Mathematics 1c: Solutions, Homework Set 7

Due: Monday, May 24 at 10am.

1. (10 Points) **Section 7.3, Exercise 6**. *Find an expression for a unit vector normal to the surface*

$$x = 3\cos\theta\sin\phi, \quad y = 2\sin\theta\sin\phi, \quad z = \cos\phi$$

for θ *in* $[0, 2\pi]$ *and* ϕ *in* $[0, \pi]$.

Solution. Here,

$$\mathbf{T}_\theta = (-3\sin\theta\sin\phi, 2\cos\theta\sin\phi, 0)$$

and

$$\mathbf{T}_\phi = (3\cos\theta\cos\phi, 2\sin\theta\cos\phi, -\sin\phi).$$

Thus,

$$\mathbf{T}_\theta \times \mathbf{T}_\phi = (-2\cos\theta\sin^2\phi, -3\sin\theta\sin^2\phi, -6\sin\phi\cos\phi)$$

and

$$\|\mathbf{T}_\theta \times \mathbf{T}_\phi\| = \sin\phi(5\sin^2\theta\sin^2\phi + 32\cos^2\phi + 4)^{1/2}.$$

Hence a unit normal vector is

$$\mathbf{n} = \frac{\mathbf{T}_\theta \times \mathbf{T}_\phi}{\|\mathbf{T}_\theta \times \mathbf{T}_\phi\|} = \frac{1}{\sin\phi\sqrt{5\sin^2\theta\sin^2\phi + 32\cos^2\phi + 4}} \times (-2\cos\theta\sin^2\phi, -3\sin\theta\sin^2\phi, -6\sin\phi\cos\phi).$$

Since

$$\frac{x^2}{9} + \frac{y^2}{4} + z^2 = 1,$$

the surface is an ellipsoid.

2. (15 Points) **Section 7.3, Exercise 15**

   (a) *Find a parameterization for the hyperboloid* $x^2 + y^2 - z^2 = 25$.

   (b) *Find an expression for a unit normal to this surface.*

   (c) *Find an equation for the plane tangent to the surface at* $(x_0, y_0, 0)$, *where* $x_0^2 + y_0^2 = 25$.

   (d) *Show that the pair of lines* $(x_0, y_0, 0) + t(-y_0, x_0, 5)$ *and* $(x_0, y_0, 0) + t(y_0, -x_0, 5)$ *lie in the surface and as well as in the tangent plane found in part* (c).

Solution. This solution uses hyperbolic functions!

(a)

$$\begin{aligned} x &= 5\cosh u \cos\theta \\ y &= 5\cosh u \sin\theta \\ z &= 5\sinh u \end{aligned}$$

(b) Since $f(x, y, z) = x^2 + y^2 - z^2 = 25$, the unit normal is

$$\mathbf{n} = \frac{\nabla f}{\|\nabla f\|}, \quad \nabla f = (2x, 2y, -2z).$$

Thus,

$$\begin{aligned} \mathbf{n} &= \frac{2(x, y, -z)}{2\sqrt{x^2 + y^2 + z^2}} \\ &= \frac{1}{\sqrt{\cosh(2u)}}(\cosh u \cos\theta, \cosh u \sin\theta, -\sinh u). \end{aligned}$$

(c) Since the normal vector of the tangent plane is parallel to the gradient $\nabla f(x_0, y_0, 0) = (2x_0, 2y_0, 0)$, an equation of the plane is

$$(x_0, y_0, 0) \cdot (x - x_0, y - y_0, z - 0) = 0,$$

i.e.,

$$x_0(x - x_0) + y_0(y - y_0) = 0.$$

(d) One simply substitutes into the equation of the surface and the tangent plane and verifies that they are satisfied.

Note. The property in (d) is of course very special and the surface is called ***ruled*** since this argument actually shows that the surface is a union of straight lines. If one looks at some cooling towers, one sees that architects make them a hyperbolic shape partly because their ruled nature allows for structural strength as well as some construction advantages.

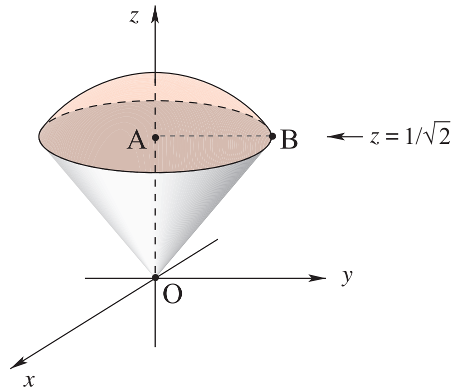
3. (10 Points) **Section 7.4, Exercise 6**. *Find the area of the portion of the unit sphere that is cut out by the cone*

$$z \geq \sqrt{x^2 + y^2}.$$

Solution. The intersection of the unit sphere and the cone $z = \sqrt{x^2 + y^2}$ is found by solving the equations

$$x^2 + y^2 + z^2 = 1 \quad \text{and} \quad x^2 + y^2 - z^2 = 0$$

(with $z \geq 0$), which is easily done by subtracting these two equations. This gives the circle described by $z = 1/\sqrt{2}$ and $x^2 + y^2 = 1/2$, as in the Figure. We are to find the area of the surface above this circle.



Notice that the triangle AOB has two sides of length $1/\sqrt{2}$, and hypotenuse of length 1, so the vertex angle AOB is $\pi/4$. Using this geometry and spherical coordinates, we find that a parametrization is

$$\begin{aligned} x &= \sin\phi\cos\theta \\ y &= \sin\phi\sin\theta \\ z &= \cos\phi, \end{aligned}$$

for $0 \leq \theta \leq 2\pi, 0 \leq \phi \leq \frac{\pi}{4}$. We find that

$$\begin{aligned} \text{Area} &= \int_0^{2\pi}\int_0^{\frac{\pi}{4}} \sin\phi \, d\phi \, d\theta \\ &= \left(1 - \frac{\sqrt{2}}{2}\right) 2\pi \\ &= (2 - \sqrt{2})\pi. \end{aligned}$$

4. (10 Points) **Section 7.5, Exercise 2**. Evaluate

$$\iint_S xyz \, dS$$

where S is the triangle with vertices $(1,0,0), (0,2,0)$ and $(0,1,1)$.

Solution. The triangle is contained in a plane whose equation is of the form $ax+by+cz+d=0$. Since $(1,0,0)$ lies on it, $a+d=0$, so $a=-d$. Since $(0,2,0)$ is on it, $b=-\frac{1}{2}d$. Since $(0,1,1)$ is on it, $b+c=-d$, so $c=-d+\frac{1}{2}d=-\frac{1}{2}d$. Letting $d=-2$, we get $2x+y+z-2=0$ *i.e.,* the equation of the plane is given by

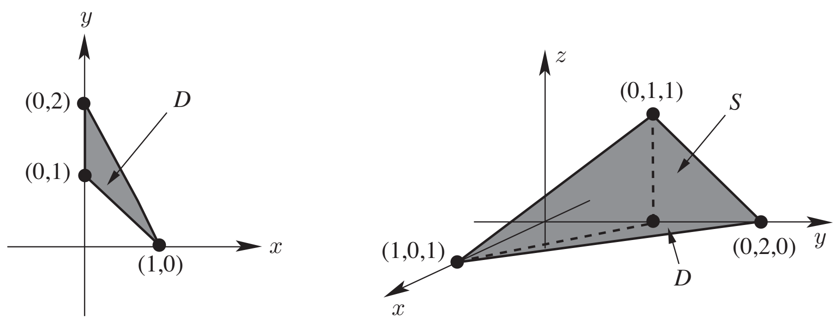
$$2x+y+z=2,$$

as in the Figure.

A normal vector is obtained from the coefficients as $(2,1,1)$, so a unit normal is

$$\mathbf{n} = \frac{1}{\sqrt{6}}(2\mathbf{i} + \mathbf{j} + \mathbf{k}).$$

The domain D in the xy plane is the triangle with vertices $(1,0), (0,2)$ and $(0,1)$, as in the Figure. It can be regarded as a graph: $z = 2 - 2x - y$



Now

$$dS = \frac{dx\,dy}{\mathbf{n}\cdot\mathbf{k}} = \sqrt{6}\,dx\,dy,$$

and so

$$\begin{aligned}\iint_S f\,dS &= \iint_D xy(2-2x-y)\sqrt{6}\,dx\,dy \\ &= \sqrt{6}\int_0^1 \int_{1-x}^{2(1-x)} [2(x-x^2)y - xy^2]dy\,dx\end{aligned}$$

Carrying out the y-integration gives

$$\begin{aligned}\iint_S f\,dS &= \sqrt{6}\int_0^1 \left(2(x-x^2)\frac{y^2}{2} - \frac{xy^3}{3}\right)\Bigg|_{1-x}^{2(1-x)} dx \\ &= \sqrt{6}\int_0^1 \left[2x(1-x)\left(\frac{[2(1-x)]^2}{2} - \frac{[1-x]^2}{2}\right)\right. \\ &\quad \left. -\frac{x}{3}([2(1-x)]^3 - (1-x)^3)\right] dx \\ &= \sqrt{6}\int_0^1 \frac{2}{3}x(1-x)^3 dx = \sqrt{6}\int_0^1 \frac{2}{3}\cdot\frac{1}{4}(1-x)^4 dx = \frac{\sqrt{6}}{30},\end{aligned}$$

where the last steps were done using integration by parts.

5. (10 Points) **Section 7.6, Exercise 7**. *Calculate the integral* $\iint_S \mathbf{F}\cdot d\mathbf{S}$, *where* S *is the surface of the half-ball* $x^2+y^2+z^2 \le 1, z \ge 0$, *and where*

$$\mathbf{F} = (x + 3y^5)\mathbf{i} + (y + 10xz)\mathbf{j} + (z - xy)\mathbf{k}.$$

Solution. Since S is the surface of the half ball $x^2 + y^2 + z^2 \leq 1, z \geq 0$, $S = H \cup D$, where H is the upper hemisphere and D is the disk. Hence

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_H \mathbf{F} \cdot d\mathbf{S} + \iint_D \mathbf{F} \cdot d\mathbf{S}.$$

(i) In spherical coordinates on the unit sphere,

$$\mathbf{r} = (\sin\phi\cos\theta)\mathbf{i} + (\sin\phi\sin\theta)\mathbf{j} + (\cos\phi)\mathbf{k},$$

where, for the hemisphere H, $0 \leq \phi \leq \frac{\pi}{2}, 0 \leq \theta \leq 2\pi$. The vector area element on the unit sphere is

$$d\mathbf{S} = \mathbf{r}\sin\phi\, d\phi\, d\theta$$

and therefore,

$$\begin{aligned}
\iint_H \mathbf{F} \cdot d\mathbf{S} &= \int_0^{\frac{\pi}{2}} \int_0^{2\pi} \mathbf{F}(\mathbf{r}) \cdot d\mathbf{S} \\
&= \int_0^{\frac{\pi}{2}} \int_0^{2\pi} (1 + 3\sin^6\phi\cos\theta\sin^5\theta \\
&\qquad + 9\sin^2\phi\cos\phi\cos\theta\sin\theta)\sin\phi\, d\theta\, d\phi \\
&= \int_0^{\frac{\pi}{2}} (2\pi + 0 + 0)\sin\phi\, d\phi = 2\pi.
\end{aligned}$$

(ii) In the case of the disk D, we have $z = 0, x^2 + y^2 \leq 1$. Since we want the unit normal to point outward from the half ball, we should use $\mathbf{n} = -\mathbf{k}$. So

$$\begin{aligned}
\iint_D \mathbf{F} \cdot d\mathbf{S} &= \iint_D (x + 3y^5, y, -xy) \cdot (0, 0, -1) dx\, dy \\
&= \iint_D xy\, dx\, dy = 0.
\end{aligned}$$

Therefore,

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = 2\pi.$$

6. (10 Points) **Section 7.6, Exercise 15.** *Let the velocity field of a fluid be given by* $\mathbf{v} = \mathbf{i} + x\mathbf{j} + z\mathbf{k}$ *in meters/second. How many cubic meters of fluid per second are crossing the surface* $x^2 + y^2 + z^2 = 1, z \geq 0$*? (Distances are in meters.)*

Solution. Here, $\mathbf{v} \cdot d\mathbf{S} = \mathbf{v} \cdot \mathbf{n}\, dS$ and $\mathbf{n} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, so

$$\mathbf{v} \cdot \mathbf{n} = x + xy + z^2.$$

By symmetry, the integrals of x and of xy vanish. Thus, the flux is

$$\iint_S \mathbf{v} \cdot d\mathbf{S} = \iint_S z^2 dS.$$

Using spherical coordinates, $z = \cos\phi$, so we get

$$\int_{\phi=0}^{\pi/2} \int_{\theta=0}^{2\pi} \cos^2\phi \sin\phi \, d\theta \, d\phi = -2\pi \left. \frac{\cos^3\phi}{3} \right|_0^{\pi/2} = \frac{2\pi}{3}.$$

7. (15 Points) **Section 7.6, Exercise 18**. *If S is the upper hemisphere*

$$\{(x, y, z) \mid x^2 + y^2 + z^2 = 1, z \geq 0\}$$

oriented by the normal pointing out of the sphere, compute

$$\iint_S \mathbf{F} \cdot d\mathbf{S}$$

for parts (a) *and* (b)*.*

(a) $\mathbf{F}(x, y, z) = x\mathbf{i} + y\mathbf{j}$

(b) $\mathbf{F}(x, y, z) = y\mathbf{i} + x\mathbf{j}$

(c) *for each of the vector fields above, compute*

$$\iint_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S} \quad \text{and} \quad \int_C \mathbf{F} \cdot d\mathbf{S},$$

where C is the unit circle in the xy plane traversed in the counterclockwise direction (as viewed from the positive z axis). (Notice that C is the boundary of S. The phenomenon illustrated here will be studied more thoroughly in the next chapter, using Stokes' theorem.)

Solution.

(a) The vector area element on the unit sphere is

$$d\mathbf{S} = \mathbf{r} \sin\phi \, d\phi \, d\theta$$

and therefore,

$$\begin{aligned}
\iint_S \mathbf{F} \cdot d\mathbf{S} &= \int_0^{\frac{\pi}{2}} \int_0^{2\pi} \mathbf{F}(\mathbf{r}) \cdot d\mathbf{S} \\
&= \int_0^{\frac{\pi}{2}} \int_0^{2\pi} \sin^3\phi \, d\theta \, d\phi \\
&= 2\pi \int_0^{\frac{\pi}{2}} (1 - \cos^2\phi) \sin\phi \, d\phi \\
&= 2\pi \cdot \frac{2}{3} = \frac{4\pi}{3}.
\end{aligned}$$

(b) Similarly,

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \int_0^{\frac{\pi}{2}} \int_0^{2\pi} 2\sin^3\phi \cos\theta \sin\theta \, d\theta \, d\phi = 0.$$

(c) The curl of the vector field in (a) is given by

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & 0 \end{vmatrix} = 0$$

and that for the vector field in (b) is

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & x & 0 \end{vmatrix} = 0.$$

Therefore,

$$\iint_S \nabla \times \mathbf{F} \cdot d\mathbf{S} = 0$$

in each case. As for

$$\int_C \mathbf{F} \cdot d\mathbf{S},$$

we can use $c(t) = (\cos t, \sin t, 0)$ for the parametrization of C. Then in each case of (a) and (b), we have

$$\begin{aligned} \int_C \mathbf{F} \cdot d\mathbf{S} &= \int_0^{2\pi} (\cos t, \sin t, 0)(-\sin t, \cos t, 0) dt = 0 \\ \int_C \mathbf{F} \cdot d\mathbf{S} &= \int_0^{2\pi} (\sin t, \cos t, 0)(-\sin t, \cos t, 0) dt = 0. \end{aligned}$$