Mathematics 1c: Solutions, Homework Set 4

Due: Monday, April 26 at 10am.

1. (10 Points) **Section 4.1, Exercise 14** *Show that, at a local maximum or minimum of the quantity* $\|\mathbf{r}(t)\|$, $\mathbf{r}'(t)$ *is perpendicular to* $\mathbf{r}(t)$.

   Solution. Notice first that at the time t where a local maximum or minimum for $\|\mathbf{r}(t)\|$ occurs, a local maximum or minimum for $\|\mathbf{r}(t)\|^2 = \mathbf{r}(t) \cdot \mathbf{r}(t)$ also occurs. And at those particular t's, the first derivative of $\|\mathbf{r}(t)\|^2$ is equal to zero. Therefore

   $$0 = (\mathbf{r}(t) \cdot \mathbf{r}(t))' = \mathbf{r}'(t) \cdot \mathbf{r}(t) + \mathbf{r}(t) \cdot \mathbf{r}'(t) = 2\mathbf{r}'(t) \cdot \mathbf{r}(t),$$

   which means that $\mathbf{r}'(t)$ is perpendicular to $\mathbf{r}(t)$.

2. (10 Points)**Section 4.1, Exercise 18.** *Let* $\mathbf{c}$ *be a path in* $\mathbb{R}^3$ *with zero acceleration. Prove that* $\mathbf{c}$ *is a straight line or a point.*

   Solution. Write $\mathbf{c}(t) = (x(t), y(t), z(t))$. If $\mathbf{c}''(t) = \mathbf{0}$, then $x''(t) = y''(t) = z''(t) \equiv 0$. These equations imply that $x'(t) = c_1, y'(t) = c_2$, and $z'(t) = c_3$, where c_1, c_2, c_3 are all constants. Continuing, we see that this implies that $x(t) = c_1 t + b_1, y(t) = c_2 t + b_2, z(t) = c_3 t + b_3$, where b_1, b_2, b_3 are all constants. If at least one c_i is nonzero, this is the equation of a straight line. If all the c_i's are zero then $(x(t), y(t), z(t)) = (b_1, b_2, b_3)$ a point.

3. (10 Points) **Section 4.2, Exercise 4.** *Compute the arc length of the curve described by*

   $$\left(t + 1, \frac{2\sqrt{2}}{3} t^{3/2} + 7, \frac{1}{2} t^2\right)$$

   on the interval $1 \leq t \leq 2$.

   Solution. Using the arc length formula,

   $$l = \int_1^2 \|\mathbf{c}'(t)\| dt = \int_1^2 \sqrt{1 + 2t + t^2} dt = \int_1^2 (t + 1) dt = \frac{5}{2}. \quad \Diamond$$

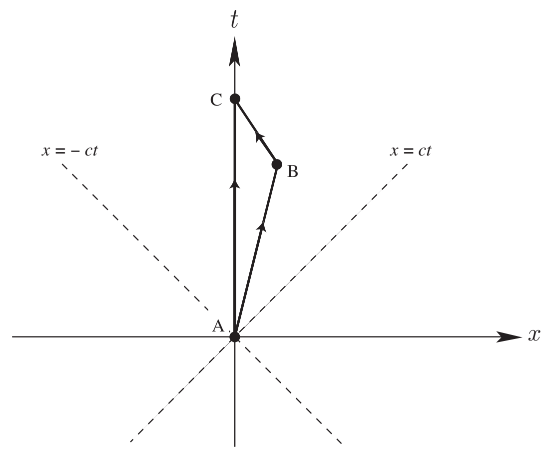
4. (20 points) **Exercise 18.** *In special relativity, the* ***proper time*** *of a path* $\mathbf{c} : [a, b] \to \mathbb{R}^4$ *with* $\mathbf{c}(\lambda) = (x(\lambda), y(\lambda), z(\lambda), t(\lambda))$ *is defined to be the quantity*

   $$\frac{1}{c} \int_a^b \sqrt{-[x'(\lambda)]^2 - [y'(\lambda)]^2 - [z'(\lambda)]^2 + c^2 [t'(\lambda)]^2} d\lambda,$$

   where c is the velocity of light, a constant. Referring to the Figure, show that, using self-explanatory notation,

   proper time (AB) + proper time (BC) < proper time (AC).

   (*This inequality is a special case of what is known as the* ***twin paradox***.)



Solution. We proceed in three steps. First we parametrize the paths.

i Let $\mathrm{A} = (0,0,0,0), \mathrm{B} = (x_{\mathrm{B}},0,0,t_{\mathrm{B}}), \mathrm{C} = (0,0,0,t_{\mathrm{C}})$. Let c_1, c_2, c_3 be the paths from A to B, B to C, A to C, respectively. Then

$$\begin{aligned} c_1(\lambda) &= (1-\lambda)(0,0,0,0) + \lambda(x_{\mathrm{B}},0,0,t_{\mathrm{B}}) \\ c_2(\lambda) &= (1-\lambda)(x_{\mathrm{B}},0,0,t_{\mathrm{B}}) + \lambda(0,0,0,t_{\mathrm{C}}) \\ c_3(\lambda) &= (1-\lambda)(0,0,0,0) + \lambda(0,0,0,t_{\mathrm{C}}) \end{aligned}$$

ii Denote the proper time of AB, BC, AC by T_{AB}, etc., then

$$T_{\mathrm{AB}} = \frac{1}{c}\int_0^1 \sqrt{-x_{\mathrm{B}}^2 + c^2 t_{\mathrm{B}}^2}\, d\lambda = \frac{1}{c}\sqrt{-x_{\mathrm{B}}^2 + c^2 t_{\mathrm{B}}^2}.$$

Similarly, we have

$$\begin{aligned} T_{\mathrm{BC}} &= \frac{1}{c}\sqrt{-x_{\mathrm{B}}^2 + c^2(t_{\mathrm{C}} - t_{\mathrm{B}})^2} \\ T_{\mathrm{AC}} &= \frac{1}{c}\sqrt{c^2 t_{\mathrm{C}}^2} = \frac{1}{c}(ct_{\mathrm{C}}). \end{aligned}$$

iii It is sufficient to show that

$$\sqrt{-x_{\mathrm{B}}^2 + c^2 t_{\mathrm{B}}^2} + \sqrt{-x_{\mathrm{B}}^2 + c^2(t_{\mathrm{C}} - t_{\mathrm{B}})^2} < ct_{\mathrm{C}}.$$

But the above is true if and only if

$$\sqrt{-x_{\mathrm{B}}^2 + c^2(t_{\mathrm{C}} - t_{\mathrm{B}})^2} < ct_{\mathrm{C}} - \sqrt{-x_{\mathrm{B}}^2 + c^2 t_{\mathrm{B}}^2}$$

if and only if

$$-x_{\mathrm{B}}^2 + c^2(t_{\mathrm{C}} - t_{\mathrm{B}})^2 < c^2 t_{\mathrm{C}}^2 - x_{\mathrm{B}}^2 + c^2 t_{\mathrm{B}}^2 - 2ct_{\mathrm{C}}\sqrt{-x_{\mathrm{B}}^2 + c^2 t_{\mathrm{B}}^2}$$

if and only if

$$ct_{\mathrm{B}} > \sqrt{-x_{\mathrm{B}}^2 + c^2 t_{\mathrm{B}}^2}.$$

Since the last inequality is clearly true, the claim is proved.

5. (10 points) **Section 4.3, Exercise 14.** *Show that the curve*

$$\mathbf{c}(t) = (t^2, 2t - 1, \sqrt{t}), t > 0$$

is a flow line of the velocity vector field

$$\mathbf{F}(x, y, z) = (y + 1, 2, 1/2z).$$

Solution. We must verify that $\mathbf{c}'(t) = \mathbf{F}(\mathbf{c}(t))$. The left hand side is $(2t, 2, 1/(2\sqrt{t}))$, while the right side is $\mathbf{F}(t^2, 2t-1, \sqrt{t}) = (2t, 2, 1/(2\sqrt{t}))$. Since we have equality, we have shown that $\mathbf{c}(t)$ is a flow line of $\mathbf{F}$.

6. (10 points) **Section 4.4, Exercise 10.** *Find the divergence of the vector field* $\mathbf{v}(x, y, z) = y\mathbf{i} - x\mathbf{j}$.

Solution.

$$\operatorname{div} \mathbf{F} = \nabla \cdot \mathbf{F} = \frac{\partial}{\partial x}(y) - \frac{\partial}{\partial y}(x) = 0. \quad \Diamond$$

7. (10 Points) **Exercise 16.** *Find the curl of the vector field*

$$\mathbf{F}(x, y, z) = \frac{yz\mathbf{i} - xz\mathbf{j} + xy\mathbf{k}}{x^2 + y^2 + z^2}.$$

Solution. We take the formal cross product of the ∇ operator with the given vector field to calculate the curl:

$$\begin{aligned}
\nabla \times \mathbf{F} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \dfrac{yz}{r^2} & \dfrac{-xz}{r^2} & \dfrac{xy}{r^2} \end{vmatrix} \\
&= \left[\frac{\partial}{\partial y}\left(\frac{xy}{r^2}\right) + \frac{\partial}{\partial z}\left(\frac{xz}{r^2}\right)\right]\mathbf{i} - \left[\frac{\partial}{\partial x}\left(\frac{xy}{r^2}\right) - \frac{\partial}{\partial z}\left(\frac{yz}{r^2}\right)\right]\mathbf{j} \\
&\quad + \left[\frac{\partial}{\partial x}\left(\frac{-xz}{r^2}\right) - \frac{\partial}{\partial y}\left(\frac{yz}{r^2}\right)\right]\mathbf{k} \\
&= \left[\frac{xr^2 - 2xy^2}{r^4} + \frac{xr^2 - 2xz^2}{r^4}\right]\mathbf{i} - \left[\frac{yr^2 - 2yx^2}{r^4} - \frac{yr^2 - 2yz^2}{r^4}\right]\mathbf{j} \\
&\quad - \left[\frac{zr^2 - 2zx^2}{r^4} + \frac{zr^3 - 2zy^2}{r^4}\right]\mathbf{k} \\
&= \frac{2}{r^4}\left(x^3\mathbf{i} + (yx^2 - yz^2)\mathbf{j} + z^3\mathbf{k}\right).
\end{aligned}$$

Therefore,

$$\nabla \times \mathbf{F} = \left(\frac{2x^3}{(x^2 + y^2 + z^2)^2}, \frac{2y(x^2 - z^2)}{(x^2 + y^2 + z^2)^2}, \frac{-2z^3}{(x^2 + y^2 + z^2)^2}\right)$$