Mathematics 1c: Homework Set 2

Due: Monday, April 12th by 10am.

1. (10 Points) **Section 2.5, Exercise 8** *Suppose that a function is given in terms of rectangular coordinates by* $u = f(x, y, z)$. *If* $x = \rho \cos\theta \sin\phi, y = \rho \sin\theta \sin\phi, z = \rho\cos\phi$, *express the partial derivatives* $\partial u/\partial\rho, \partial u/\partial\theta$, *and* $\partial u/\partial\phi$ *in terms of* $\partial u/\partial x, \partial u/\partial y$, *and* $\partial u/\partial z$.

   Solution. By the chain rule,

$$\begin{aligned}
\frac{\partial u}{\partial \rho} &= \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial \rho} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial \rho} + \frac{\partial u}{\partial z}\cdot\frac{\partial z}{\partial \rho} \\
&= \frac{\partial u}{\partial x}\cdot\cos\theta\sin\phi + \frac{\partial u}{\partial y}\cdot\sin\theta\sin\phi + \frac{\partial u}{\partial z}\cdot\cos\phi \\
\frac{\partial u}{\partial \theta} &= \frac{\partial u}{\partial x}\cdot(-\rho\sin\theta\sin\phi) + \frac{\partial u}{\partial y}\cdot\rho\cos\theta\sin\phi + \frac{\partial u}{\partial z}\cdot 0 \\
&= -\sin\theta\sin\phi\rho\frac{\partial u}{\partial x} + \cos\theta\sin\phi\rho\frac{\partial u}{\partial y} \\
\frac{\partial u}{\partial \phi} &= \frac{\partial u}{\partial x}\cdot\rho\cos\theta\cos\phi + \frac{\partial u}{\partial y}\cdot\rho\sin\theta\cos\phi + \frac{\partial u}{\partial z}\cdot(-\rho\sin\phi) \\
&= \rho\cos\theta\cos\phi\frac{\partial u}{\partial x} + \rho\sin\theta\cos\phi\frac{\partial u}{\partial y} - \rho\sin\phi\frac{\partial u}{\partial z}. \quad \Diamond
\end{aligned}$$

2. (10 Points) **Section 2.5, Exercise 12** *Suppose that the temperature at the point* (x, y, z) *in space is* $T(x, y, z) = x^2 + y^2 + z^2$. *Let a particle follow the right circular helix* $\sigma(t) = (\cos t, \sin t, t)$ *and let* $T(t)$ *be its temperature at time* t.

   (a) *What is* $T'(t)$*?*

   (b) *Find an approximate value for the temperature at* $t = (\pi/2) + 0.01$.

   Solution.

   (a) $T(t) = T(\sigma(t)) = \cos^2 t + \sin^2 t + t^2 = 1 + t^2$. $T'(t) = 2t$.

   (b) By the linear approximation, an approximate value is

$$T\left(\frac{\pi}{2}\right) + T'\left(\frac{\pi}{2}\right)\cdot\left(\frac{\pi}{2} + 0.01 - \frac{\pi}{2}\right) = 1 + \left(\frac{\pi}{2}\right)^2 + 2\cdot\frac{\pi}{2}\cdot 0.01 \approx 3.4988 \quad \Diamond$$

3. (10 Points) **Section 2.6, Exercise 3(c)** *Compute the directional derivative of the function*

$$f(x, y, z) = xyz$$

   at the point $(x_0, y_0, z_0) = (1, 0, 1)$ *in the direction of the unit vector parallel to the vector* $\mathbf{d} = (1, 0, -1)$.

Solution. The function is a polynomial in x, y, z and so is smooth and in particular is differentiable. Thus, we compute the gradient by differentiating each component:

$$\nabla f(x, y, z) = (yz, xz, xy).$$

The directional derivative is therefore

$$\begin{aligned}\nabla f(1,0,1) \cdot \frac{\mathbf{d}}{\|\mathbf{d}\|} &= (0,1,0) \cdot \frac{1}{\sqrt{2}}(1,0,-1) \\ &= \frac{1}{\sqrt{2}}(0+0+0) = 0. \quad \diamondsuit\end{aligned}$$

4. (20 Points) **Section 2.6, Exercise 16**. *Captain Ralph is in trouble near the sunny side of Mercury and notices that the hull of his ship is beginning to melt. The temperature in his vicinity is given by*

$$T = e^{-x^2} + e^{-2y^2} + e^{-3z^2}.$$

   (a) *If he is at the point* $(1, 1, 1)$*, in what direction should he proceed in order to cool fastest?*

   (b) *If the ship travels at* e^8 *meters per second, how fast will the temperature decrease if he proceeds in that direction?*

   (c) *Unfortunately, the metal of the hull will crack if cooled at a rate greater than* $\sqrt{14}\mathrm{e}^2$ *degrees per second. Describe the set of possible directions in which he may proceed to bring the temperature down at no more than that rate.*

Solution.

   (a) In order to cool the fastest, the captain should proceed in the direction in which T is decreasing the fastest; that is, in the direction of the *negative* gradient at the point $(1, 1, 1)$, namely

$$\begin{aligned}-\nabla T(1,1,1) &= -\frac{\partial T}{\partial x}\mathbf{i} - \frac{\partial T}{\partial y}\mathbf{j} - \frac{\partial T}{\partial z}\mathbf{k}\bigg|_{(1,1,1)} \\ &= 2e^{-1}\mathbf{i} + 4e^{-1}\mathbf{j} + 6e^{-1}\mathbf{k} \\ &= \frac{2}{e}(1,2,3)\end{aligned}$$

   Thus,

$$-\nabla T(1,1,1) = \frac{2}{e}\left(\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}\right),$$

   which is the (non-normalized) direction required.

(b) As we have seen, to cool the fastest, the ship should move in the *direction* of the negative gradient of T, namely $\frac{2}{e}\,(\mathbf{i}+2\mathbf{j}+3\mathbf{k})$; however, the ship is moving at a velocity e^8 meters per second. The actual velocity of the ship is a vector in the same direction as the negative of the gradient, namely $\mathbf{v} = \frac{2a}{e}\,(\mathbf{i}+2\mathbf{j}+3\mathbf{k})$ where $a > 0$ for a positive constant a. We can solve for a by setting the velocity equal to e^8, namely

$$e^8 = \frac{2a}{e}\sqrt{(1+4+9)} = \frac{2a}{e}\sqrt{14}.$$

Thus, $a = \frac{e^9}{2\sqrt{14}}$ and so the velocity of the ship is $\mathbf{v} = \frac{e^8}{\sqrt{14}}\,(\mathbf{i}+2\mathbf{j}+3\mathbf{k})$. Note that, as a check on our arithmetic, that the length of this vector is e^8, as it should be. The rate of change of cooling of the temperature as the ship moves in this direction is given by (for example, the chain rule)

$$\nabla T(1,1,1)\cdot\mathbf{v} = \frac{2}{e}\,(\mathbf{i}+2\mathbf{j}+3\mathbf{k})\cdot\mathbf{v} = 2e^7\sqrt{14}.$$

(c) Suppose that the ship is proceeding in the direction of a unit vector $\mathbf{n}$ with a velocity of e^8; that is it is proceeding with the velocity vector $\mathbf{u} = e^8\mathbf{n}$. We need to see which of these directions gives a rate of change of temperature that does not exceed $\sqrt{14}e^2$ degrees per second. The negative rate of change of temperature at the point $(1,1,1)$ in the direction is given by the chain rule to be

$$-\nabla T(1,1,1)\cdot\mathbf{u} = \frac{2}{e}\,(\mathbf{i}+2\mathbf{j}+3\mathbf{k})\cdot e^8\mathbf{n}.$$

Let $\mathbf{w} = \frac{1}{\sqrt{14}}\,(\mathbf{i}+2\mathbf{j}+3\mathbf{k})$ be a unit vector in the direction of the negative direction of the gradient $-\nabla T(1,1,1)$; then we can write the preceding equation for the rate of change of temperature in the direction $e^8\mathbf{n}$ as

$$\mathbf{w}\sqrt{14}\cdot e^8\mathbf{n} = \sqrt{14}\cdot e^8\cos\theta$$

where $\cos\theta$ is the angle between the unit vectors $\mathbf{w}$ and $\mathbf{n}$. This not to exceed $\sqrt{14}e^2$ degrees per second, we require that

$$\sqrt{14}\cdot e^8\cos\theta \le \sqrt{14}e^2$$

that is, $\cos\theta \le 1/e^6$. In other words, the allowed set of directions in which the spaceship goes should form a cone about the negative gradient and making an angle of no less than $\cos^{-1}(1/e^6)$ with it.

5. (10 Points) **Review Exercise 20 for Chapter 2**. *Let $(x(t), y(t))$ be a path in the plane, $0 \le t \le 1$, and let $f(x,y)$ be a C^1 function of two variables and let f_x and f_y denote the two partial derivatives. Assume that*

$$\left(\frac{dx}{dt}\right)f_x + \left(\frac{dy}{dt}\right)f_y \le 0.$$

Show that $f(x(1),y(1)) \le f(x(0),y(0))$.

Solution. Let $g(t) = f(x(t), y(t))$. By the chain rule and our assumption,

$$g'(t) = \left(\frac{dx}{dt}\right) f_x + \left(\frac{dy}{dt}\right) f_y \leq 0.$$

By the fundamental theorem of calculus

$$g(1) - g(0) = \int_0^1 g'(t)dt \leq 0.$$

Therefore,

$$f(x(1), y(1)) = g(1) \leq g(0) = f(x(0), y(0)).$$

6. (10 Points) **Review Exercise 22 for Chapter 2**. *Find the direction in which the function*

$$w = x^2 + xy$$

increases most rapidly at the point $(-1, 1)$*. What is the magnitude of* ∇w *at this point? Interpret this magnitude geometrically.*

Solution. We compute

$$\frac{\partial w}{\partial x} = 2x + y, \quad \frac{\partial w}{\partial y} = x,$$

so $\nabla w(-1, 1) = (-1, -1)$. Therefore, since the gradient is the direction of fastest increase, the required direction is $(-1, -1)$. The magnitude of ∇w at this point is

$$\|\nabla w(-1, 1)\| = \|(-1, -1)\| = \sqrt{2},$$

which is equal to the rate of change of w in the direction $(-1, -1)$. If $\mathbf{n} = (-1, -1)/\sqrt{2}$ is the unit vector in this direction, note that at the point $(-1, 1)$,

$$\nabla w \cdot \mathbf{n} = (-1, -1) \cdot \frac{1}{\sqrt{2}}(-1, -1) = \sqrt{2}$$

verifying the general fact that the magnitude of a gradient of a function equals the rate of change of the function in the direction of the normalized gradient.

7. (10 Points) **Review Exercise 42 for Chapter 2**. *Use the chain rule to find a formula for*

$$\frac{d}{dt}(f(t)^{g(t)}).$$

Solution. Let $w = u^v, u = f(t)$, and $v = g(t)$. Then

$$\begin{aligned} \frac{d}{dt}(f(t)^{g(t)}) &= \frac{dw}{dt} = \frac{\partial w}{\partial u}\frac{du}{dt} + \frac{\partial w}{\partial v}\frac{dv}{dt} \\ &= vu^{v-1} f'(t) + u^v \log u g'(t) \\ &= g(t) f(t)^{g(t)-1} f'(t) + (f(t)^{g(t)} \log f(t)) g'(t). \end{aligned}$$