Mathematics 1c: Solutions, Homework Set 5

Due: Monday, May 10th at 10am.

1. (10 Points) **Section 5.1, Exercise 4** *Using Cavalieri's principle, compute the volume of the structure shown in Figure* 5.1.11 *of the textbook; each section is a rectangle of length* 5 *and width* 3*.*

   Solution. By Cavalieri's principle the volume of the solid in Figure 5.1.11 is the same as that of a rectangular parallelepiped of dimensions $3 \times 5 \times 7$ or $(3)(5)(7) = 105$.

2. (20 Points) **Section 5.2, Exercise 8** *Let f be continuous on $R = [a, b] \times [c, d]$. For $a < x < b$, and $c < y < d$, define*

$$F(x, y) = \int_a^x \int_c^y f(u, v) dv\, du.$$

   Show that

$$\frac{\partial^2 F}{\partial x \partial y} = \frac{\partial^2 F}{\partial y \partial x} = f(x, y).$$

   Use this example to discuss the relationship between Fubini's Theorem and the equality of mixed partial derivatives.

   Solution. By the Fundamental Theorem of Calculus

$$\frac{\partial F}{\partial x}(x, y) = \int_c^y f(x, v) dv$$

   and applying it once again, we have

$$\frac{\partial^2 F}{\partial y \partial x}(x, y) = f(x, y).$$

   In the reverse order, we first apply Fubini's Theorem and then the Fundamental Theorem of Calculus twice. Thus

$$F(x, y) = \int_c^y \int_a^x f(u, v) du\, dv$$

   and

$$\frac{\partial F}{\partial y}(x, y) = \int_a^x f(u, y) du$$

   and then

$$\frac{\partial^2 F}{\partial x \partial y}(x, y) = f(x, y).$$

   Fubini's Theorem is, in a sense, the integral version of the theorem on the equality of mixed partials. If

$$\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial^2 F}{\partial x \partial y} \quad \text{and} \quad \frac{\partial^2 F}{\partial y \partial x}$$

are all continuous, then Fubini's Theorem, and the Fundamental Theorem of Calculus imply that

$$\begin{aligned}\int_a^x \int_c^y \frac{\partial^2 F}{\partial x \partial y}(u,v)dv\,du &= \int_c^y \int_a^x \frac{\partial^2 F}{\partial x \partial y}(u,v)du\,dv \\ &= \int_c^y \left[\frac{\partial F}{\partial y}(x,v) - \frac{\partial F}{\partial y}(a,v)\right] dv \\ &= F(x,y) - F(x,c) - F(a,y) + F(a,c).\end{aligned}$$

A similar calculation of the iterated integral

$$\int_a^x \int_c^y \frac{\partial^2 F}{\partial y \partial x}(u,v)dv\,du$$

gives the same answer and thus

$$\int_a^x \int_c^y \frac{\partial^2 F}{\partial y \partial x}(u,v)dv\,du = \int_a^x \int_c^y \frac{\partial^2 F}{\partial x \partial y}(u,v)dv\,du.$$

This in turn implies that

$$\iint_R \frac{\partial^2 F}{\partial x \partial y} dA = \iint_R \frac{\partial^2 F}{\partial y \partial x} dA$$

for all rectangles R. Since R is arbitrary, this implies that

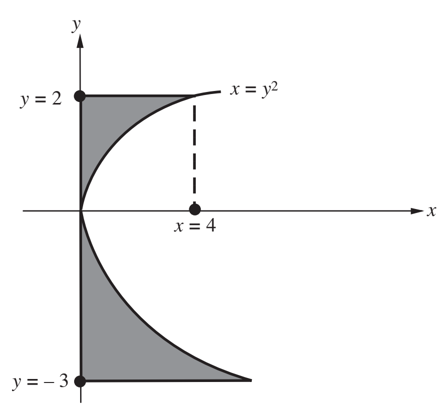
$$\frac{\partial^2 F}{\partial x \partial y} = \frac{\partial^2 F}{\partial y \partial x}.$$

This shows how one may deduce equality of mixed partials assuming Fubini's Theorem has been proved.

3. (10 Points)**Section 5.3, Exercise 2(a).** *Evaluate and sketch the region of integration*

$$\int_{-3}^{2} \int_0^{y^2} (x^2 + y)dx\,dy.$$

Solution. The region of integration is shown in the Figure.



This region is x-simple but not y-simple, since it is bounded on the left and right by graphs, but not top and bottom (unless we broke it into two pieces, one above the x-axis and one below. The integral is

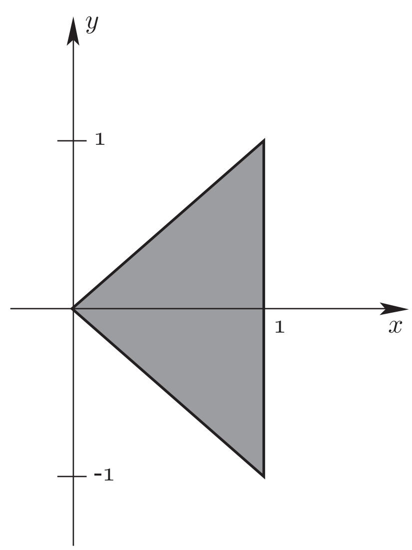
$$\int_{-3}^{2}\left(\frac{x^3}{3}+xy\bigg|_{x=0}^{x=y^2}\right)dy=\int_{-3}^{2}\left(\frac{y^6}{3}+y^3\right)dy=\frac{y^7}{21}+\frac{y^4}{4}\bigg|_{-3}^{2}$$
$$=\frac{2^7}{21}+\frac{3^7}{21}+\frac{2^4}{4}-\frac{3^4}{4}=\frac{2^7+3^7}{21}-\frac{65}{4}.$$

4. (10 Points)**Section 5.4, Exercise 2(a).** *Find*

$$\int_{-1}^{1}\int_{|y|}^{1}(x+y)^2 dx\, dy.$$

Solution. Changing the order of integration (see the Figure) we see that this iterated integral is equal to

$$\int_0^1\int_{-x}^{x}(x+y)^2 dy\, dx = \frac{1}{3}\int_0^1\left[(x+y)^3\right]_{-x}^{x}dx$$
$$= \frac{8}{3}\int_0^1 x^3 dx = \frac{2}{3}.$$



5. (10 Points)**Section 5.4, Exercise 8.** *Compute the double integral*

$$\iint_D f(x,y)dA$$

where

$$f(x,y)=y^2\sqrt{x}$$

and D is the set of (x,y) where $x>0, y>x^2$, and $y<10-x^2$.

Solution. This integral is equal to the iterated integral

$$\int_0^{\sqrt{5}}\int_{x^2}^{10-x^2} y^2\sqrt{x}\,dy\,dx = \frac{1}{3}\int_0^{\sqrt{5}} \left[y^3x^{1/2}\right]\Big|_{x^2}^{10-x^2} dx$$
$$= \frac{1}{3}\int_0^{\sqrt{5}}(10-x^2)^3x^{1/2}dx - \frac{1}{3}\int_0^{\sqrt{5}} x^{13/2}dx.$$

The second integral equals

$$-\frac{1}{3}(\frac{2}{15})(\sqrt{5})^{15/2}.$$

To slightly simplify the first integral set $u^2 = x$ so that $2udu = dx$. Changing variables, the first integral equals

$$\frac{2}{3}\int^{5^{1/4}}(10-u^4)^3u^2du$$
$$= \frac{2}{3}\int^{5^{1/4}}[1000u^2 - 100u^6 + 10u^{10} - u^{14}]du$$
$$= \frac{2}{3}\left[1000\frac{u^3}{3} - 100\frac{u^7}{7} + \frac{10}{11}u^{11} - \frac{u^{15}}{15}\right]_0^{5^{1/4}}$$
$$= \frac{2}{3}\left[\frac{1000}{3}5^{3/4} - \frac{100}{7}5^{7/4} + \frac{10}{11}5^{11/4} - \frac{5^{15/4}}{15}\right].$$

6. (10 Points)**Section 5.5, Exercise 15.** *Evaluate*

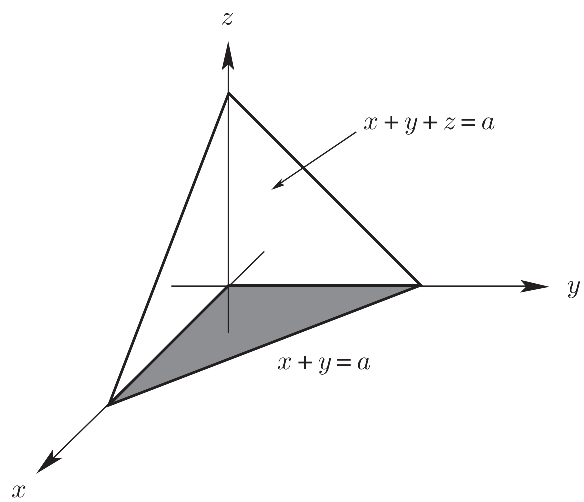
$$\iiint_W (x^2+y^2+z^2)dx\,dy\,dz,$$

where W *is the region bounded by* $x+y+z=a$ *(where* $a>0$ *is a given constant),* $x=0, y=0,$ *and* $z=0$.

Solution. The set W is defined by the inequalities

$$x\geq 0,\quad y\geq 0,\quad z\geq 0,\quad \text{and}\quad 0\leq x+y+z\leq a$$

and is shown in the Figure.



Thus,

$$\begin{aligned}
&\iiint_W (x^2+y^2+z^2)dx\,dy\,dz \\
&\quad = \int_0^a dx \int_0^{a-x} dy \int_0^{a-x-y} dz (x^2+y^2+z^2) \\
&\quad = \int_0^a dx \left(\int_0^{a-x} \left[(x^2+y^2)(a-x-y) + \frac{1}{3}(a-x-y)^3 \right] dy \right) \\
&\quad = \int_0^a \left[x^2(a-x)^2 + \frac{1}{12}(a-x)^4 - \frac{1}{2}x^2(a-x)^2 + \frac{1}{12}(a-x)^4 \right] dx \\
&\quad = \frac{1}{2}\int_0^a (a^2x^2 - 2ax^3 + x^4)dx + \frac{1}{60}(x-a)^5\Big|_0^a + \frac{1}{60}(x-a)^5\Big|_0^a \\
&\quad = \frac{1}{6}a^5 - \frac{1}{4}a^5 + \frac{1}{10}a^5 + \frac{1}{60}a^5 + \frac{1}{60}a^5 = \frac{1}{20}a^5. \quad \Diamond
\end{aligned}$$

7. (10 Points)**Section 5.5, Exercise 16.** *Evaluate*

$$\iiint_W z\,dx\,dy\,dz$$

where W is the region bounded by the planes $x=0, y=0, z=0, z=1$, and the cylinder $x^2+y^2=1$, with $x \geq 0, y \geq 0$.

Solution. Since W is defined by the inequalities

$$x^2+y^2 \leq 1, \quad x \geq 0, \quad y \geq 0, \quad \text{and} \quad 0 \leq z \leq 1,$$

we have

$$\iiint_W z\,dx\,dy\,dz = \iint_{\substack{x^2+y^2\leq 1 \\ x\geq 0, y\geq 0}} dx\,dy \int_0^1 z dz = \frac{\pi}{4} \cdot \frac{1}{2}z^2\Big|_0^1 = \frac{\pi}{8}. \quad \Diamond$$