let $\lambda \in \mathbb{R}$. Then $\nabla f(x, y) = \lambda \nabla g(x, y)$ if and only if $2(2x(y - \frac{1}{2}), x^2 - y) = 2\lambda(x, y)$, if and only if $x(2y - \lambda - 1) = 0$ and $x^2 = (1 + \lambda)y$. Thus, we must solve the following system of equations:

$$\begin{aligned} x(2y - \lambda - 1) &= 0 \\ x^2 &= (1 + \lambda)y \\ x^2 + y^2 &= 1. \end{aligned}$$

Now, by the first equation, there are two possible cases, namely, $x = 0$ or $y = \frac{1+\lambda}{2}$. First, suppose that $x = 0$. Then by the third equation, $y^2 = 1$, so that $y = \pm 1$, and hence, by the second equation, $\lambda = -1$. Hence, we obtain the candidate points $(0, \pm 1)$. Now, suppose that $y = \frac{1+\lambda}{2}$. Then, by the second equation, $x^2 = \frac{(1+\lambda)^2}{2}$, so that $x = \pm \frac{|1+\lambda|}{\sqrt{2}}$, and hence, by the third equation,

$$1 = \tfrac{(1+\lambda)^2}{2} + \tfrac{(1+\lambda)^2}{4} = \tfrac{3}{4}(1 + \lambda)^2,$$

so that $1 + \lambda = \pm \frac{2}{\sqrt{3}}$. Hence, we obtain the candidate points $(\frac{\sqrt{2}}{\sqrt{3}}, \frac{1}{\sqrt{3}})$, $(\frac{\sqrt{2}}{\sqrt{3}}, -\frac{1}{\sqrt{3}})$, $(-\frac{\sqrt{2}}{\sqrt{3}}, \frac{1}{\sqrt{3}})$, and $(-\frac{\sqrt{2}}{\sqrt{3}}, -\frac{1}{\sqrt{3}})$.

Thus, the possible global maxima and minima of f on $D \cup \partial D$ are $(0, 0)$, $(\frac{1}{\sqrt{2}}, \frac{1}{2})$, $(0, \pm 1)$, $(\frac{\sqrt{2}}{\sqrt{3}}, \frac{1}{\sqrt{3}})$, $(\frac{\sqrt{2}}{\sqrt{3}}, -\frac{1}{\sqrt{3}})$, $(-\frac{\sqrt{2}}{\sqrt{3}}, \frac{1}{\sqrt{3}})$, and $(-\frac{\sqrt{2}}{\sqrt{3}}, -\frac{1}{\sqrt{3}})$. Since

$$\begin{aligned} f(0, 0) &= 0 \\ f(\tfrac{1}{\sqrt{2}}, \tfrac{1}{2}) &= -\tfrac{1}{2} \\ f(0, \pm 1) &= -1 \\ f(\tfrac{\sqrt{2}}{\sqrt{3}}, \tfrac{1}{\sqrt{3}}) &= \tfrac{4}{3\sqrt{3}} - 1 \\ f(\tfrac{\sqrt{2}}{\sqrt{3}}, -\tfrac{1}{\sqrt{3}}) &= -\tfrac{4}{3\sqrt{3}} - 1 \\ f(-\tfrac{\sqrt{2}}{\sqrt{3}}, \tfrac{1}{\sqrt{3}}) &= \tfrac{4}{3\sqrt{3}} - 1 \\ f(-\tfrac{\sqrt{2}}{\sqrt{3}}, -\tfrac{1}{\sqrt{3}}) &= -\tfrac{4}{3\sqrt{3}} - 1, \end{aligned}$$

it follows that f attains a global maximum at $(0,0)$, which is *not* on the boundary circle ∂D. $\square$

(c) Consider the function $f(x,y) = a\,x^2 + 2b\,xy + c\,y^2$ defined on the whole xy-plane. Suppose that the eigenvalues of the matrix

$$\begin{bmatrix} a & b \\ b & c \end{bmatrix}$$

are both positive. Is the origin a local minimum of f? Must it be a global minimum of f?

Solution. Yes and Yes.

We calculate the partial derivatives

$$\frac{\partial f}{\partial x} = 2ax + 2by$$
$$\frac{\partial f}{\partial y} = 2bx + 2cy.$$

These both vanish at the origin so the origin is indeed a critical point. Further,

$$\frac{\partial^2 f}{\partial x^2} = 2a$$
$$\frac{\partial^2 f}{\partial y^2} = 2c$$
$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x} = 2b,$$

so the Hessian is twice the matrix given in the problem. If the eigenvalues are both positive then the origin is a local minimum.

While the fact that the origin is a local minimum does not imply that it is a global minimum, it turns out that in this case the origin is also a global minimum. If both eigenvalues are positive then the diagonal derivatives must be positive. That is, $a > 0$ and $ac - b^2 > 0$ implying as well that $c > 0$. Now for fixed (x,y), we have either

$$ax^2 + 2bxy + cy^2 > ax^2 + 2\sqrt{ac}xy + cy^2 = (\sqrt{a}x + \sqrt{c}y)^2$$

or

$$ax^2 + 2bxy + cy^2 > ax^2 - 2\sqrt{ac}xy + cy^2 = (\sqrt{a}x - \sqrt{c}y)^2$$

depending on whether the middle term is positive or negative. Either way we have expressed $f(x,y)$ as a square so it is nonnegative, and we know that $f(0,0) = 0$. $\square$

5. (a) Let $\mathbf{F}(x,y) = f(x^2+y^2)[-y\mathbf{i} + x\mathbf{j}]$ for a given function f of one variable. Find an equation that $g(t)$ should satisfy so that

$$\mathbf{c}(t) = [\cos g(t)]\mathbf{i} + [\sin g(t)]\mathbf{j}$$

will be a flow line for $\mathbf{F}$.

Solution. By definition, $\mathbf{c}$ is a flow line of $\mathbf{F}$ if $\mathbf{F}(\mathbf{c}(t)) = \mathbf{c}'(t)$. We calculate that

$$\begin{aligned}\mathbf{F}(\mathbf{c}(t)) &= f\big(\cos^2(g(t)) + \sin^2(g(t))\big)\big(-\sin(g(t))\mathbf{i} + \cos(g(t))\mathbf{j}\big)\\ &= f(1)\big(-\sin(g(t))\mathbf{i} + \cos(g(t))\mathbf{j}\big)\end{aligned}$$

and

$$\begin{aligned}\mathbf{c}'(t) &= \frac{d}{dt}\cos(g(t))\mathbf{i} + \frac{d}{dt}\sin(g(t))\mathbf{j}\\ &= g'(t)\big(-\sin(g(t))\mathbf{i} + \cos(g(t))\mathbf{j}\big).\end{aligned}$$

Thus the condition that $\mathbf{c}(t)$ is a flow line of $\mathbf{F}$ means that

$$f(1)\big(-\sin(g(t))\mathbf{i} + \cos(g(t))\mathbf{j}\big) = g'(t)\big(-\sin(g(t))\mathbf{i} + \cos(g(t))\mathbf{j}\big).$$

Since $-\sin(g(t))\mathbf{i} + \cos(g(t))\mathbf{j}$ is always a nonzero vector, we may divide through by it to obtain $f(1) = g'(t)$. Therefore we conclude that $g(t) = f(1)t + C$ for some constant C. □

(b) Let $\mathbf{F} = (x^2 + y - 4)\mathbf{i} + 3xy\mathbf{j} + (2xz + z^2)\mathbf{k}$. Calculate the divergence and curl of $\mathbf{F}$.

Solution. We compute that

$$\begin{aligned}\operatorname{div}\mathbf{F} &= \frac{\partial}{\partial x}(x^2 + y - 4) + \frac{\partial}{\partial y}3xy + \frac{\partial}{\partial z}(2xz + z^2)\\ &= (2x) + (3x) + (2x + 2z)\\ &= 7x + 2z.\end{aligned}$$

and that

$$\begin{aligned}\operatorname{curl}\mathbf{F} &= (\partial_y(2xz + z^2) - \partial_z 3xy)\mathbf{i} + (\partial_z(x^2 + y - 4) - \partial_x(2xz + z^2))\mathbf{j}\\ &\quad + (\partial_x 3xy - \partial_y(x^2 + y - 4))\mathbf{k}\\ &= (0 - 0)\mathbf{i} + (0 - 2z)\mathbf{j} + (3y - 1)\mathbf{k}\\ &= -2z\mathbf{j} + (3y - 1)\mathbf{k}.\end{aligned}$$

□