Mathematics 1c: Solutions, Homework Set 3

Due: Monday, April 19th by 10am.

1. (10 Points) **Section 3.1, Exercise 16** *Let $w = f(x, y)$ be a function of two variables, and let*

$$x = u + v, \quad y = u - v.$$

Show that

$$\frac{\partial^2 w}{\partial u \partial v} = \frac{\partial^2 w}{\partial x^2} - \frac{\partial^2 w}{\partial y^2}.$$

Solution. By the chain rule,

$$\frac{\partial w}{\partial v} = \frac{\partial w}{\partial x} \cdot \frac{\partial x}{\partial v} + \frac{\partial w}{\partial y} \cdot \frac{\partial y}{\partial v} = w_x - w_y.$$

Thus,

$$\begin{aligned}\frac{\partial^2 w}{\partial u \partial v} &= \frac{\partial}{\partial u}\left(\frac{\partial w}{\partial v}\right) = \frac{\partial}{\partial u}(w_x - w_y) = \frac{\partial}{\partial u} w_x - \frac{\partial}{\partial u} w_y \\ &= \frac{\partial w_x}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial w_x}{\partial y} \cdot \frac{\partial y}{\partial u} - \left(\frac{\partial w_y}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial w_y}{\partial y} \cdot \frac{\partial y}{\partial u}\right) \\ &= w_{xx} + w_{xy} - (w_{yx} + w_{yy}) = w_{xx} - w_{yy}\end{aligned}$$

i.e.,

$$\frac{\partial^2 w}{\partial u \partial v} = \frac{\partial^2 w}{\partial x^2} - \frac{\partial^2 w}{\partial y^2}.$$

2. (10 Points) **Section 3.1, Exercise 22**

(a) *Show that the function*

$$g(x, t) = 2 + e^{-t} \sin x$$

satisfies the heat equation: $g_t = g_{xx}$. [Here $g(x, t)$ represents the temperature in a metal rod at position x and time t.]

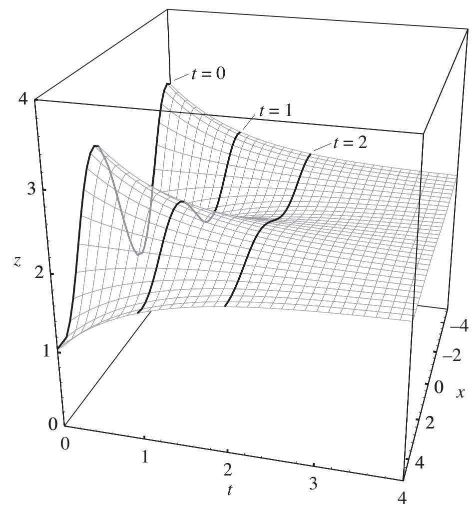
(b) *Sketch the graph of g for $t \geq 0$. (Hint: Look at sections by the planes $t = 0, t = 1,$ and $t = 2$.)*

(c) *What happens to $g(x, t)$ as $t \to \infty$? Interpret this limit in terms of the behavior of heat in the rod.*

Solution.

(a) Since $g(x, y) = 2 + e^{-t} \sin x$, then $g_t = -e^{-t} \sin x, g_x = e^{-t} \cos x$, and $g_{xx} = -e^{-t} \sin x$. Therefore, $g_t = g_{xx}$.

(b) The graph of g and the times $t = 0, 1,$ and 2 is shown in the figure—try this yourself on the computing site.



(c) Note that

$$\lim_{t\to\infty} g(x,t) = \lim_{t\to\infty}(2 + e^{-t}\sin x) = 2$$

This means that the temperature in the rod at position x tends to be a constant ($= 2$) as the time t is large enough.

3. (10 Points) **Section 3.2, Exercise 6** *Determine the second-order Taylor formula for the function*

$$f(x,y) = e^{(x-1)^2}\cos y$$

expanded about the point $x_0 = 1, y_0 = 0$.

Solution. The ingredients needed in the second-order Taylor formula are computed as follows:

$$\begin{aligned}
f_x &= 2(x-1)e^{(x-1)^2}\cos y \\
f_y &= -e^{(x-1)^2}\sin y \\
f_{xx} &= 2e^{(x-1)^2}\cos y + 4(x-1)^2 e^{(x-1)^2}\cos y \\
f_{xy} &= -2(x-1)e^{(x-1)^2}\sin y = f_{yx} \\
f_{yy} &= -e^{(x-1)^2}\cos y.
\end{aligned}$$

Evaluating the function and these derivatives at the point $(1,0)$ gives

$$\begin{aligned}
f(1,0) &= 1 \\
f_x(1,0) &= f_y(1,0) = 0 \\
f_{xx}(1,0) &= 2 \\
f_{xy}(1,0) &= f_{yx}(1,0) = 0 \quad \text{and} \\
f_{yy}(1,0) &= -1.
\end{aligned}$$

Consequently, the second order Taylor formula is

$$f(\mathbf{h}) = 1 + h_1^2 - \frac{1}{2}h_2^2 + R_2((1,0),\mathbf{h}),$$

where $\mathbf{h} = (h_1, h_2)$ and where

$$\frac{R_2((1,0),\mathbf{h})}{\|\mathbf{h}\|} \to 0 \quad \text{as} \quad \|\mathbf{h}\| \to 0.$$

4. (10 Points) **Section 3.3, Exercise 7** *Find the critical points for the function*

$$f(x,y) = 3x^2 + 2xy + 2x + y^2 + y + 4$$

and determine if they are maxima, minima or saddle points.

Solution. Here,

$$\frac{\partial f}{\partial x} = 6x + 2y + 2, \quad \frac{\partial f}{\partial y} = 2x + 2y + 1.$$

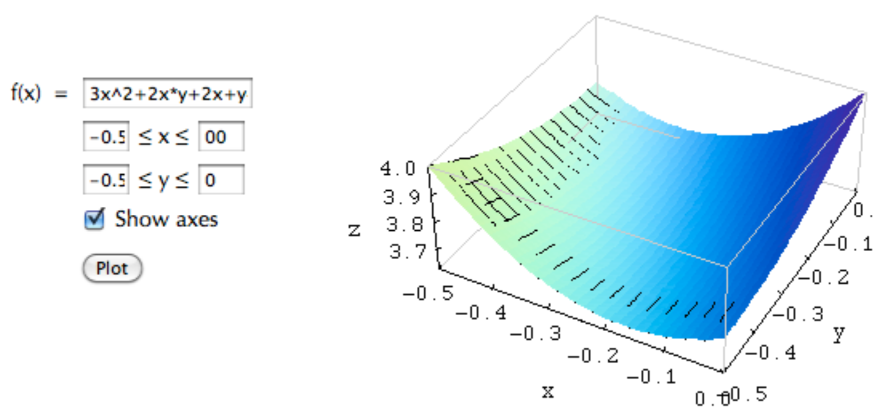
We have

$$\frac{\partial f}{\partial x} = 0, \quad \frac{\partial f}{\partial y} = 0$$

when $x = y = -1/4$. Therefore, the only critical point is $(-1/4, -1/4)$. The second derivative matrix at this point is

$$\begin{bmatrix} 6 & 2 \\ 2 & 2 \end{bmatrix}$$

The diagonal determinants are both positive and so the matrix is positive definite and we have a local minimum. This is confirmed by the figure.



5. (10 Points) **Section 3.3, Exercise 25** *Write the number* 120 *as a sum of three non-negative numbers so that the sum of the products taken two at a time is a maximum.*

Solution. Let the three numbers be x, y, z. Thus,

$$x + y + z = 120, \quad z = 120 - x - y.$$

We want to find the maximum value for

$$\begin{aligned} S(x, y) &= xy + yz + xz = xy + (x + y)(120 - x - y) \\ &= -x^2 - xy - y^2 + 120x + 120y. \end{aligned}$$

The maximum must exist by Theorem 7 on page 220. To locate it, we use the first derivative test; we differentiate to get

$$\frac{\partial S}{\partial x} = -2x - y + 120, \quad \frac{\partial S}{\partial y} = -x - 2y + 120.$$

These partial derivatives vanish when $x = y = 40$, in which case $z = 120 - (x + y) = 40$. Therefore, when $x = y = z = 40$ is the only critical point.

We must examine the extreme, or boundary, cases in which either $x = 0, x = 120, y = 0, y = 120, z = 0$, or $z = 120$ separately. For example, if $x = 0$ then $S(x, y) = -y^2 + 120y$ which has a maximum (from one variable calculus) at $y = 60$ in which case we also have $z = 60$ and so $S = 60 \times 60 = 3,600$. However, at the point $x = y = z = 40$, S has the value $3 \times 40 \times 40 = 4,800$, which is greater. Likewise one checks that the other boundary values also lead to smaller values, so the global maximum is at the critical point where $x = y = z = 40$.

6. (10 Points) **Section 3.4, Exercise 2** *Find the extrema of* $f(x, y) = x - y$ *subject to the constraint* $x^2 - y^2 = 2$.

Solution. By the method of Lagrange multipliers, we write the constraint as $g = 0$, where $g(x, y) = x^2 - y^2 - 2$ and then write the Lagrange multiplier equations as $\nabla f = \lambda \nabla g$. Thus, we get

$$\begin{aligned} 1 &= \lambda \cdot 2x \\ 1 &= \lambda \cdot 2y \\ x^2 - y^2 - 2 &= 0. \end{aligned}$$

First of all, the first two equations imply that $x \neq 0$ and $y \neq 0$. Hence we can eliminate λ, giving $x = y$. From the last equation this would imply that $2 = 0$. Hence there are no extrema.

7. (10 Points) **Section 3.4, Exercise 20** *A light ray travels from point* A *to point* B *crossing a boundary between two media* (*see Figure* 3.4.7 *of the text*). *In the first medium its speed is* v_1 *and in the second* v_2. *Show that the trip is made in minimum time when Snell's law holds:*

$$\frac{\sin \theta_1}{\sin \theta_2} = \frac{v_1}{v_2}.$$

Solution. This is a one variable minimization problem. Let (p, q) denote the coordinates of point A and (l, m) those of point B. Then the total time for the light ray to go from A to B after being refracted at a point x on the x-axis is

$$f(x) = \frac{1}{v_1}\sqrt{(x-p)^2+q^2} + \frac{1}{v_2}\sqrt{(l-x)^2+m^2}.$$

There must be a global minimum on the interval $[p, l]$ by Theorem 7 on page 220. If the minimum occurs at a point x_0 satisfying $p < x_0 < l$, then $f'(x_0) = 0$ or

$$\frac{1}{v_1}\frac{x_0-p}{\sqrt{(x_0-p)^2+q^2}} - \frac{1}{v_2}\frac{l-x_0}{\sqrt{(l-x_0)^2+m^2}} = 0.$$

Rearranging the terms, we obtain:

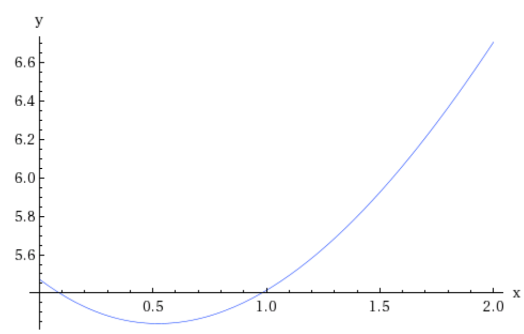
$$\frac{x_0-p}{\sqrt{(x_0-p)^2+q^2}} \Big/ \frac{l-x_0}{\sqrt{(l-x_0)^2+m^2}} = \frac{v_1}{v_2}$$

or

$$\frac{\sin\theta_1}{\sin\theta_2} = \frac{v_1}{v_2}.$$

This equation shows that there is only one critical point in the open interval where $p < x_0 < l$ and that Snell's law holds.

However, we need to make sure that the minimum does not occur at the endpoints. If you don't see a way to do this in general, you can always give a partial answer by choosing values for the velocities and the two endpoints and drawing a graph of f, as in the following figure with $p = 1, q = 1, v_1 = 1, v_2 = 1/2, l = 2, m = -2$:



It is clear from this *example* that the minimum is not at an endpoint. To conclude this more generally, we compute the second derivative of the function $f(x)$ to get (after a little algebra):

$$f''(x) = \frac{1}{v_1}\frac{q^2}{[(x-p)^2+q^2]^{3/2}} + \frac{1}{v_2}\frac{m^2}{[(l-x)^2+m^2]^{3/2}}$$

The thing to notice about this expression is that (barring the degenerate case in which both q and m vanish and the velocities are assumed to be non-zero) each term is positive and so the function $f(x)$ is convex upwards (consistent with the figure) and thus the minimum cannot occur at the endpoints, so must occur in the interior of the interval.

8. (10 Points) **Section 3.4, Exercise 22** *Let* P *be a point on a surface* S *in* $\mathbb{R}^3$ *defined by the equation* $f(x, y, z) = 1$*, where* f *is of class* C^1*. Suppose that* P *is a point where the distance from the origin to* S *is maximized. Show that the vector emanating from the origin and ending at* P *is perpendicular to* S*.*

Solution. We want to maximize the function $g(x, y, z) = x^2 + y^2 + z^2$ subject to the constraint $f(x, y, z) = 1$. Suppose this maximum occurs at $\mathrm{P} = (x_0, y_0, z_0)$, then by the method of Lagrange multipliers we have the equations

$$\begin{aligned} 2x_0 &= \lambda \{\nabla f(x_0, y_0, z_0)\}_1 \\ 2y_0 &= \lambda \{\nabla f(x_0, y_0, z_0)\}_2 \\ 2z_0 &= \lambda \{\nabla f(x_0, y_0, z_0)\}_3 \end{aligned}$$

where $\{\nabla f(x_0, y_0, z_0)\}_i$ denotes the ith component of $\nabla f(x_0, y_0, z_0), 1 \leq i \leq 3$. If $\mathbf{v} = (x_0, y_0, z_0)$ is the vector from the origin ending at P, then these equations say that $\mathbf{v} = \left(\frac{\lambda}{2}\right) \cdot \nabla f(x_0, y_0, z_0)$. But $\nabla f(x_0, y_0, z_0)$ is perpendicular to S at P, and since $\mathbf{v}$ is a scalar multiple of $\nabla f(x_0, y_0, z_0)$ it is also perpendicular to S at P.