1. (10 Points) **Section 8.1, Exercises 3c and 3d.** Verify Green’s theorem for the disk $D$ with center $(0,0)$ and radius $R$ and $P(x,y) = xy = Q(x,y)$ and the same disk for $P = 2y, Q = x$.

**Solution.** For 3c, let $c(t) = (R \cos t, R \sin t)$ be the parameterization of $\partial D$. Then

$$
\int_{\partial D} P \, dx + Q \, dy = \int_0^{2\pi} (R^2 \cos t \sin t, R^2 \cos t \sin t) \cdot (-R \sin t, R \cos t) \, dt
$$

$$
= -R^3 \int_0^{2\pi} \sin^2 t \cos t \, dt + R^3 \int_0^{2\pi} \cos^2 t \sin t \, dt
$$

$$
= 0 + 0 = 0.
$$

Also,

$$
\iint_D \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \, dx \, dy = \iint_D (y-x) \, dx \, dy
$$

$$
= \int_0^R \int_0^{2\pi} (r \sin \theta - r \cos \theta) r \, d\theta \, dr
$$

$$
= \int_0^R (0+0)r^2 \, dr = 0.
$$

Hence, Green’s theorem for 3c is verified.

For 3d, note that Green’s theorem

$$
\int_{\partial D} P \, dx + Q \, dy = \iint_D \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \, dx \, dy
$$

becomes

$$
\int_{\partial D} 2y \, dx + x \, dy = \iint_D (1-2) \, dx \, dy = -\iint_D \, dx \, dy
$$

The right side is $-\pi R^2$ while the left side is, since $x = R \cos \theta$ and $y = R \sin \theta$,

$$
\int_0^{2\pi} (2R \sin \theta)(-R \sin \theta) d\theta + (R \cos \theta)(R \cos \theta) d\theta
$$

$$
= -2R^2 \int_0^{2\pi} \sin^2 \theta d\theta + R^2 \int_0^{2\pi} \cos^2 \theta d\theta.
$$

Using the fact that $\sin^2 \theta$ and $\cos^2 \theta$ have averages $\frac{1}{2}$, namely

$$
\frac{1}{2\pi} \int_0^{2\pi} \sin^2 \theta \, d\theta = \frac{1}{2}
$$

(this is one way of remembering the formula for the integrals of $\sin^2 \theta$ and $\cos^2 \theta$ on $[0,2\pi]$ and $[0,\pi]$), we get $-2R^2 \cdot \pi + R^2 \cdot \pi = -\pi R^2$. Thus, Green’s theorem checks.
2. (10 Points) Section 8.2, Exercise 3. Verify Stokes’ theorem for \( z = \sqrt{1 - x^2 - y^2} \), the upper hemisphere, with \( z \geq 0 \), and the radial vector field \( \mathbf{F}(x, y, z) = xi + yj + zk \).

**Solution.** Let \( H \) be the upper hemisphere.

(i) Since \( \mathbf{F}(x, y, z) = (x, y, z) \), we have \( \nabla \times \mathbf{F} = 0 \), so
\[
\int \int _{H} (\nabla \times \mathbf{F}) \cdot d\mathbf{S} = 0.
\]

(ii) Notice that the tangent to \( \partial H \) at the point \((x, y, 0)\) is the vector \((-y, x, 0)\) which is perpendicular to \( \mathbf{F} = (x, y, z) \). So
\[
\int \int _{\partial H} \mathbf{F} \cdot d\mathbf{S} = 0.
\]

Hence, Stokes’ theorem is verified.

3. (10 Points) Section 8.2, Exercise 16. For a surface \( S \) and a fixed vector \( \mathbf{v} \), prove that
\[
2 \int _{S} \mathbf{v} \cdot \mathbf{n} \, dS = \int _{\partial S} (\mathbf{v} \times \mathbf{r}) \cdot d\mathbf{S},
\]
where \( \mathbf{r}(x, y, z) = (x, y, z) \).

**Solution.** Let \( \mathbf{v} = (a, b, c) \) and \( \mathbf{r} = (x, y, z) \). Then
\[
\mathbf{v} \times \mathbf{r} = \begin{vmatrix}
\mathbf{i} & \mathbf{j} & \mathbf{k} \\
a & b & c \\
x & y & z
\end{vmatrix}
= (bz - cy, cx - az, ay - bx)
\]
and
\[
\nabla \times (\mathbf{v} \times \mathbf{r}) = \begin{vmatrix}
\mathbf{i} & \mathbf{j} & \mathbf{k} \\
\frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\
(bz - cy) & (cx - az) & (ay - bx)
\end{vmatrix}
= (2a, 2b, 2c)
\]
\[
= 2\mathbf{v}.
\]

Therefore, by Stokes’ theorem, we have
\[
\int _{\partial S} (\mathbf{v} \times \mathbf{r}) \cdot d\mathbf{S} = \int \int _{S} (\nabla \times (\mathbf{v} \times \mathbf{r})) \cdot d\mathbf{S} = 2 \int \int _{S} \mathbf{v} \cdot \mathbf{n} \, dS.
\]

4. (15 Points) Section 8.2, Exercise 23. Let \( \mathbf{F} = x^2 \mathbf{i} + (2xy + x) \mathbf{j} + z \mathbf{k} \). Let \( C \) be the circle \( x^2 + y^2 = 1 \) in the plane \( z = 0 \) oriented counterclockwise and \( S \) the disk \( x^2 + y^2 \leq 1 \) oriented with the normal vector \( \mathbf{k} \). Determine:
(a) The integral of $\mathbf{F}$ over $S$.
(b) The circulation of $\mathbf{F}$ around $C$.
(c) Find the integral of $\nabla \times \mathbf{F}$ over $S$. Verify Stokes' theorem directly in this case.

Solution.

(a) Notice that $\mathbf{F} = (x^2, 2xy + x, 0)$ on $S$. Hence
$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_S (x^2, 2xy + x, 0) \cdot (0, 0, 1) \, dS = 0.$$ 

(b) Let $c(t) = (\cos t, \sin t, 0)$ be the parameterization of $C$. Then
$$\int_C \mathbf{F} \cdot d\mathbf{S} = \int_0^{2\pi} (\cos^2 t, 2 \cos t \sin t + \cos t, 0) \cdot (-\sin t, \cos t, 0) \, dt = \int_0^{2\pi} (\cos^2 t \sin t + \cos^2 t) \, dt = \pi.$$ 

(c) Routine computation shows that $\nabla \times \mathbf{F} = (0, 0, 2y + 1)$. Hence
$$\iint_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S} = \int_0^1 \int_0^{2\pi} (0, 0, 2r \sin \theta + 1) \cdot (0, 0, 1) \, r \, d\theta \, dr = \pi.$$ 

Combining the results in (b) and (c), Stokes' theorem is verified.

5. (15 Points) Section 8.3, Exercise 14. Determine which of the following vector fields $\mathbf{F}$ in the plane is the gradient of a scalar function $f$. If such an $f$ exists, find it.

(a) $\mathbf{F}(x, y) = (\cos xy - xy \sin xy)i - (x^2 \sin xy)j$
(b) $\mathbf{F}(x, y) = (x \sqrt{x^2 y^2 + 1})i + (y \sqrt{x^2 y^2 + 1})j$
(c) $\mathbf{F}(x, y) = (2x \cos y + \cos y)i - (x^2 \sin y + x \sin y)j$.

Solution. In this problem, we apply the cross-derivative test. For example, for problem (a),
$$\frac{\partial F_1}{\partial y} - \frac{\partial F_2}{\partial x} = (x \sin xy - x \sin xy - x^2 y \cos xy) - (-2x \sin xy - x^2 y \cos xy) = 0,$$
so $\mathbf{F}$ is indeed the gradient of some function on the plane. To find such a function, we seek $f$ satisfying
$$\frac{\partial f}{\partial y} = F_2 = x^2 \sin xy,$$
for example, \( f(x, y) = x \cos xy \). (Of course, \( f \) is unique only up to an additive constant). Part (b) and (c) proceed similarly. (b) is not a gradient field. For part (c), \( f(x, y) = x^2 \cos y + x \cos y \) is a function whose gradient is the given field.

6. (10 Points) **Section 8.4, Exercise 2.** Let \( \mathbf{F} = x^3 \mathbf{i} + y^3 \mathbf{j} + z^3 \mathbf{k} \). Evaluate the surface integral of \( \mathbf{F} \) over the unit sphere.

**Solution.** Rather than integrating \( \mathbf{F} = x^3 \mathbf{i} + y^3 \mathbf{j} + z^3 \mathbf{k} \) over the sphere directly, we apply Gauss’ theorem and integrate

\[
\nabla \cdot \mathbf{F} = 3(x^2 + y^2 + z^2) = 3\rho^2
\]

over the unit ball:

\[
\iiint_B \nabla \cdot \mathbf{F} \rho^2 \sin \phi \, d\rho \, d\theta \, d\phi = 3 \int_0^\pi \int_0^{2\pi} \int_0^1 \rho^4 \sin \phi \, d\rho \, d\theta \, d\phi = \frac{12\pi}{5}.
\]

7. (10 Points) **Section 8.4, Exercise 14.** Fix \( k \) vectors \( \mathbf{v}_1, \ldots, \mathbf{v}_k \) in space and numbers (“charges”) \( q_1, \ldots, q_k \). Define

\[
\phi(x, y, z) = \sum_{i=1}^k \frac{q_i}{4\pi \| \mathbf{r} - \mathbf{v}_i \|},
\]

where \( \mathbf{r} = (x, y, z) \). Show that for a closed surface \( S \) and \( \mathbf{e} = -\nabla \phi \),

\[
\int_S \mathbf{e} \cdot d\mathbf{S} = Q,
\]

where \( Q = q_1 + \cdots + q_k \) is the total charge inside \( S \). Assume that none of the charges are on \( S \).

**Solution.** Surround each charge at vector \( \mathbf{v}_i \) by a small ball \( B_i \) in such a way that the \( B_i \) are mutually disjoint and do not intersect \( S \). Assume that \( B_1, \ldots, B_n \), (where \( n \leq k \)) are those balls contained within \( S \). Then since \( \text{div} \, \mathbf{e} = 0 \), and as in Theorem 10,

\[
\int_S \mathbf{e} \cdot d\mathbf{S} = \sum_{i=1}^n \int_{\partial B_i} \mathbf{e} \cdot d\mathbf{S}
\]

where \( \partial B_i \) is given the outward orientation. But again, as in Theorem 10,

\[
\int_{\partial B_i} \mathbf{e} \cdot d\mathbf{S} = q_i.
\]

Thus,

\[
\int_S \mathbf{e} \cdot d\mathbf{S} = \sum_{i=1}^n q_i = Q,
\]

the total charge inside \( S \).