Mathematics 1c: Solutions to homework Set 1

1. (10 Points) Using the computing site or otherwise, draw the graphs of the following functions:

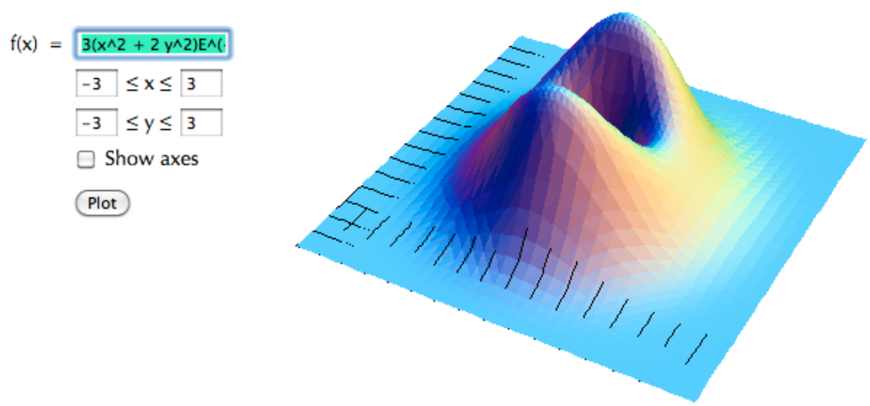
   (a) $f(x, y) = 3(x^2 + 2y^2)e^{-x^2-y^2}$; ***Tip:*** On the computing site use E to take the exponent and there is no need to type a * for multiplication; we suggest taking x and y between -2 and 2.

   (b) $f(x, y) = (x^3 - 3x)/(1 + y^2)$

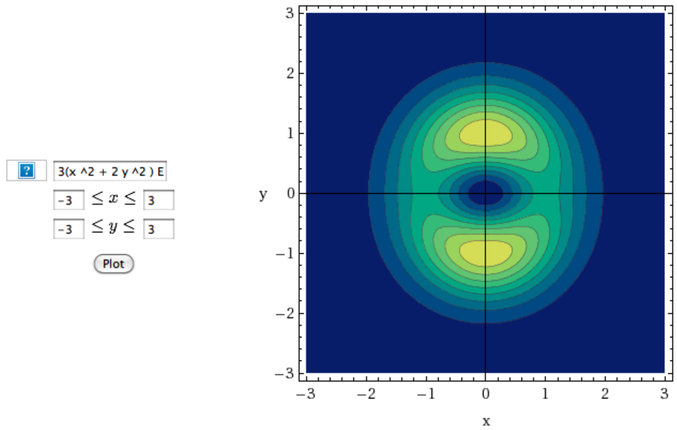
   Indicate some key features of these graphs, such as the location of the maxima and minima, important sections, etc

   Solutions.

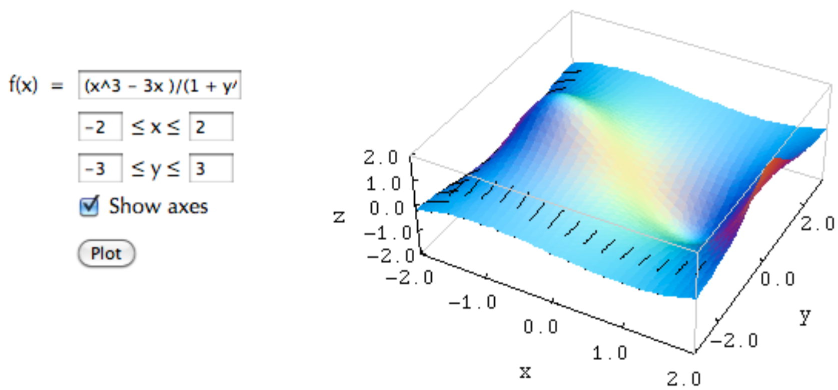
   (a) The graph is shown in the accompanying figure:



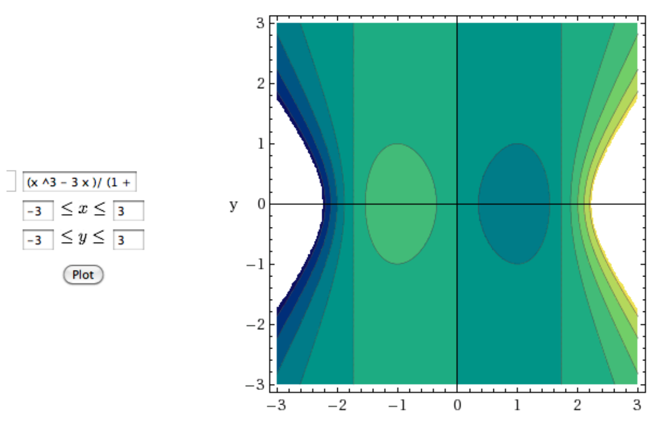
   As we see this function has one minimum at the center, two maxima and two saddle points. Interesting sections would be obtained by slicing the graph with planes parallel to the two axes. Interesting level sets are obtained by cutting the graph with horizontal planes at various heights. The level sets, computed using the computing site are shown in the next figure.



(b) The graph is shown in the accompanying figure:



Perhaps the most interesting section is obtained by cutting the graph using the vertical plane $y = 0$. Level curves are obtained using horizontal planes and these are distorted circles surrounding the maximum and minimum.



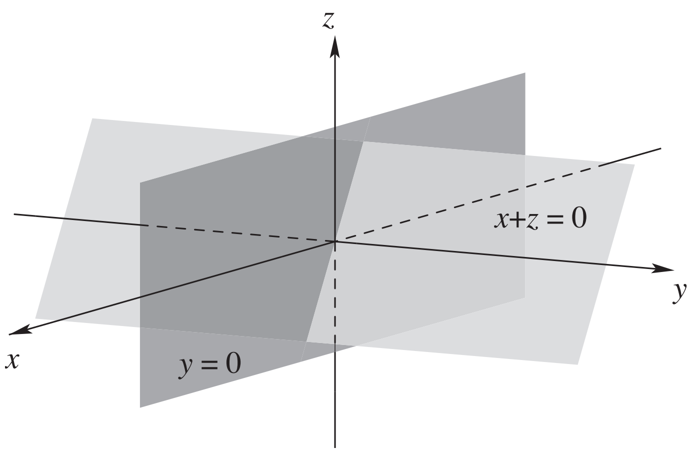
2. (10 Points) **Section 2.1, parts of Exercises 15, 18.** *Sketch the zero level set*

of the function $f(x, y, z) = xy + yz$ *and the level set for* $c = 1$ *of the function* $f(x, y) = \max(|x|, |y|)$.

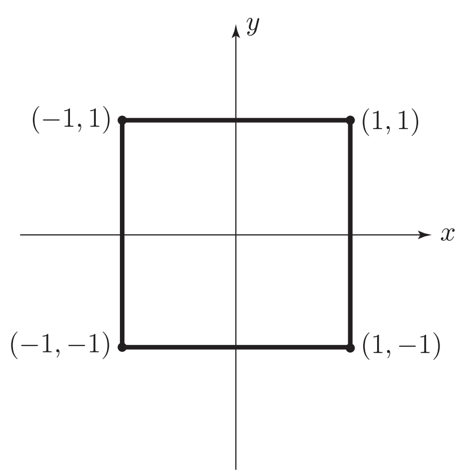
Solutions. Setting $f(x, y, z) = xy + yz = c = 0$, we have

$$(x + z)y = 0, \quad \text{so that either} \quad x + z = 0, \quad \text{or} \quad y = 0.$$

The level set consists of the two planes shown in shown in the Figure.



Notice that $f(-1, 1) = 1$, $f(1, 1) = 1$, $f(-1, -1) = 1$ and $f(-1, 1) = 1$that $f(1, -1) = 1$, so these 4 points are on the level set $f = 1$; they form the vertices of a square. The straight line joining the points $(-1, 1)$ and $(1, 1)$ is the line $(x, y) = (t, 1)$, $-1 \leq t \leq 1$. The value of f along this line is also 1, as is the value of f along the line segments joining the other vertices. This forms the level set as in the figure.



3. (15 Points) **Section 2.2, Exercise 12.** *Compute the following limits, if they exist*

   (a) $\displaystyle\lim_{x\to 0}\frac{\sin 2x - 2x}{x^3}$.

   (b) $\displaystyle\lim_{(x,y)\to(0,0)}\frac{\sin 2x - 2x + y}{x^3 + y}$.

   (c) $\displaystyle\lim_{(x,y,z)\to(0,0,0)}\frac{2x^2 y\cos z}{x^2 + y^2}$.

Solution. (a) By l'Hopital's rule,

$$\begin{aligned}\lim_{x\to 0}\frac{\sin 2x - 2x}{x^3} &= \lim_{x\to 0}\frac{2\cos 2x - 2}{3x^2}\\ &= \lim_{x\to 0}\frac{-4\sin 2x}{6x} \quad \text{(again by l'Hopital)}\\ &= -\left(\frac{4}{3}\right)\lim_{x\to 0}\frac{\sin 2x}{2x} = -\frac{4}{3}\end{aligned}$$

since

$$\lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1.$$

(b) From (a), for $y = 0$

$$\frac{\sin 2x - 2x}{x^3} \quad \text{approaches} \quad -\frac{4}{3}$$

as $(x, 0)$ approaches $(0, 0)$. Setting $y = 2x$, we see that

$$\frac{\sin 2x - 2x + y}{x^3 + y} = \frac{\sin 2x}{x^3 + 2x} = \frac{\sin 2x}{2x}\left(\frac{2}{x^2 + 2}\right)$$

and hence

$$\lim_{(x,2x)\to 0}\frac{\sin 2x - 2x + y}{x^3 + y} = \lim_{x\to 0}\frac{\sin 2x}{2x}\cdot\frac{2}{x^2 + 2} = 1\cdot 1 = 1.$$

Therefore,

$$\frac{\sin 2x - 2x + y}{x^3 + y}$$

has *two different* limits along two different rays approaching the origin $(0, 0)$, and consequently

$$\lim_{(x,y)\to(0,0)}\frac{\sin 2x - 2x + y}{x^3 + y}$$

does *not* exist.

(c) We have

$$\lim_{(x,y,z)\to(0,0,0)} \frac{2x^2 y\cos z}{x^2+y^2} = \lim_{(x,y)\to(0,0)} \frac{2xy}{x^2+y^2}x = 0$$

To demonstrate this last limit, lets us first show that the following inequality holds:

$$\frac{|2xy|}{x^2+y^2} \le 1. \tag{0.1}$$

This inequality (0.1) holds by the following reasoning: first, $x^2+y^2 \pm 2xy = (x\pm y)^2 \ge 0$ which we can rewrite as $x^2+y^2 \ge |2xy|$. Thus, assuming that $(x,y) \neq (0,0)$, and dividing $x^2+y^2 \ge |2xy|$ by x^2+y^2 gives the required relation (0.1).

This inequality shows that $\left|\frac{2xy}{x^2+y^2}x\right|$ lies between 0 and $|x|$. Thus as x tends to zero, so does $\frac{2xy}{x^2+y^2}x$ (explicitly, given $\epsilon > 0$, choose $\delta = \epsilon$ and so if $|x| < \delta$, then $\left|\frac{2xy}{x^2+y^2}x\right| \le |x| < \delta = \epsilon$. (See the related two dimensional Example 12 on page 121.)

4. (10 Points) **Section 2.3, Exercise 4(d)** *Show that the following function is differentiable at each point in its domain. Determine if the function is C^1.*

$$f(x,y) = \frac{xy}{\sqrt{x^2+y^2}}.$$

Solution. The domain of the function $f(x,y) = xy/\sqrt{x^2+y^2}$ consists of all points $(x,y) \neq (0,0)$. We have the partial derivatives

$$\begin{aligned}
\frac{\partial f}{\partial x} &= \frac{\frac{\partial(xy)}{\partial x}\sqrt{x^2+y^2} - xy \cdot \frac{\partial}{\partial x}\sqrt{x^2+y^2}}{x^2+y^2} \\
&= \frac{y\sqrt{x^2+y^2} - x^2 y(x^2+y^2)^{-1/2}}{x^2+y^2} \\
&= \frac{y^3}{(x^2+y^2)^{3/2}} \\
\frac{\partial f}{\partial y} &= \frac{x^3}{(x^2+y^2)^{3/2}}
\end{aligned}$$

Observe that these partial derivatives are all continuous in the domain of f. Therefore f is C^1 and the function is differentiable by Theorem 9 on page 137.

5. (10 Points) **Section 2.3, Exercise 8(c).** *Compute the matrix of partial derivatives of the function $f(x,y) = (x+y, x-y, xy)$.*

Solution. Taking the partial derivatives of each of the three components of f and arranging them as the three rows of the derivative matrix, we get

$$Df(x,y) = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ y & x \end{bmatrix}.$$

6. (10 Points) **Section 2.3 Exercise 10.** *Why should the graphs of* $f(x,y) = x^2 + y^2$, *and* $g(x,y) = -x^2 - y^2 + xy^3$ *be called "tangent" at* $(0,0)$*?*

Solution. At $(0,0)$,

$$\frac{\partial f}{\partial x}(0,0) = 0 = \frac{\partial g}{\partial x}(0,0)$$

and

$$\frac{\partial f}{\partial y}(0,0) = 0 = \frac{\partial g}{\partial y}(0,0).$$

Therefore the graphs of both f and g have the same tangent plane at $(0,0,0)$, namely the plane $z = 0$; *i.e.,* the xy plane.

7. (15 Points) **Section 2.4, Exercise 18.** *Suppose that a particle following the path*

$$\mathbf{c}(t) = (e^t, e^{-t}, \cos(t))$$

flies off on a tangent at $t_0 = 1$. *Compute the position of the particle at time* $t_1 = 2$.

Solution. The velocity vector is $(e^t, -e^{-t}, -\sin t)$, which at $t_0 = 1$ is the vector $(e, -e^{-1}, -\sin 1)$. The particle is at $(e, e^{-1}, \cos 1)$ at $t_0 = 1$. Hence the tangent line placing the particle at its "take off" point at $t = 1$ is

$$\ell(t) = (e, e^{-1}, \cos 1) + (t-1)(e, -e^{-1}, -\sin 1).$$

At $t = 2$, the position of the particle is on the line and is at

$$\ell(2) = (e, e^{-1}, \cos 1) + (e, -e^{-1}, -\sin 1) = (2e, 0, \cos 1 - \sin 1).$$