1. (10 Points) **Section 4.1, Exercise 14** Show that, at a local maximum or minimum of the quantity \(|r(t)|\), \(r'(t)\) is perpendicular to \(r(t)\).

**Solution.** Notice first that at the time \(t\) where a local maximum or minimum for \(|r(t)|\) occurs, a local maximum or minimum for \(|r(t)|^2 = r(t) \cdot r(t)\) also occurs. And at those particular \(t\)'s, the first derivative of \(|r(t)|^2\) is equal to zero. Therefore

\[
0 = (r(t) \cdot r(t))' = r'(t) \cdot r(t) + r(t) \cdot r'(t) = 2r'(t) \cdot r(t),
\]

which means that \(r'(t)\) is perpendicular to \(r(t)\).

2. (10 Points) **Section 4.1, Exercise 18.** Let \(c\) be a path in \(\mathbb{R}^3\) with zero acceleration. Prove that \(c\) is a straight line or a point.

**Solution.** Write \(c(t) = (x(t), y(t), z(t))\). If \(c''(t) = 0\), then \(x''(t) = y''(t) = z''(t) \equiv 0\). These equations imply that \(x'(t) = c_1, y'(t) = c_2, \) and \(z'(t) = c_3, \) where \(c_1, c_2, c_3\) are all constants. Continuing, we see that this implies that \(x(t) = c_1 t + b_1, y(t) = c_2 t + b_2, z(t) = c_3 t + b_3, \) where \(b_1, b_2, b_3\) are all constants. If at least one \(c_i\) is nonzero, this is the equation of a straight line. If all the \(c_i\)'s are zero then \((x(t), y(t), z(t)) = (b_1, b_2, b_3)\) a point.

3. (10 Points) **Section 4.2, Exercise 4.** Compute the arc length of the curve described by

\[
\left( t + 1, \frac{2\sqrt{2}}{3} t^{3/2} + 7, \frac{1}{2} t^2 \right)
\]

on the interval \(1 \leq t \leq 2\).

**Solution.** Using the arc length formula,

\[
l = \int_1^2 \|c'(t)\| dt = \int_1^2 \sqrt{1 + 2t + t^2} dt = \int_1^2 (t + 1) dt = \frac{5}{2}. \quad \Diamond
\]

4. (20 points) **Exercise 18.** In special relativity, the **proper time** of a path \(c: [a, b] \to \mathbb{R}^4\) with \(c(\lambda) = (x(\lambda), y(\lambda), z(\lambda), t(\lambda))\) is defined to be the quantity

\[
\frac{1}{c} \int_a^b \sqrt{-[x'(\lambda)]^2 - [y'(\lambda)]^2 - [z'(\lambda)]^2 + c^2 [t'(\lambda)]^2} d\lambda,
\]

where \(c\) is the velocity of light, a constant. Referring to the Figure, show that, using self-explanatory notation,

\[
\text{proper time (AB) + proper time (BC) < proper time (AC)}.
\]

(This inequality is a special case of what is known as the **twin paradox**.)
Solution. We proceed in three steps. First we parametrize the paths.

i Let $A = (0, 0, 0, 0), B = (x_B, 0, 0, t_B), C = (0, 0, 0, t_C)$. Let $c_1, c_2, c_3$ be the paths from $A$ to $B$, $B$ to $C$, $A$ to $C$, respectively. Then
\[
\begin{align*}
    c_1(\lambda) & = (1 - \lambda)(0, 0, 0, 0) + \lambda(x_B, 0, 0, t_B) \\
    c_2(\lambda) & = (1 - \lambda)(x_B, 0, 0, t_B) + \lambda(0, 0, 0, t_C) \\
    c_3(\lambda) & = (1 - \lambda)(0, 0, 0, 0) + \lambda(0, 0, 0, t_C)
\end{align*}
\]

ii Denote the proper time of $AB$, $BC$, $AC$ by $T_{AB}$, etc., then
\[
T_{AB} = \frac{1}{c} \int_0^1 \sqrt{-x_B^2 + c^2 t_B^2} \, d\lambda = \frac{1}{c} \sqrt{-x_B^2 + c^2 t_B^2}.
\]
Similarly, we have
\[
\begin{align*}
    T_{BC} & = \frac{1}{c} \sqrt{-x_B^2 + c^2 (t_C - t_B)^2} \\
    T_{AC} & = \frac{1}{c} \sqrt{c^2 t_C^2} = \frac{1}{c} (ct_C).
\end{align*}
\]

iii It is sufficient to show that
\[
\sqrt{-x_B^2 + c^2 t_B^2} + \sqrt{-x_B^2 + c^2 (t_C - t_B)^2} < ct_C.
\]
But the above is true if and only if
\[
\sqrt{-x_B^2 + c^2 (t_C - t_B)^2} < ct_C - \sqrt{-x_B^2 + c^2 t_B^2}
\]
if and only if
\[
-x_B^2 + c^2 (t_C - t_B)^2 < c^2 t_C^2 - x_B^2 + c^2 t_B^2 - 2ct_C \sqrt{-x_B^2 + c^2 t_B^2}
\]
if and only if
\[
ct_B > \sqrt{-x_B^2 + c^2 t_B^2}.
\]
Since the last inequality is clearly true, the claim is proved.
5. (10 points) **Section 4.3, Exercise 14.** Show that the curve
c(t) = (t^2, 2t - 1, \sqrt{t}), t > 0

is a flow line of the velocity vector field

\[ F(x, y, z) = (y + 1, 2, 1/2z). \]

**Solution.** We must verify that \( c'(t) = F(c(t)) \). The left hand side is \( (2t, 2, 1/(2\sqrt{t})) \), while the right side is \( F(t^2, 2t - 1, \sqrt{t}) = (2t, 2, 1/(2\sqrt{t})) \). Since we have equality, we have shown that \( c(t) \) is a flow line of \( F \).

6. (10 points) **Section 4.4, Exercise 10.** Find the divergence of the vector field

\[ \mathbf{v}(x, y, z) = yi - xj. \]

**Solution.**

\[ \text{div } \mathbf{F} = \nabla \cdot \mathbf{F} = \frac{\partial}{\partial x}(y) - \frac{\partial}{\partial y}(x) = 0. \]

6. (10 Points) **Exercise 16.** Find the curl of the vector field

\[ \mathbf{F}(x, y, z) = \frac{yz \mathbf{i} - xz \mathbf{j} + xy \mathbf{k}}{x^2 + y^2 + z^2}. \]

**Solution.** We take the formal cross product of the \( \nabla \) operator with the given vector field to calculate the curl:

\[
\nabla \times \mathbf{F} = \begin{vmatrix}
i & j & k \\
\frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\
yz & -xz & xy \\
\frac{r^2}{r^2} & \frac{r^2}{r^2} & \frac{r^2}{r^2}
\end{vmatrix}
\]

\[
= \left[ \frac{\partial}{\partial y} \left( \frac{xy}{r^2} \right) + \frac{\partial}{\partial z} \left( \frac{xz}{r^2} \right) \right] \mathbf{i} - \left[ \frac{\partial}{\partial x} \left( \frac{xy}{r^2} \right) - \frac{\partial}{\partial z} \left( \frac{yz}{r^2} \right) \right] \mathbf{j}
\]

\[
+ \left[ \frac{\partial}{\partial x} \left( -xz \right) - \frac{\partial}{\partial y} \left( \frac{yz}{r^2} \right) \right] \mathbf{k}
\]

\[
= \left[ \frac{2xz - 2xy^2}{r^4} + \frac{2x - 2xz^2}{r^4} \right] \mathbf{i} - \left[ \frac{2yz - 2y^2x^2}{r^4} - \frac{2y - 2y^2z^2}{r^4} \right] \mathbf{j}
\]

\[
- \left[ \frac{2z^3 - 2z^3x^2}{r^4} + \frac{2z^3 - 2z^3y^2}{r^4} \right] \mathbf{k}
\]

\[
= \frac{2}{r^4} \left( x^3 \mathbf{i} + (yx^2 - yz^2) \mathbf{j} + z^3 \mathbf{k} \right).
\]

Therefore,

\[
\nabla \times \mathbf{F} = \left( \frac{2x^3}{(x^2 + y^2 + z^2)^2}, \frac{2y(x^2 - z^2)}{(x^2 + y^2 + z^2)^2}, \frac{-2z^3}{(x^2 + y^2 + z^2)^2} \right).
\]