Mathematics 1c: Solutions, Homework Set 6

Due: Monday, May 17 at 10am.

1. (10 Points) **Section 6.1, Exercise 6** *Let D^* be the parallelogram with vertices*

$$(-1,3), \quad (0,0), \quad (2,-1) \quad \textit{and} \quad (1,2)$$

and D be the rectangle $D = [0,1] \times [0,1]$. Find a transformation T such that D is the image set of D^* under T.

Solution. We are required to find a linear mapping T with $T(D^*) = D$. To do this, we seek a linear mapping $T(u,v) = (x,y)$ of the form

$$x = au + bv \quad \text{and} \quad y = cu + dv.$$

We require vertices to be mapped to vertices in the same clockwise order and observe that we already have $T(0,0) = (0,0)$. Thus, we suppose $T(1,2) = (1,1), T(-1,3) = (1,0)$ and $T(2,-1) = (0,1)$. This gives us three sets of equations

$$\begin{aligned} 1 &= a + 2b \quad \text{and} \quad 1 = c + 2d \\ 1 &= -a + 3b \quad \text{and} \quad 0 = -c + 3d \\ 0 &= 2a - b \quad \text{and} \quad 1 = 2c - d. \end{aligned}$$

From the last line, $b = 2a$ and so from the first equation, we find $a = 1/5, b = 2/5$ and similarly from the second line, $c = 3d$ and so from one of the other two equations for c, d, we get $c = 3/5, d = 1/5$. Therefore, we conclude that T is given by $T(u,v) = (u + 2v, 3u + v)/5$.

2. (10 Points) **Section 6.2, Exercise 6** *Define $T(u,v) = (u^2 - v^2, 2uv)$. Let D^* be the set of (u,v) with $u^2 + v^2 \le 1, u \ge 0, v \ge 0$. Find $T(D^*) = D$ and evaluate*

$$\iint_D dx\, dy.$$

Solution. One trick to finding D is to use the fact that the boundary of D^* gets mapped into the boundary of D (assuming that T is one to one). Thus, let us first show that T is one to one. There are two ways to do this, one using polar coordinates, or the other by algebraic brute force. Taking the brute force route, assume that $u^2 - v^2 = x, 2uv = y$. We must show that there is a unique $(u,v) \in D^*$ solving this equation. Squaring and adding we get that

$$(u^2 + v^2)^2 = (u^2 - v^2)^2 + 4u^2v^2 = x^2 + y^2.$$

Therefore $u^2 + v^2 = \sqrt{x^2 + y^2}$. But $u^2 - v^2 = x$ and thus

$$u^2 = \frac{1}{2}\left(x + \sqrt{x^2 + y^2}\right) \ge 0$$

and clearly there is *only one* $u \geq 0$ solving this equation. Similarly

$$v^2 = \frac{1}{2}\left(\sqrt{x^2+y^2} - x\right) \geq 0$$

and there is only one $v \geq 0$ that solves this equation. Thus, T is one to one.

The set D^* is the 1st quadrant of the unit circle. The interval $[0,1]$ on the u axis ($v = 0$) is mapped onto the interval $[0,1]$ on the x axis. The interval $u = 0, 0 \leq v \leq 1$ on the v axis is mapped onto the interval $[-1,0]$ on the x axis. We claim that the circle $u^2 + v^2 = 1$ gets mapped onto the circle $x^2 + y^2 = 1, y \geq 0$. Perhaps the simplest way to see this is to use polar coordinates. Let $u = \cos\theta, v = \sin\theta, 0 \leq \theta \leq \pi/2$. Then

$$(x, y) = (\cos^2\theta \sin^2\theta, 2\cos\theta\sin\theta) = (\cos 2\theta, \sin 2\theta)$$

which, as θ varies between 0 and $\pi/2$, traces out the unit circle in the 1st and 2nd quadrants of the xy plane. Therefore, the image of D^* is the region D bounded by the semi circle $x^2 + y^2 = 1, y \geq 0$ and the x axis.

Aside: For those students familiar with complex numbers, this transformation is the mapping which sends $z = x+iy$ (where $i = \sqrt{-1}$) to z^2. From properties of complex multiplication, one sees that T "opens" the wedge D^* into the half disk D.

Computing the Jacobian determinant, we get

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{bmatrix} 2u & -2v \\ 2v & 2u \end{bmatrix} = 4(u^2+v^2)$$

which is zero at $(0,0)$. Therefore

$$\iint_D dx\,dy = 4\iint_{D^*} (u^2+v^2)du\,dv.$$

Introducing polar coordinates $u = r\cos\theta, v = r\sin\theta$, this integral is equal to the iterated integral

$$4\int_0^{\pi/2}\int_0^1 r^2 \cdot r\,dr\,d\theta = 4\int_0^{\pi/2}\left[\frac{r^4}{4}\right]_0^1 d\theta = \int_0^{\pi/2} d\theta = \pi/2.$$

3. (10 Points) **Section 6.2, Exercise 8** *Calculate*

$$\iint_R \frac{dx\,dy}{x+y},$$

where R is the region bounded by $x = 0, y = 0, x + y = 1$, and $x + y = 4$ by using the mapping $T(u,v) = (u - uv, uv)$.

Solution. The region D, bounded by $x = 0, y = 0, x + y = 1$ and $x + y = 4$, is a quadrilateral. If $x = u - uv$ and $y = uv$, then $u = x + y$, and $v = y/(x+y)$. Therefore, away from the v axis ($u = 0$), the mapping is one to one. (Note that any portion of the v axis is mapped to $(0, 0)$.) The pre-image of the interval $y = 0, 1 \leq x \leq 4$ is the interval $1 \leq u \leq 4, v = 0$. The pre-image of the line $x + y = 4$, where $0 \leq x \leq 4$ and $0 \leq y \leq 4$, is the line described by $u = 4, 0 \leq v \leq 1$. The pre-image of the line $x + y = 1, 0 \leq x \leq 1, 0 \leq y \leq 1$ is the line $u = 1, 0 \leq v \leq 1$ and finally, the pre-image of the interval $x = 0, 1 \leq y \leq 4$, is the line $v = 1, 1 \leq u \leq 4$. Thus, D^* is the rectangle $[1, 4] \times [0, 1]$ in the uv plane, and $T : D^* \to D$ is one to one. The Jacobian determinant is given by

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{bmatrix} 1 - v & -u \\ v & u \end{bmatrix} = u.$$

Therefore,

$$\iint_R \frac{dx\, dy}{x + y} = \iint_{D^*} \frac{u}{u} du\, dv = \int_1^4 \int_0^1 dv\, du = 3.$$

4. (10 Points) **Section 6.3, Exercise 4** *Find the center of mass of the region between $y = 0$ and $y = x^2$, where $0 \leq x \leq 1/2$.*

Solution. We assume the material is uniform, so $\delta =$ constant. This constant cancels in the center of mass formulas, so we can assume that $\delta = 1$. The formula for the center of mass then gives

$$\bar{x} = \frac{\iint x\, dA}{\iint dA} = \frac{\int_0^{1/2} \int_0^{x^2} x\, dy\, dx}{\int_0^{1/2} \int_0^{x^2} dy\, dx}.$$

The numerator is

$$\int_0^{1/2} \int_0^{x^2} x\, dy\, dx = \int_0^{1/2} (xy)|_0^{x^2} dx = \int_0^{1/2} x^3\, dx = \frac{1}{4}\left(\frac{1}{2}\right)^4 = \frac{1}{2^6} = \frac{1}{64},$$

while the denominator is

$$\int_0^{1/2} \int_0^{x^2} dy\, dx = \int_0^{1/2} x^2\, dx = \left(\frac{1}{3}x^3\right)\Big|_0^{1/2} = \frac{1}{24}.$$

Therefore, $\bar{x} = 24/64 = 3/8$. Similarly,

$$\bar{y} = \frac{\int_0^{1/2} \int_0^{x^2} y\, dy\, dx}{\int_0^{1/2} \int_0^{x^2} dy\, dx} = 24 \int_0^{1/2} \left(\frac{y^2}{2}\right)\Big|_0^{x^2} dx = 12 \int_0^{1/2} x^4\, dx = \frac{3}{40}.$$

Thus, the center of mass is located at the point $\left(\frac{3}{8}, \frac{3}{40}\right)$.

5. (10 Points) **Section 6.4, Exercise 8** *Show that the integral*

$$\int_0^1 \int_0^a \frac{x}{\sqrt{a^2 - y^2}} dy\, dx$$

exists, and compute its value. (You may assume that a is a positive constant).

Solution. We write

$$\int_0^1 \int_0^a \frac{x}{\sqrt{a^2-y^2}} dy\, dx = \lim_{\epsilon\to 0} \int_0^1 \int_0^{a-\epsilon} \frac{x}{\sqrt{a^2-y^2}} dy\, dx$$
$$= \lim_{\epsilon\to 0} \int_0^{a-\epsilon} \frac{1}{\sqrt{a^2-y^2}} \left\{ \int_0^1 x\, dx \right\} dy = \lim_{\epsilon\to 0} \frac{1}{2} \int_0^{a-\epsilon} \frac{dy}{\sqrt{a^2-y^2}}.$$

Let $y = a\sin\theta$, so that $dy = a\cos\theta d\theta$. Substituting,

$$\int \frac{dy}{\sqrt{a^2-y^2}} = \int \frac{a\cos\theta d\theta}{a\cos\theta} = \int d\theta = \theta = \sin^{-1}\left(\frac{y}{a}\right).$$

Consequently,

$$\int_0^{a-\epsilon} \frac{dy}{\sqrt{a^2-y^2}} = \left[\sin^{-1}\frac{y}{a}\right]_0^{a-\epsilon} = \sin^{-1}\left(\frac{a-\epsilon}{a}\right).$$

Thus,

$$\lim_{\epsilon\to 0} \frac{1}{2} \int_0^{a-\epsilon} \frac{dy}{\sqrt{a^2-y^2}} = \lim_{\epsilon\to 0} \frac{1}{2} \sin^{-1}\left(\frac{a-\epsilon}{a}\right) = \frac{1}{2}\sin^{-1}(1) = \frac{\pi}{4}.$$

6. (10 Points) **Review Exercise 4b for Chaper 6** *Perform a change of variables to cylindrical coordinates for*

$$\int_{-1}^1 \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \int_{-\sqrt{4-x^2-y^2}}^{\sqrt{4-x^2-y^2}} xyz\, dz\, dx\, dy.$$

Solution. Using cylindrical coordinates,

$$\int_{-1}^1 \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \int_{-\sqrt{4-x^2-y^2}}^{\sqrt{4-x^2-y^2}} xyz\, dz\, dx\, dy$$
$$= \int_0^{2\pi} \int_0^1 \int_{-\sqrt{4-r^2}}^{\sqrt{4-r^2}} r\cdot r^2 \cos\theta \sin\theta z\, dz\, dr\, d\theta$$
$$= \int_0^{2\pi} \int_0^1 \int_{-\sqrt{4-r^2}}^{\sqrt{4-r^2}} r^3 \cos\theta \sin\theta z\, dz\, dr\, d\theta.$$

7. (10 Points) **Section 7.1, Exercise 4(a)** *Evaluate the path integral of* $f(x,y,z) = x\cos z$ *along the path* $\mathbf{c} : t \mapsto t\mathbf{i} + t^2\mathbf{j}, t \in [0,1]$.

Solution. The path integral is

$$\int_{\mathbf{c}} f\, ds = \int_0^1 (t\cos 0)\sqrt{1+4t^2} dt$$
$$= \int_0^1 t(1+4t^2)^{1/2}\, dt$$

This is readily integrated using the substitution $u = 1 + 4t^2$ and we get

$$\frac{1}{12}\left[\left(1+4t^2\right)^{3/2}\right]\Big|_0^1 = \frac{1}{12}\left(5^{3/2}-1\right).$$

8. (10 Points) **Section 7.2, Exercise 2** *Evaluate each of the following integrals:*

(a) $\displaystyle\int_{\mathbf{c}} x\,dy - y\,dx, \quad \mathbf{c}(t) = (\cos t, \sin t), \quad 0 \le t \le 2\pi$

(b) $\displaystyle\int_{\mathbf{c}} x\,dx + y\,dy, \quad \mathbf{c}(t) = (\cos \pi t, \sin \pi t), \quad 0 \le t \le 2$

(c) $\displaystyle\int_{\mathbf{c}} yz\,dx + xz\,dy + xy\,dz$, *where* $\mathbf{c}$ *consists of straight-line segments joining* $(1,0,0)$ *to* $(0,1,0)$ *to* $(0,0,1)$

(d) $\displaystyle\int_{\mathbf{c}} x^2\,dx - xy\,dy + dz$, *where* $\mathbf{c}$ *is the parabola* $z = x^2, y = 0$ *from* $(-1,0,1)$ *to* $(1,0,1)$.

Solution. (a) By definition,

$$\begin{aligned}\int_{\mathbf{c}} x\,dy - y\,dx &= \int_0^{2\pi} [\cos t(\cos t\,dt) - (\sin t)(-\sin t\,dt)] \\ &= \int_0^{2\pi} \left[\cos^2 t + \sin^2 t\right] dt = \int_0^{2\pi} dt = 2\pi.\end{aligned}$$

(b) Here,

$$\begin{aligned}\int_{\mathbf{c}} x\,dx + y\,dy &= \int_0^2 (\cos \pi t)(-\pi \sin \pi t)dt + \int_0^2 (\sin \pi t)(\pi \cos \pi t)dt \\ &= \left[\frac{\cos^2 \pi t}{2}\right]\Big|_0^2 + \left[\frac{\sin^2 \pi t}{2}\right]\Big|_0^2 = \frac{1}{2}\left[\cos^2 \pi t + \sin^2 \pi t\right]\Big|_0^2 \\ &= \frac{1}{2}\,[1]\big|_0^2 = 0.\end{aligned}$$

(c) First we write

$$\int_{\mathbf{c}} yz\,dx + xz\,dy + xy\,dz = \sum_{i=1}^{2} \int_{\mathbf{c}_i} yz\,dx + xz\,dy + xy\,dz$$

where $\mathbf{c}_1$ is the straight line path from $(1,0,0)$ to $(0,1,0)$ and $\mathbf{c}_2$ is the straight line path from $(0,1,0)$ to $(0,0,1)$.

We can parametrize $\mathbf{c}_1$ by $\mathbf{c}_1(t) = (1-t, t, 0), 0 \le t \le 1$, and $\mathbf{c}_2$ by $\mathbf{c}_2(t) = (0, 1-t, t), 0 \le t \le 1$. Therefore

$$\begin{aligned}\int_{\mathbf{c}_1} yz\,dx + xz\,dy + xy\,dz &= \int_0^1 0\,dt = 0 \\ \int_{\mathbf{c}_2} yz\,dx + xz\,dy + xy\,dz &= \int_0^1 0\,dt = 0.\end{aligned}$$

Thus, the given integral is zero.

(d) This parabola may be parameterized by $x = t, z = t^2, y = 0$, where $-1 \le t \le 1$. Therefore,

$$\begin{aligned}\int_{\mathbf{c}} x^2 dx - xy\, dy + dz &= \int_{-1}^{1} (t^2 dt + 2t\, dt)\\ &= \int_{-1}^{1} (t^2 + 2t) dt = \left[\frac{t^3}{3} + t^2\right]\Bigg|_{-1}^{1} = \frac{2}{3}.\end{aligned}$$