Mathematics 1c: Solutions, Homework Set 8

Due: Tuesday, June 1 at 10am.

1. (10 Points) **Section 8.1, Exercises 3c and 3d**. *Verify Green's theorem for the disk D with center $(0,0)$ and radius R and $P(x,y) = xy = Q(x,y)$ and the same disk for $P = 2y, Q = x$. .*

   Solution. For 3c, let $\mathbf{c}(t) = (R\cos t, R\sin t)$ be the parameterization of ∂D. Then

$$\begin{aligned}\int_{\partial D} P\,dx + Q\,dy &= \int_0^{2\pi} (R^2\cos t\sin t, R^2\cos t\sin t)\cdot(-R\sin t, R\cos t)dt \\ &= -R^3\int_0^{2\pi}\sin^2 t\cos t\,dt + R^3\int_0^{2\pi}\cos^2 t\sin t\,dt \\ &= 0+0=0.\end{aligned}$$

Also,

$$\begin{aligned}\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dx\,dy &= \iint_D (y-x)dx\,dy \\ &= \int_0^R\int_0^{2\pi}(r\sin\theta - r\cos\theta)r\,d\theta\,dr \\ &= \int_0^R (0+0)r^2 dr = 0.\end{aligned}$$

Hence, Green's theorem for 3c is verified.

For 3d, note that Green's theorem

$$\int_{\partial D} Pdx + Qdy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dx\,dy$$

becomes

$$\int_{\partial D} 2y\,dx + x\,dy = \iint_D (1-2)dx\,dy = -\iint_D dx\,dy$$

The right side is $-\pi R^2$ while the left side is, since $x = R\cos\theta$ and $y = R\sin\theta$,

$$\begin{aligned}&\int_0^{2\pi}(2R\sin\theta)(-R\sin\theta)d\theta + (R\cos\theta)(R\cos\theta)d\theta \\ &\qquad = -2R^2\int_0^{2\pi}\sin^2\theta d\theta + R^2\int_0^{2\pi}\cos^2\theta d\theta.\end{aligned}$$

Using the fact that $\sin^2\theta$ and $\cos^2\theta$ have averages $\frac{1}{2}$, namely

$$\frac{1}{2\pi}\int_0^{2\pi}\sin^2\theta d\theta = \frac{1}{2}$$

(this is one way of remembering the formula for the integrals of $\sin^2\theta$ and $\cos^2\theta$ on $[0,2\pi]$ and $[0,\pi]$), we get $-2R^2\cdot\pi + R^2\cdot\pi = -\pi R^2$. Thus, Green's theorem checks.

2. (10 Points) **Section 8.2, Exercise 3.** *Verify Stokes' theorem for* $z = \sqrt{1 - x^2 - y^2}$*, the upper hemisphere, with* $z \geq 0$*, and the radial vector field* $\mathbf{F}(x, y, z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$*.*

   Solution. Let H be the upper hemisphere.

   (i) Since $\mathbf{F}(x, y, z) = (x, y, z)$, we have $\nabla \times \mathbf{F} = \mathbf{0}$, so
   $$\iint_H (\nabla \times \mathbf{F}) \cdot d\mathbf{S} = 0.$$

   (ii) Notice that the tangent to ∂H at the point $(x, y, 0)$ is the vector $(-y, x, 0)$ which is perpendicular to $\mathbf{F} = (x, y, z)$. So
   $$\int_{\partial H} \mathbf{F} \cdot d\mathbf{S} = 0.$$

   Hence, Stokes' theorem is verified.

3. (10 Points) **Section 8.2, Exercise 16.** *For a surface* S *and a fixed vector* $\mathbf{v}$*, prove that*
   $$2 \iint_S \mathbf{v} \cdot \mathbf{n} \, dS = \int_{\partial S} (\mathbf{v} \times \mathbf{r}) \cdot d\mathbf{S},$$
   where $\mathbf{r}(x, y, z) = (x, y, z)$*.*

   Solution. Let $\mathbf{v} = (a, b, c)$ and $\mathbf{r} = (x, y, z)$. Then
   $$\begin{aligned} \mathbf{v} \times \mathbf{r} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a & b & c \\ x & y & z \end{vmatrix} \\ &= (bz - cy, cx - az, ay - bx) \end{aligned}$$
   and
   $$\begin{aligned} \nabla \times (\mathbf{v} \times \mathbf{r}) &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ bz - cy & cx - az & ay - bx \end{vmatrix} \\ &= (2a, 2b, 2c) \\ &= 2\mathbf{v}. \end{aligned}$$
   Therefore, by Stokes' theorem, we have
   $$\int_{\partial S} (\mathbf{v} \times \mathbf{r}) \cdot d\mathbf{S} = \iint_S (\nabla \times (\mathbf{v} \times \mathbf{r})) \cdot d\mathbf{S} = 2 \iint_S \mathbf{v} \cdot \mathbf{n} \, dS.$$

4. (15 Points) **Section 8.2, Exercise 23.** *Let* $\mathbf{F} = x^2\mathbf{i} + (2xy + x)\mathbf{j} + z\mathbf{k}$*. Let* C *be the circle* $x^2 + y^2 = 1$ *in the plane* $z = 0$ *oriented counterclockwise and* S *the disk* $x^2 + y^2 \leq 1$ *oriented with the normal vector* $\mathbf{k}$*. Determine:*

(a) *The integral of* $\mathbf{F}$ *over* S.

(b) *The circulation of* $\mathbf{F}$ *around* C.

(c) *Find the integral of* $\nabla \times \mathbf{F}$ *over* S. *Verify Stokes' theorem directly in this case.*

Solution.

(a) Notice that $F = (x^2, 2xy + x, 0)$ on S. Hence

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_S (x^2, 2xy + x, 0) \cdot (0, 0, 1)\, dS = 0.$$

(b) Let $c(t) = (\cos t, \sin t, 0)$ be the parameterization of C. Then

$$\begin{aligned} \int_C \mathbf{F} \cdot d\mathbf{S} &= \int_0^{2\pi} (\cos^2 t, 2\cos t \sin t + \cos t, 0) \cdot (-\sin t, \cos t, 0) dt \\ &= \int_0^{2\pi} (\cos^2 t \sin t + \cos^2 t) dt = \pi. \end{aligned}$$

(c) Routine computation shows that $\nabla \times \mathbf{F} = (0, 0, 2y + 1)$. Hence

$$\begin{aligned} \iint_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S} &= \int_0^1 \int_0^{2\pi} (0, 0, 2r \sin\theta + 1) \cdot (0, 0, 1) r\, d\theta\, dr \\ &= \int_0^1 \int_0^{2\pi} (2r \sin\theta + 1) r\, d\theta\, dr = \pi. \end{aligned}$$

Combining the results in (b) and (c), Stokes' theorem is verified.

5. (15 Points) **Section 8.3, Exercise 14.** *Determine which of the following vector fields* $\mathbf{F}$ *in the plane is the gradient of a scalar function* f. *If such an* f *exists, find it.*

(a) $\mathbf{F}(x, y) = (\cos xy - xy \sin xy)\mathbf{i} - (x^2 \sin xy)\mathbf{j}$

(b) $\mathbf{F}(x, y) = (x\sqrt{x^2y^2 + 1})\mathbf{i} + (y\sqrt{x^2y^2 + 1})\mathbf{j}$

(c) $\mathbf{F}(x, y) = (2x \cos y + \cos y)\mathbf{i} - (x^2 \sin y + x \sin y)\mathbf{j}$.

Solution. In this problem, we apply the cross-derivative test. For example, for problem (a),

$$\frac{\partial F_1}{\partial y} - \frac{\partial F_2}{\partial x} = (x \sin xy - x \sin xy - x^2 y \cos xy) - (-2x \sin xy - x^2 y \cos xy) = 0,$$

so $\mathbf{F}$ is indeed the gradient of some function on the plane. To find such a function, we seek f satisfying

$$\frac{\partial f}{\partial y} = F_2 = x^2 \sin xy,$$

for example, $f(x,y) = x \cos xy$. (Of course, f is unique only up to an additive constant). Part (b) and (c) proceed similarly. (b) is not a gradient field. For part (c), $f(x,y) = x^2 \cos y + x \cos y$ is a function whose gradient is the given field.

6. (10 Points) **Section 8.4, Exercise 2.** *Let* $\mathbf{F} = x^3\mathbf{i} + y^3\mathbf{j} + z^3\mathbf{k}$. *Evaluate the surface integral of* $\mathbf{F}$ *over the unit sphere.*

Solution. Rather than integrating $\mathbf{F} = x^3\mathbf{i} + y^3\mathbf{j} + z^3\mathbf{k}$ over the sphere directly, we apply Gauss' theorem and integrate

$$\nabla \cdot \mathbf{F} = 3(x^2 + y^2 + z^2) = 3\rho^2$$

over the unit ball:

$$\iiint_B \nabla \cdot \mathbf{F} \rho^2 \sin\phi \, d\rho \, d\theta \, d\phi = 3 \int_0^{\pi} \int_0^{2\pi} \int_0^1 \rho^4 \sin\phi \, d\rho \, d\theta \, d\phi = \frac{12\pi}{5}.$$

7. (10 Points) **Section 8.4, Exercise 14.** *Fix* k *vectors* $\mathbf{v}_1, \ldots, \mathbf{v}_k$ *in space and numbers ("charges")* $q_1, \ldots, q_k$. *Define*

$$\phi(x,y,z) = \sum_{i=1}^{k} \frac{q_i}{4\pi \|\mathbf{r} - \mathbf{v}_i\|},$$

where $\mathbf{r} = (x,y,z)$. *Show that for a closed surface* S *and* $\mathbf{e} = -\nabla\phi$,

$$\iint_S \mathbf{e} \cdot d\mathbf{S} = Q,$$

where $Q = q_1 + \cdots + q_k$ *is the total charge inside* S. *Assume that none of the charges are on* S.

Solution. Surround each charge at vector $\mathbf{v}_i$ by a small ball B_i in such a way that the B_i are mutually disjoint and do not intersect S. Assume that $B_1, \ldots, B_n$, (where $n \leq k$) are those balls contained within S. Then since div $\mathbf{e} = 0$, and as in Theorem 10,

$$\iint_S \mathbf{e} \cdot d\mathbf{S} = \sum_{i=1}^{n} \iint_{\partial B_i} \mathbf{e} \cdot d\mathbf{S}$$

where ∂B_i is given the outward orientation. But again, as in Theorem 10,

$$\iint_{\partial B_i} \mathbf{e} \cdot d\mathbf{S} = q_i.$$

Thus,

$$\iint_S \mathbf{e} \cdot d\mathbf{S} = \sum_{i=1}^{n} q_i = Q,$$

the total charge inside S.