Mathematics 1c: Homework Set 8

Due: Tuesday, June 1 at 10am.

1. (10 Points) **Section 8.1, Exercises 3c and 3d**. *Verify Green's theorem for the disk D with center $(0,0)$ and radius R and $P(x,y) = xy = Q(x,y)$ and the same disk for $P = 2y, Q = x$. .*

2. (10 Points) **Section 8.2, Exercise 3.** *Verify Stokes' theorem for $z = \sqrt{1 - x^2 - y^2}$, the upper hemisphere, with $z \geq 0$, and the radial vector field $\mathbf{F}(x,y,z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$.*

3. (10 Points) **Section 8.2, Exercise 16.** *For a surface S and a fixed vector $\mathbf{v}$, prove that*
$$2 \iint_S \mathbf{v} \cdot \mathbf{n}\, dS = \int_{\partial S} (\mathbf{v} \times \mathbf{r}) \cdot d\mathbf{S},$$
where $\mathbf{r}(x,y,z) = (x,y,z)$.

4. (15 Points) **Section 8.2, Exercise 23.** *Let $\mathbf{F} = x^2\mathbf{i} + (2xy + x)\mathbf{j} + z\mathbf{k}$. Let C be the circle $x^2 + y^2 = 1$ in the plane $z = 0$ oriented counterclockwise and S the disk $x^2 + y^2 \leq 1$ oriented with the normal vector $\mathbf{k}$. Determine:*
   - (a) *The integral of $\mathbf{F}$ over S.*
   - (b) *The circulation of $\mathbf{F}$ around C.*
   - (c) *Find the integral of $\nabla \times \mathbf{F}$ over S. Verify Stokes' theorem directly in this case.*

5. (15 Points) **Section 8.3, Exercise 14.** *Determine which of the following vector fields $\mathbf{F}$ in the plane is the gradient of a scalar function f. If such an f exists, find it.*
   - (a) $\mathbf{F}(x,y) = (\cos xy - xy \sin xy)\mathbf{i} - (x^2 \sin xy)\mathbf{j}$
   - (b) $\mathbf{F}(x,y) = (x\sqrt{x^2y^2 + 1})\mathbf{i} + (y\sqrt{x^2y^2 + 1})\mathbf{j}$
   - (c) $\mathbf{F}(x,y) = (2x \cos y + \cos y)\mathbf{i} - (x^2 \sin y + x \sin y)\mathbf{j}$.

6. (10 Points) **Section 8.4, Exercise 2.** *Let $\mathbf{F} = x^3\mathbf{i} + y^3\mathbf{j} + z^3\mathbf{k}$. Evaluate the surface integral of $\mathbf{F}$ over the unit sphere.*

7. (10 Points) **Section 8.4, Exercise 14.** *Fix k vectors $\mathbf{v}_1, \ldots, \mathbf{v}_k$ in space and numbers ("charges") $q_1, \ldots, q_k$. Define*
$$\phi(x,y,z) = \sum_{i=1}^{k} \frac{q_i}{4\pi \|\mathbf{r} - \mathbf{v}_i\|},$$
where $\mathbf{r} = (x,y,z)$. Show that for a closed surface S and $\mathbf{e} = -\nabla\phi$,
$$\iint_S \mathbf{e} \cdot d\mathbf{S} = Q,$$
where $Q = q_1 + \cdots + q_k$ is the total charge inside S. Assume that none of the charges are on S.