Mathematics 1c: Solutions, Final Examination

Due: Wednesday, June 9, at 10am

1. (a) [7 points] Let $f : \mathbb{R}^3 \to \mathbb{R}^2$ be defined by

$$f(x, y, z) = \left(e^{-2xy}, x^2 - z^2 - 4x + \sin(x + y + z)\right)$$

and let $g : \mathbb{R}^2 \to \mathbb{R}$ be a function such that $g(1, 0) = -1$, and $\nabla g(1, 0) = \mathbf{i} - 3\mathbf{j}$. Calculate the gradient of the composition $g \circ f$ at the point $(0, 0, 0)$.

Solution. Let $f(x, y, z) = (u(x, y, z), v(x, y, z))$ and $h(x, y, z) = g(f(x, y, z))$. By the chain rule,

$$\frac{\partial h}{\partial x} = \frac{\partial g}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial g}{\partial v}\frac{\partial v}{\partial x},$$

$$\frac{\partial h}{\partial y} = \frac{\partial g}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial g}{\partial v}\frac{\partial v}{\partial y},$$

and

$$\frac{\partial h}{\partial z} = \frac{\partial g}{\partial u}\frac{\partial u}{\partial z} + \frac{\partial g}{\partial v}\frac{\partial v}{\partial z}.$$

Thus, at $(0, 0, 0)$,

$$\begin{aligned}
\frac{\partial h}{\partial x} &= (1)(0) + (-3)(-4 + 1) = 9 \\
\frac{\partial h}{\partial y} &= (1)(0) + (-3)(1) = -3 \\
\frac{\partial h}{\partial z} &= (1)(0) + (-3)(1) = -3.
\end{aligned}$$

Therefore the gradient of $g \circ f$ at $(0, 0, 0)$ is $9\mathbf{i} - 3\mathbf{j} - 3\mathbf{k}$. □

(b) [5 points] Find the equation of the tangent plane to the level set $g \circ f = -1$ at the point $(0, 0, 0)$, where g and f are defined in part (a).

Solution. The gradient from (a) is orthogonal to the level set $g \circ f = -1$, so therefore the tangent plane is given by

$$(x - 0)(9) + (y - 0)(-3) + (z - 0)(-3) = 0.$$

This may be rewritten as

$$3x - y - z = 0.$$ □

(c) Consider the function $f(x, y) = x^2 + 3xy + y^2 + 16$.

i. [4 points] Show that f has a minimum at the origin along the x-axis and the y-axis.

Solution. Note that $f(x, 0) = x^2 + 16$. Since this is smallest when $x^2 = 0$, which only happens at $x = 0$, the origin is a minimum along the x-axis. Note also that $f(0, y) = y^2 + 16$, which is smallest at $y = 0$. Thus the origin is a minimum along the y-axis. □

ii. [4 points] Show that the origin is not a minimum of f by computing the eigenvalues of the second derivative matrix of f evaluated at the origin.

Solution. The matrix of second partial derivatives of f is

$$Hf(x, y) = \begin{pmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 3 & 2 \end{pmatrix}.$$

The characteristic polynomial of this matrix is

$$\det \begin{pmatrix} 2-\lambda & 3 \\ 3 & 2-\lambda \end{pmatrix} = \lambda^2 - 4\lambda + 4 - 9 = \lambda^2 - 4\lambda - 5 = (\lambda - 5)(\lambda + 1).$$

Thus the eigenvalues of $Hf(0, 0)$ are 5 and -1. Since one of these is positive and the other negative, the origin is a saddle point, not a minimum. □

2. Answer each of the following three questions:

(a) [7 points] Let D be the parallelogram in the xy-plane with vertices

$$(0, 0), (1, 1), (1, 3), (0, 2).$$

Using a suitable change of variables, write an expression for the integral

$$\iint_D f(x, y)\, dxdy$$

of a function $f(x, y)$, as an integral over the rectangle $[0, 1] \times [0, 1]$.

Solution. The parallelogram D is spanned by the vectors $\mathbf{v}_1 = (1, 1)$ and $\mathbf{v}_2 = (0, 2)$. The linear transformation taking the standard basis $\mathbf{i}$ and $\mathbf{j}$ to the vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ has the 2×2 matrix given by putting the components of the latter into the columns of the matrix. That is,

$$\begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}.$$

Thus, the transformation taking the unit square $[0,1] \times [0,1]$ to the parallelogram is given by $x = u, y = u + 2v$. The determinant of the above matrix (which is the same as the determinant of the Jacobian matrix) is 2. This both shows that this transformation is 1-1 and finds the appropriate change of variables factor. Therefore, by the change of variables theorem, we have that

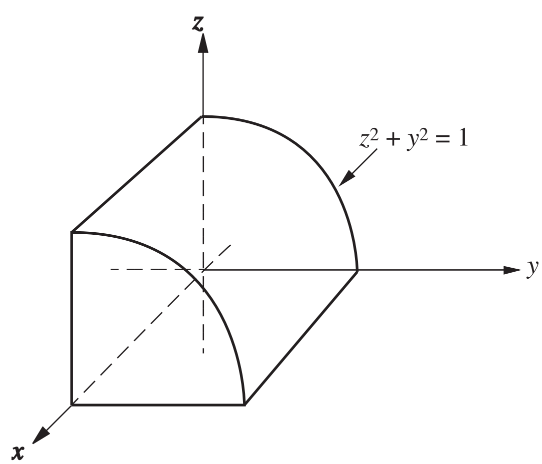
$$\iint_D f(x,y)\,dxdy = \int_0^1\int_0^1 2f(u, u+2v)\,dxdy. \qquad \square$$

(b) [6 points] Rewrite the integral

$$\int_0^1 \int_0^1 \int_0^{\sqrt{1-y^2}} f(x,y,z)\,dz\,dy\,dx$$

in the order $dy\,dz\,dx$, including a sketch of the region of integration.

Solution. The region of integration is shown in the following figure.



Since, for each fixed x, the corresponding region is both y-simple and z-simple, we may change the order of integration using Fubini's Theorem. In the order $dy\,dz\,dx$, we get

$$\int_0^1 \int_0^1 \int_0^{\sqrt{1-z^2}} f(x,y,z)\,dy\,dz\,dx. \qquad \square$$

(c) [7 points] Let S be the surface $x^2 + 2y^2 + 2z^2 = 1$. Write down an integral expression for its surface area.

Solution. First, we need to find a parametrization of the surface. An appropriate choice would be the following modification of spherical coordinates:

$$\mathbf{\Phi}(\theta, \phi) = \left(\cos\theta\sin\phi, \frac{1}{\sqrt{2}}\sin\theta\sin\phi, \frac{1}{\sqrt{2}}\cos\phi\right).$$

We calculate that

$$\mathbf{T}_\theta = \left(-\sin\theta\sin\phi, \frac{1}{\sqrt{2}}\cos\theta\sin\phi, \frac{1}{\sqrt{2}}\cos\phi\right)$$

and

$$\mathbf{T}_\phi = \left(\cos\theta\cos\phi, \frac{1}{\sqrt{2}}\sin\theta\cos\phi, \frac{1}{\sqrt{2}} - \sin\phi\right).$$

Then we calculate that

$$\begin{aligned}
\text{Area} &= \int_0^\pi \int_0^{2\pi} \|\mathbf{T}_\theta \times \mathbf{T}_\phi\| d\theta d\phi \\
&= \int_0^\pi \int_0^{2\pi} \sin\phi \sqrt{\frac{1}{4}\sin^2\phi\cos^2\theta + \frac{1}{2}\sin^2\phi\sin^2\theta + \frac{1}{2}\cos^2\phi}\, d\theta\, d\phi \\
&= \frac{1}{2}\int_0^\pi \int_0^{2\pi} \sin\phi \sqrt{1 + \sin^2\phi\sin^2\theta + \cos^2\phi}\, d\theta\, d\phi.
\end{aligned}$$

□

3. If true, ***justify***, and if false, give a ***counterexample***, or explain why.

(a) [3 points] The path integral $\int_{\mathbf{c}} 2\pi\, ds$ is the surface area of a cylinder of radius 1 and height 2π where the curve is defined by $\mathbf{c} = (\cos t, \sin t, 0)$, and $0 \le t \le 2\pi$.

Solution. True. The surface area is (circumference) $\times$ (height) $= (2\pi) \times (2\pi) = (2\pi)^2$. The value of the path integral is also $(2\pi)^2$. □

(b) [4 points] If $f(x, y)$ is a smooth function defined on the disk $x^2 + y^2 < 1$ and has a strict minimum at the origin $(0, 0)$, then the matrix of second partial derivatives of f at $(0, 0)$ is *positive definite*.

Solution. False. For example, let $f(x, y) = x^4 + y^4$. Then the matrix of second partial derivatives at $(0, 0)$ is the zero matrix, and hence not positive definite, but the origin is nevertheless a minimum of f. □

(c) [3 points] If $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$ on the disk $x^2 + y^2 < 1$, then

$$\int_C \frac{\partial u}{\partial y} dx - \frac{\partial u}{\partial x} dy = 0,$$

where C is the circle of radius $\frac{1}{2}$ centered at the origin.

Solution. True. By Green's theorem,

$$\begin{aligned}
\int_C \frac{\partial u}{\partial y} dx - \frac{\partial u}{\partial x} dy &= \iint_D \left(-\frac{\partial}{\partial x}\frac{\partial u}{\partial x} - \frac{\partial}{\partial y}\frac{\partial u}{\partial y} \right) dx\, dy \\
&= -\iint_D \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) dx\, dy \\
&= 0.
\end{aligned}$$

□

(d) [3 points] There is no vector field $\mathbf{F}$ such that $\nabla \times \mathbf{F} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$.

Solution. True. If there were such an $\mathbf{F}$, then we would have $\nabla \cdot (\nabla \times \mathbf{F}) = 0$. However, $\nabla \cdot (x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) = 3$. Therefore there is no such $\mathbf{F}$. □

(e) [3 points] The flux of a (smooth) vector field $\mathbf{F}$ out of the unit sphere $x^2 + y^2 + z^2 = 1$ equals $(4\pi/3)\operatorname{div} \mathbf{F}(P)$ for some point P inside the sphere.

Solution. True. By Gauss' Theorem, the flux of $\mathbf{F}$ through the unit sphere is equal to $\iiint_W \operatorname{div} \mathbf{F}\, dx\, dy\, dz$, where W is the unit ball. The mean value theorem says that there is some point P in W at which $\operatorname{div} \mathbf{F}$ takes its average value on W. Since the volume of the unit ball is $4\pi/3$, this means $\operatorname{div} \mathbf{F}(P)$ equals the flux of F through the unit sphere divided by $4\pi/3$. Thus the flux of F through the unit sphere is equal to $(4\pi/3)\operatorname{div} \mathbf{F}(P)$. □

(f) [4 points] If f is a smooth function of (x, y), then there is at least one point (x_0, y_0) on the circle $x^2 + y^2 = 1$ such that $\nabla f(x_0, y_0) = \lambda(x_0\mathbf{i} + y_0\mathbf{j})$ for some constant λ.

Solution. True. Let $g(x, y) = x^2 + y^2$, and let $S = \{(x, y) : g(x, y) = 1\}$. Note that

- S is the level set of a continuous function, hence closed.
- S is a circle, hence bounded.
- f is a continuous function.

Hence f restricted to S achieves maximum and minimum values. At points where this happens, the Lagrange multiplier theorem says that $\nabla f = \lambda_1 \nabla g$ for some constant λ_1. Since $\nabla g(x, y) = (2x, 2y)$, this means that $\nabla f(x_0, y_0) = \lambda(x_0\mathbf{i} + y_0\mathbf{j})$ for $\lambda = 2\lambda_1$. □

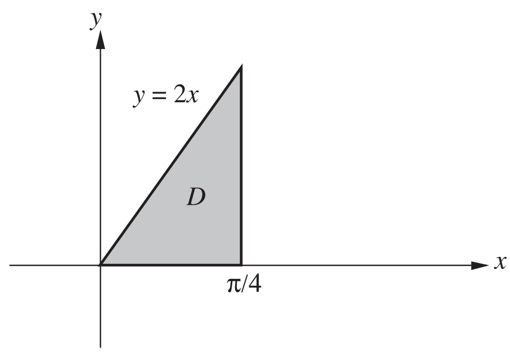
4. Let W be the region in space under the graph of

$$f(x, y) = (\cos y)\exp(1 - \cos 2x) + xy$$

over the region in the xy plane bounded by the line $y = 2x$, the x axis, and the line $x = \pi/4$.

(a) [10 points] Find the volume of W.

Solution. The region in the xy-plane is as shown in the following figure.



The volume of W is

$$\begin{aligned}
\iint_D f(x, y)dy\,dx &= \int_0^{\pi/4}\int_0^{2x} [(\cos y)\exp(1 - \cos 2x) + xy]dy\,dx \\
&= \int_0^{\pi/4} \left[(\sin 2x)\exp(1 - \cos 2x) + x \cdot \frac{(2x)^2}{2}\right] dx \\
&= \left[\frac{1}{2}\exp(1 - \cos 2x) + \frac{x^4}{2}\right]\Bigg|_0^{\pi/4} \\
&= \frac{1}{2}(e - 1) + \frac{\pi^4}{512}.
\end{aligned}$$

□

(b) [10 points] Let $\mathbf{F} = 5x\mathbf{i} + 5y\mathbf{j} + 5z\mathbf{k}$ be the velocity field of a fluid in space. Calculate the rate at which fluid is leaving the region W in part (a).

Solution. By the divergence theorem, the flux is

$$\iiint_W \text{div } \mathbf{F}\, dx\, dy\, dz = \iiint_W 15\, dx\, dy\, dz,$$

so the flux is $15\left[\frac{1}{2}(e-1) + \frac{\pi^4}{512}\right]$ by (a). □

5. Consider the vector field on $\mathbb{R}^3$ given by

$$\mathbf{F}(x, y, z) = -y\mathbf{i} + x\mathbf{j} + \frac{xyz^2}{x^2 + z^2 + 1}\mathbf{k}$$

and the two surfaces S_1 and S_2 defined by

$$S_1 : \quad x^2 + y^2 + z^2 = 1, \quad \text{and} \quad z \le 0$$

$$S_2 : \quad x^2 + y^2 + \frac{1}{2}z^2 = 1, \quad \text{and} \quad z \le 0.$$

(a) [5 points] Draw a sketch of the surfaces, indicating compatible orientations of the surfaces and that of the curve C given by $x^2 + y^2 = 1$ in the xy plane.

Solution. S_1 is the lower hemisphere and S_2 the lower half of an ellipsoid. If C is oriented in the usual counterclockwise direction, then S_1 and S_2 must be oriented by the choice of upward normal vector. Sketch omitted. □

(b) [5 points] Using Gauss' theorem, show that the surface integrals of the curl of $\mathbf{F}$ are equal:

$$\iint_{S_1} \nabla \times \mathbf{F} \cdot d\mathbf{S} = \iint_{S_2} \nabla \times \mathbf{F} \cdot d\mathbf{S}.$$

Solution. Let W be the bounded region between S_1 and S_2. In Gauss' theorem we want to use the outward normal vector from W, which means the upward normal vector for S_1 and the downward normal vector for S_2. The fact that we are using the opposite normal vector for S_2 from above corresponds to a change of sign, so therefore we have

$$\begin{aligned} \iint_{S_1} \nabla \times \mathbf{F} \cdot d\mathbf{S} - \iint_{S_2} \nabla \times \mathbf{F} \cdot d\mathbf{S} &= \iiint_W \nabla \cdot (\nabla \times \mathbf{F})\, dW \\ &= \iiint_W 0\, dW \\ &= 0. \end{aligned}$$

Thus $\iint_{S_1} \nabla \times \mathbf{F} \cdot d\mathbf{S} = \iint_{S_2} \nabla \times \mathbf{F} \cdot d\mathbf{S}$. □

(c) [5 points] Demonstrate the same equality using Stokes' theorem.

Solution. Using Stokes' theorem, we have that

$$\iint_{S_1} \nabla \times \mathbf{F} \cdot d\mathbf{S} = \int_{\partial S_1} \mathbf{F} \cdot d\mathbf{s} = \int_{\partial S_2} \mathbf{F} \cdot d\mathbf{s} = \iint_{S_2} \nabla \times \mathbf{F} \cdot d\mathbf{S}. \quad \square$$

(d) [5 points] Evaluate the two given integrals.

Solution. By Stokes' theorem, we may integrate the vector field along the boundary. Parametrizing the boundary by $\mathbf{c}(t) = (\cos t, \sin t, 0)$, we have that

$$\begin{aligned}
\int_C \mathbf{F} \cdot d\mathbf{s} &= \int_0^{2\pi} F(\mathbf{c}(t)) \cdot \mathbf{c}'(t)\, dt \\
&= \int_0^{2\pi} (-\sin t, \cos t, 0) \cdot (-\sin t, \cos t, 0)\, dt \\
&= \int_0^{2\pi} (\sin^2 t + \cos^2 t)\, dt \\
&= \int_0^{2\pi} dt \\
&= 2\pi.
\end{aligned}$$

$\square$