1
The Geometry of Euclidean Space

1.1 Vectors in Two and Three-Dimensional Space

Key Points in this Section.

1. Addition and scalar multiplication for three-tuples are defined by

$$(a_1, a_2, a_3) + (b_1, b_2, b_3) = (a_1 + b_1, a_2 + b_2, a_3 + b_2)$$

and

$$\alpha(a_1, a_2, a_3) = (\alpha a_1, \alpha a_2, \alpha a_3).$$

There are similar definitions for pairs of real numbers (just leave off the third component).

2. A ***vector*** (in the plane or space) is a directed line segment with a specified tail (with the default being the origin) and an arrow at its head.

3. Vectors are added by the parallelogram law and scalar multiplication by α stretches the vector by this amount (in the opposite direction if α is negative).

4. If a vector has its tail at the origin, the coordinates of its tip are its ***components***.

5. Addition and scalar multiplication of vectors (geometric) corresponds to the same operations on the components (algebraic).

6. **Standard Bases**: Unit vectors **i**, **j**, **k** along the x, y, and z-axes.

7. A vector **a** (a_1, a_2, a_3) is written

$$\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}.$$

8. The ***vector joining two points*** $\mathrm{P} = (x, y, z)$ and $\mathrm{P}' = (x', y', z')$ is the vector $\overrightarrow{\mathrm{PP}'}$, represented as an arrow from P to P′, and has components

$$\overrightarrow{\mathrm{PP}'} = (x - x', y - y', z - z').$$

9. The equation of the line through the point **a** (regarded as a vector from the origin) in the direction of the vector **v** (regarded as a vector based at the point **a**) is

$$\ell(t) = \mathbf{a} + t\mathbf{v},$$

where t ranges over all real numbers.

10. The equations of the straight line through the points $P_1 = (x_1, y_1, z_1)$ and $P_2 = (x_2, y_2, z_2)$ are

$$x = x_1 + t(x_1 - x_2)$$
$$y = y_1 + t(y_1 - y_2)$$
$$z = z_1 + t(z_1 - z_2)$$

11. The plane through the origin containing the vectors $\mathbf{v}$ and $\mathbf{w}$ consists of all points of the form

$$s\mathbf{v} + t\mathbf{w}$$

where s and t range over all real numbers.

1.2 The Inner Product, Length, and Distance

Key Points in Section 1.2.

1. The ***inner product*** of the vectors $\mathbf{a} = (a_1, a_2, a_3)$ and $\mathbf{b} = (b_1, b_2, b_3)$ is defined as
$$\mathbf{a} \cdot \mathbf{b} = (a_1 b_1 + a_2 b_2 + a_3 b_3);$$
this inner product is sometimes denoted $\langle \mathbf{a}, \mathbf{b} \rangle$.

2. The ***length*** or ***norm*** of $\mathbf{a} = (a_1, a_2, a_3)$ is
$$\|\mathbf{a}\| = \sqrt{\mathbf{a} \cdot \mathbf{a}} = \sqrt{a_1^2 + a_2^2 + a_3^2}.$$

3. To ***normalize*** a nonzero vector $\mathbf{a}$, form the *unit* vector
$$\frac{\mathbf{a}}{\|\mathbf{a}\|}.$$

4. The ***distance*** between two points P and Q is $\|\overrightarrow{\mathrm{PQ}}\|$.

5. The angle θ between two vectors $\mathbf{a}$ and $\mathbf{b}$ satisfies
$$\mathbf{a} \cdot \mathbf{b} = \|\mathbf{a}\| \|\mathbf{b}\| \cos\theta.$$

6. The ***Cauchy-Schwarz Inequality:***
$$|\mathbf{a} \cdot \mathbf{b}| \leq \|\mathbf{a}\| \|\mathbf{b}\|.$$

7. The ***orthogonal projection*** of the vector $\mathbf{v}$ on the nonzero vector $\mathbf{a}$ is
$$\mathbf{p} = \frac{\mathbf{a} \cdot \mathbf{v}}{\|\mathbf{a}\|^2} \mathbf{a}.$$
Note that this is unchanged is $\mathbf{a}$ is multiplied by any nonzero scalar.

8. ***Triangle Inequality***:
$$\|\mathbf{a} + \mathbf{b}\| \leq \|\mathbf{a}\| + \|\mathbf{b}\|.$$

9. If an object has a constant velocity vector $\mathbf{v}$, then after t units of time, the object is moved by the ***displacement vector*** $\mathbf{d} = t\mathbf{v}$.

1.3 Matrices, Determinants and the Cross Product

Key Points in this Section.

1. ***Matrices*** are arrays of numbers, such as the 2×2 matrix
$$\begin{bmatrix} 1 & 3 \\ -1 & 4 \end{bmatrix}$$
and the general 3×3 matrix
$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

2. The ***determinant*** of a 2×2 matrix is
$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{21}a_{12}.$$

3. The ***determinant*** of a 3×3 matrix is
$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

4. Determinants may be expanded along any column or any row using the following checkerboard pattern
$$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

5. Any multiple of one row can be added to another row with out changing the determinant. Same for columns, but you cannot mix rows and columns.

6. The ***cross product*** of the vectors $\mathbf{a}$ and $\mathbf{b}$ is the vector
$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

7. The length of $\mathbf{a} \times \mathbf{b}$ is
$$\|\mathbf{a} \times \mathbf{b}\| = \|\mathbf{a}\|\|\mathbf{b}\| \sin\theta,$$
where θ is the angle (with $0 \le \theta \le \pi$) between the vectors $\mathbf{a}$ and $\mathbf{b}$, and equals the area of the parallelogram spanned by these vectors.

8. The ***triple product***

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

   is the volume of the parallelogram spanned by the three vectors **a**, **b**, and **c**.

9. The equation of the plane through the point (x_0, y_0, z_0) and normal to the vector $\mathbf{n} = A\mathbf{i} + B\mathbf{j} + C\mathbf{k}$ is

$$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0.$$

   that is,

$$Ax + By + Cz + D = 0,$$

   where $D = -(Ax_0 + By_0 + Cz_0)$.

10. The distance from the *point* (x_1, y_1, z_1) to the *plane*

$$Ax + By + Cz + D = 0$$

   is

$$\text{Distance} = \frac{|Ax_1 + By_1 + Cz_1 + D}{\sqrt{A^2 + B^2 + C^2}}.$$

1.4 Cylindrical and Spherical Coordinates

Key Points in this Section.

1. The ***polar coordinates*** (r, θ) of a point (x, y) in the xy-plane are determined by
$$x = r\cos\theta \quad \text{and} \quad y = r\sin\theta.$$

2. The ***cylindrical coordinates*** (r, θ, z) of a point (x, y, z) in $\mathbb{R}^3$ are determined by
$$x = r\cos\theta, \quad y = r\sin\theta, \quad \text{and} \quad z = z.$$

3. The ***spherical coordinates*** (ρ, θ, ϕ) of a point (x, y, z) in $\mathbb{R}^3$ are determined by
$$x = \rho\sin\phi\cos\theta, \quad y = \rho\sin\phi\sin\theta, \quad \text{and} \quad z = \rho\cos\phi.$$

4. The equations of geometric objects can sometimes be easiest to describe using one of these coordinate systems. For example, a cylinder is described by $r =$ constant and a sphere by $\rho =$ constant.

1.5 n-dimensional Euclidean Space

Key Points in this Section.

1. Euclidean n-space, denoted $\mathbb{R}^n$, consists of n-tuples of real numbers: $\mathbf{x} = (x_1, x_2, \ldots, x_n)$.
2. Addition and scalar multiplication of n-tuples is defines as we did with 2- and 3-tuples:

$$(x_1, x_2, \ldots, x_n) + (y_1, y_2, \ldots, y_n) = (x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n)$$
$$\alpha(x_1, x_2, \ldots, x_n) = (\alpha x_1, \alpha x_2, \ldots, \alpha x_n)$$

3. The ***inner, or dot product*** is defined by

$$\mathbf{x} \cdot \mathbf{y} = x_1 y_1 + x_2 y_2 + \cdot + x_n y_n,$$

   and satisfies properties as with vectors in $\mathbb{R}^2$ and $\mathbb{R}^3$.
4. In particular, the ***Cauchy-Schwarz*** and ***triangle inequalities*** hold:

$$|\mathbf{x} \cdot \mathbf{y}| \leq \|\mathbf{x}\| \|\mathbf{y}\| \quad \text{and} \quad \|\mathbf{x} + \mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|.$$

5. An $n \times n$ ***matrix*** is a square array of numbers with n rows and n columns. For instance, a 4×4 matrix has the form

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}$$

6. The determinant of a 4×4 matrix may be expanded along any row or column with a pattern of alternating $+$'s and $-$'s, as in the three by three case. For example,

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & a_{23} & a_{24} \\ a_{32} & a_{33} & a_{34} \\ a_{42} & a_{43} & a_{44} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} & a_{24} \\ a_{31} & a_{33} & a_{34} \\ a_{41} & a_{43} & a_{44} \end{vmatrix}$$
$$+ a_{13} \begin{vmatrix} a_{21} & a_{22} & a_{24} \\ a_{31} & a_{32} & a_{34} \\ a_{41} & a_{42} & a_{44} \end{vmatrix} - a_{14} \begin{vmatrix} a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ a_{41} & a_{42} & a_{43} \end{vmatrix}$$

7. If A and B are two $n \times n$ matrices, their ***matrix product*** AB is another $n \times n$ matrix, whose ij th entry (sitting in the ith row and jth column) is the inner product of the ith row of A with the jth column of B.

8. In general, matrix multiplication is associative; that is, $(AB)C = A(BC)$, but it need not be commutative; that is, $AB \neq BC$ in general.

9. The linear mapping of $\mathbb{R}^n$ to $\mathbb{R}^n$ defined by the $n \times n$ matrix A is the map

$$\mathbf{x} \mapsto A\mathbf{x}$$

where $\mathbf{x}$ is regarded as a column vector.

10. If $\det A \neq 0$, then A has an inverse, denoted A^{-1}, which has the property $AA^{-1} = A^{-1}A = I$ where I is the identity matrix (one's down the diagonal and zero's elsewhere). The solution of a linear system $\mathbf{y} = A\mathbf{x}$ is given by $\mathbf{x} = A^{-1}\mathbf{y}$.

2

Differentiation

2.1 Functions, Graphs, and Level Surfaces

Key Points in this Section.

1. A ***mapping*** or ***function*** $f : A \subset \mathbb{R}^n \to \mathbb{R}^m$ sends each point $\mathbf{x} \in A$ (the domain of f) to a specific point $f(\mathbf{x}) \in \mathbb{R}^m$. If $m = 1$, we call f a ***real valued function***.

2. The ***graph*** of $f : U \subset \mathbb{R}^2 \to \mathbb{R}$ is the set of all points of the form (x, y, z) where $(x, y) \in U$ and $z = f(x, y)$. More generally, for $f : U \subset \mathbb{R}^n \to \mathbb{R}$ the graph is the subset of $\mathbb{R}^{n+1}$ consisting of points of the form $(x_1, \ldots, x_n, z)$, where $(x_1, \ldots, x_n) \in U$ (the domain of f) and $z = f(x_1, \ldots, x_n)$.

3. A ***level set*** of a real valued function $f : U \subset \mathbb{R}^n \to \mathbb{R}$ obtained by picking a constant c and forming the set of points $(x_1, \ldots, x_n)$ in U such that $f(x_1, \ldots, x_n) = c$. For $n = 3$ we speak of them as ***level surfaces*** and for $n = 2$, ***level curves***.

4. A ***section*** of a graph is obtained by intersecting the graph with a vertical plane. For instance, for $z = f(x, y)$, setting $y = 0$ produces the section $z = f(x, 0)$ which is the graph of one function of one variable.

5. Level sets and sections are useful tools in constructing and visualizing graphs.

2.2 Limits and Continuity

Key Points in this Section.

1. A set $U \subset \mathbb{R}^n$ is ***open*** when, for every point $\mathbf{x}_0 \in U$, there is an $r > 0$ such that $D_r(\mathbf{x}_0) \subset U$. Here, $D_r(\mathbf{x}_0)$ is the ***open disk***, consisting of all points $\mathbf{x} \in \mathbb{R}^n$ such that $\| \mathbf{x} - \mathbf{x}_0 \| < r$. Open disks themselves are open sets.

2. A ***neighborhood*** of a point $\mathbf{x} \in \mathbb{R}^n$ is an open set containing $\mathbf{x}$.

3. A ***boundary point*** of a set $A \subset \mathbb{R}^n$ is a point $\mathbf{x} \in \mathbb{R}^n$ such that every neighborhood of $\mathbf{x}$ contains a point in A and a point not in A.

4. **Limits**. Let $f : A \subset \mathbb{R}^n \to \mathbb{R}^m$ and $\mathbf{x}_0$ be in A or be a boundary point of A and let $\mathbf{b} \in \mathbb{R}^m$. When we write
$$\lim_{\mathbf{x} \to \mathbf{x}_0} f(\mathbf{x}) = \mathbf{b}$$
we mean that for any neighborhood N of $\mathbf{b}$, there is a neighborhood U of $\mathbf{x}_0$ such that if $\mathbf{x} \in A \cap U$, then $f(\mathbf{x}) \in N$.

5. Limits, if they exist, are unique. Also, the properties of limits from one-variable calculus (such as: the limit of a sum is the sum of the limits) also hold for functions of several variables.

6. **Continuity**. Let $f : A \subset \mathbb{R}^n \to \mathbb{R}^m$ and $\mathbf{x}_0 \in A$. We say f is ***continuous at*** $\mathbf{x}_0$ provided
$$\lim_{\mathbf{x} \to \mathbf{x}_0} f(\mathbf{x}) = f(\mathbf{x}_0).$$
If f is continuous at every point of A, we just say f is ***continuous***.

7. The sum of continuous functions is continuous. The same is true of products and quotients of real-valued functions (if the denominator is non-zero).

8. The composition of continuous functions is continuous. ***Compositions*** $f \circ g$ are defined by $(f \circ g)(\mathbf{x}) = f(g(\mathbf{x}))$.

9. The usual functions of one-variable calculus, such as polynomials, trigonometric, and exponential functions are continuous and these can be used to build up continuous functions of several variables. For instance, $f(x, y) = e^{xy}/(1 - x^2 - y^2)$ is continuous on $\mathbb{R}^2$ minus the unit circle.

10. If $f(x, y)$ has different limits as $(0, 0)$ is approached along two different rays (such as the x- and y-axes), then f is not continuous at $(0, 0)$.

2.3 Differentiation

Key Points in this Section.

1. Given $f : U \subset \mathbb{R}^3 \to \mathbb{R}$, where U is open, the ***partial derivative with respect to*** x is defined by

$$f_x(x,y,z) = \frac{\partial f}{\partial x}(x,y,z) = \lim_{h\to 0} \frac{f(x+h,y,z) - f(x,y,z)}{h}$$

   if it exists. The partial derivatives $\partial f/\partial y$ and $\partial f/\partial z$ are defined similarly and the extension to function of n variables is analogous.

2. The ***linear approximation*** to $f(x,y)$ at (x_0,y_0) is

$$\ell_{(x_0,y_0)}(x,y) = f(x_0,y_0) + \left[\frac{\partial f}{\partial x}(x_0,y_0)\right](x-x_0) + \left[\frac{\partial f}{\partial y}(x_0,y_0)\right](y-y_0)$$

3. The function $f(x,y)$ is ***differentiable*** at (x_0,y_0) if the partials exist at (x_0,y_0) *and* if

$$\lim_{(x,y)\to(x_0,y_0)} \frac{f(x,y) - \ell_{(x_0,y_0)}(x,y)}{\| (x,y) - (x_0,y_0) \|} = 0$$

4. If f is differentiable at (x_0,y_0), the ***tangent plane*** to the graph of f at (x_0,y_0,z_0), where $z_0 = f(x_0,y_0)$ is

$$z = \ell_{(x_0,y_0)}(x,y).$$

5. The definition of differentiability is motivated by the idea that the tangent plane should give a good approximation to the function.

6. If $f : U \subset \mathbb{R}^n \to \mathbb{R}^m$ has partial derivatives at $\mathbf{x}_0 \in U$, the ***derivative matrix*** is the $m \times n$ matrix $\mathbf{D}f(\mathbf{x}_0)$ given by

$$\mathbf{D}f(\mathbf{x}_0) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \vdots & & & \vdots \\ \frac{\partial f_m}{\partial x_1} & \frac{\partial f_m}{\partial x_2} & \cdots & \frac{\partial f_m}{\partial x_n} \end{bmatrix}$$

   where the partials are all evaluated at $\mathbf{x}_0$.

7. We say $f : U \subset \mathbb{R}^n \to \mathbb{R}^m$ is ***differentiable*** at $\mathbf{x}_0$ provided the partials exist and

$$\lim_{\mathbf{x}\to\mathbf{x}_0} \frac{\|f(\mathbf{x}) - f(\mathbf{x}_0) - \mathbf{D}f(\mathbf{x}_0)\cdot(\mathbf{x}-\mathbf{x_0})\|}{\|\mathbf{x}-\mathbf{x}_0\|} = 0.$$

8. For $f : U \subset \mathbb{R}^3 \to \mathbb{R}$, its ***gradient*** is

$$\nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{i} + \frac{\partial f}{\partial z}\mathbf{k}.$$

Similarly, for $f : U \subset \mathbb{R}^n \to \mathbb{R}$, ∇f is the vector with components

$$\nabla f = \left(\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}\right).$$

9. If f is differentiable at $\mathbf{x}_0$, then it is continuous at $\mathbf{x}_0$. If the partials exist and are continuous in a neighborhood of $\mathbf{x}_0$ (that is, f is C^1), then f is differentiable at $\mathbf{x}_0$.

2.4 Introduction to Paths

Key Points in this Section.

1. A ***path*** in $\mathbb{R}^3$ is a map $\mathbf{c}$ of an interval $[a, b]$ to $\mathbb{R}^3$. The ***endpoints*** of the path are the points $\mathbf{c}(a)$ and $\mathbf{c}(b)$. The associated geometric curve C is the set of image points $\mathbf{c}(t)$ as t ranges from a to b. We say $\mathbf{c}$ is a ***parametrization*** of C. Paths in the plane are similar (leave off the last component).

2. A particle on the rim of a rolling circle of radius 1 traces out a path called a ***cycloid***:

$$\mathbf{c}(t) = (t - \sin t, 1 - \cos t).$$

3. If a path $\mathbf{c}$ is differentiable, its ***velocity*** is defined to be

$$\mathbf{c}'(t) = \lim_{h \to 0} \frac{\mathbf{c}(t+h) - \mathbf{c}(t)}{h} = x'(t)\mathbf{i} + y'(t)\mathbf{j} + z'(t)\mathbf{k},$$

where $\mathbf{c}(t)$ has components $(x(t), y(t), z(t))$.

4. The vector $\mathbf{c}'(t_0)$ is tangent to the path at the point $\mathbf{c}(t_0)$. The ***tangent line*** at this point is

$$\ell(t) = \mathbf{c}(t_0) + (t - t_0)\mathbf{c}'(t_0).$$

2.5 Properties of the Derivative

Key Points in this Section.

1. The ***constant multiple rule***, the ***sum rule***, ***product rule*** and ***quotient rule*** are all analogous to their counterparts in single-variable calculus.

2. The ***chain rule*** states that

$$\mathbf{D}(f \circ g)(\mathbf{x}_0) = \mathbf{D}f(\mathbf{y}_0)\mathbf{D}g(\mathbf{x}_0)$$

where $g : U \subset \mathbb{R}^n \to \mathbb{R}^m$ and $f : V \subset \mathbb{R}^m \to \mathbb{R}^p$ are differentiable, with $g(U) \subset V$ so that the ***composition*** $f \circ g$ is defined and where $\mathbf{D}f(\mathbf{y}_0)\mathbf{D}g(\mathbf{x}_0)$ is the $p \times n$ matrix that is the product of the $p \times m$ matrix $\mathbf{D}f(\mathbf{y}_0)$ with the $m \times n$ matrix $\mathbf{D}g(\mathbf{x}_0)$.

3. Special cases of the chain rule are, firstly,

$$\frac{dh}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}$$

where $h(t) = f(x(t), y(t), z(t))$ and secondly,

$$\frac{\partial h}{\partial x} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial x} + \frac{\partial f}{\partial w}\frac{\partial w}{\partial x},$$

where $h(x, y, z) = f(u(x, y, z), v(x, y, z), w(x, y, z))$.

2.6 Gradients and Directional Derivatives

Key Points in this Section.

1. The***gradient*** of a differentiable function $f : U \subset \mathbb{R}^3 \to \mathbb{R}$ is

$$\nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}.$$

2. The ***directional derivative*** of f in the direction of a *unit* vector $\mathbf{v}$ at the point $\mathbf{x}$ is

$$\left.\frac{d}{dt}f(\mathbf{x}+t\mathbf{v})\right|_{t=0} = \nabla f(\mathbf{x})\cdot\mathbf{v}$$

3. The direction in which f is *increasing the fastest* at $\mathbf{x}$ is the direction parallel to $\nabla f(\mathbf{x})$. The direction of fastest *decrease* is parallel to $-\nabla f(\mathbf{x})$.

4. For $f : U \subset \mathbb{R}^3 \to \mathbb{R}$ a C^1 function, with $\nabla f(x_0, y_0, z_0) \neq \mathbf{0}$, the vector $\nabla f(x_0, y_0, z_0)$ is perpendicular to the level set $f(x, y, z) = f(x_0, y_0, z_0)$. Thus, the ***tangent plane*** to this level set is

$$\nabla f(x_0, y_0, z_0)\cdot(x - x_0, y - y_0, z - z_0) = 0.$$

5. The gravitational force field

$$\mathbf{F} = -\frac{GMm}{r^3}\mathbf{r} = -\frac{GMm}{r^2}\mathbf{n}$$

(the inverse square law), where $\mathbf{n} = \mathbf{r}/r$, $\mathbf{r} = x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$ and $r = \|\mathbf{r}\|$, is a gradient. Namely,

$$\mathbf{F} = -\nabla V,$$

where

$$V = -\frac{GMm}{r}.$$

3

Higher-Order Derivatives; Maxima and Minima

3.1 Iterated Partial Derivatives

Key Points in this Section.

1. **Equality of Mixed Partials.** If $f(x, y)$ is C^2 (has continuous 2nd partial derivatives), then

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}.$$

2. The idea of the proof is to apply the mean value theorem to the "difference of differences" written in the two ways

$$\begin{aligned} S(h, k) &= \{S(x + h, y + k) - S(x + h, y)\} - \{S(x, y + k) - S(x, y)\} \\ &= \{S(x + h, y + k) - S(x, y + k)\} - \{S(x + h, y) - S(x, y)\} \end{aligned}$$

3. Higher order partials are also symmetric; for example, for $f(x, y, z)$,

$$\frac{\partial^4 f}{\partial x \partial^2 z \partial y} = \frac{\partial^4 f}{\partial x \partial y \partial^2 z}$$

4. Many important equations describing nature involve partial derivatives, such as the ***heat equation*** for the temperature $T(x, y, z, t)$:

$$\frac{\partial T}{\partial t} = k \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} \right).$$

3.2 Taylor's Theorem

Key Points in this Section.

1. The one-variable ***Taylor Theorem*** states that if f is C^{k+1}, then

$$f(x_0+h) = f(x_0)+f'(x_0)h+\frac{f''(x_0)}{2}h^2+\cdots+\frac{f^{(k)}(x_0)}{k!}h^k+R_k(x_0,h),$$

where $R_k(x_0,h)/h^k \to 0$ as $h \to 0$

2. The idea of the proof is to start with the Fundamental Theorem of Calculus

$$f(x_0+h) = f(x_0) + \int_{x_0}^{x_0+h} f'(\tau)d\tau$$

(which gives Taylors' theorem for $k = 0$) and integrating by parts.

3. For $f : U \subset \mathbb{R}^n \to \mathbb{R}$ of class C^3, the second-order ***Taylor Theorem*** states that

$$f(\mathbf{x}_0+\mathbf{h}) = f(\mathbf{x}_0)+\sum_{i=1}^{n} h_i \frac{\partial f}{\partial x_i}(\mathbf{x}_0)+\frac{1}{2}\sum_{i,j} h_i h_j \frac{\partial^2 f}{\partial x_i \partial x_j}(\mathbf{x}_0)+R_2(\mathbf{x}_0,\mathbf{h})$$

where $R_2(\mathbf{x}_0,\mathbf{h})/\|\mathbf{h}\|^2 \to 0$ as $\mathbf{h} \to \mathbf{0}$. Higher order versions are similar.

4. The idea of the proof is to apply the single-variable Taylor theorem to the function $g(t) = f(\mathbf{x}_0+t\mathbf{h})$, expanded about $t_0 = 0$ with $h = 1$.

3.3 Extrema of Real Valued Functions

Key Points in this Section.

1. **Definitions.** A ***local minimum point*** of $f : U \subset \mathbb{R}^n \to \mathbb{R}$ is a point $\mathbf{x}_0 \in U$ such that $f(\mathbf{x}_0) \leq f(\mathbf{x})$ for all $\mathbf{x}$ in some neighborhood of $\mathbf{x}_0$; we say $f(\mathbf{x}_0)$ is the corresponding ***local minimum value***. If, similarly, $f(\mathbf{x}_0) \geq f(\mathbf{x})$, then $\mathbf{x}_0$ is a ***local maximum point*** (and $f(\mathbf{x}_0)$ is the ***local maximum value***). If $\mathbf{x}_0$ is either of these, it is a ***local extremum***.

2. **First Derivative Test.** If $U \subset \mathbb{R}^n$ is open, $f : U \subset \mathbb{R}^n \to \mathbb{R}$ is differentiable and $\mathbf{x}_0$ is a local extremum, then $\mathbf{x}_0$ is a ***critical point***; that is, all the partials of f vanish at $\mathbf{x}_0$:
$$\frac{\partial f}{\partial x_1}(\mathbf{x}_0) = 0, \cdots, \frac{\partial f}{\partial x_0}(\mathbf{x}_0) = 0.$$
The idea of the proof is to apply the one-variable first derivative test to f restricted to lines through $\mathbf{x}_0$.

3. If $f : U \subset \mathbb{R}^n \to \mathbb{R}$ is C^2, the ***Hessian*** of f at $\mathbf{x}_0$ is the quadratic function of $\mathbf{h}$ given by
$$Hf(\mathbf{x}_0)(\mathbf{h}) = \frac{1}{2}[h_1, \ldots, h_n] \begin{bmatrix} \frac{\partial^2 f}{\partial x_1 \partial x_1} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\ \vdots & & \vdots \\ \frac{\partial^2 f}{\partial x_n \partial x_1} & \cdots & \frac{\partial^2 f}{\partial x_n \partial x_n} \end{bmatrix} \begin{bmatrix} h_1 \\ \vdots \\ h_n \end{bmatrix}$$
which also equals the second term in the Taylor expansion of f about $\mathbf{x}_0$.

4. **Second Derivative Test—n Variables.** If $f : U \subset \mathbb{R}^n \to \mathbb{R}$ is C^3 (and again U is open), $\mathbf{x}_0$ is a critical point, and if $Hf(\mathbf{x}_0)(\mathbf{h}) > 0$ for all $\mathbf{h} \neq \mathbf{0}$ (that is, $Hf(\mathbf{x}_0)$ is ***positive definite***), then $\mathbf{x}_0$ is a local minimum. Likewise, if $Hf(\mathbf{x}_0)(\mathbf{h}) < 0$ for all $\mathbf{h} \neq \mathbf{0}$, (that is, $Hf(\mathbf{x}_0)$ is ***negative definite***), then $\mathbf{x}_0$ is a local maximum.

5. The idea of the proof of the second derivative test is to apply the second order Taylor theorem and show that the remainder term can be ignored.

6. **Second Derivative Test—Two Variables.** Let $f : U \subset \mathbb{R}^2 \to \mathbb{R}$ (again with U open) be of class C^3. A point $(x_0, y_0) \in U$ is a local minimum if the following conditions are satisfied:

   (i) $\frac{\partial f}{\partial x}(x_0, y_0) = \frac{\partial f}{\partial y}(x_0, y_0) = 0$ (that is, (x_0, y_0) is a critical point)

(ii) $\dfrac{\partial^2 f}{\partial x^2}(x_0, y_0) > 0$

(iii) $D = \begin{vmatrix} \dfrac{\partial^2 f}{\partial x^2}(x_0, y_0) & \dfrac{\partial^2 f}{\partial x \partial y}(x_0, y_0) \\ \dfrac{\partial^2 f}{\partial x \partial y}(x_0, y_0) & \dfrac{\partial^2 f}{\partial y^2}(x_0, y_0) \end{vmatrix} > 0.$

If (i) and (iii) hold, but $\partial^2 f/\partial x^2$ at (x_0, y_0) is negative, then (x_0, y_0) is a local maximum. If the ***discriminant*** D is negative, then (x_0, y_0) is a ***saddle point*** (that is, (x_0, y_0) is neither a local maximum nor a local minimum).

7. **Global Extrema.** Let $f : A \subset \mathbb{R}^n \to \mathbb{R}$, where A need not be open. A point $\mathbf{x}_0 \in A$ is an ***absolute*** or ***global minimum*** of f if $f(\mathbf{x}_0) \leq f(\mathbf{x})$ for all $\mathbf{x} \in A$. Similarly, $\mathbf{x}_0$ is an ***absolute*** or ***global maximum*** if $f(\mathbf{x}_0) \geq f(\mathbf{x})$ for all $\mathbf{x} \in A$.

8. If $D \subset \mathbb{R}^n$ is ***closed*** (that is, all boundary points of D lie in D) and ***bounded*** (that is, D is a subset of some, perhaps large ball), and if $f : D \subset \mathbb{R}^n \to \mathbb{R}$ is continuous, then f has (at least one) absolute maximum point $\mathbf{x}_0 \in D$ and (at least one) absolute minimum point $\mathbf{x}_1 \in D$.

9. **Strategy for Global Extrema.** To find absolute extrema on a closed and bounded region $D \subset \mathbb{R}^n$ that is an open set U together with its boundary points ∂U,

   (i) find the critical points in U

   (ii) find the maximum points of f on ∂U

   (iii) compute the values of f at all the points in (i) and (ii)

   (iv) the largest such value gives the maximum and the smallest the minimum.

   If $n = 2$ and ∂U is a closed curve, step (ii) can be done by parametrizing this curve and using the methods of one-variable calculus. Alternatively, for $n = 2$ or 3, one can use the Lagrange multipliers given in the next section.

3.4 Constrained Extrema and Lagrange Multipliers

Key Points in this Section.

1. **Lagrange Multiplier Equations.** Let $f : U \subset \mathbb{R}^n \to \mathbb{R}$ and $g : U \subset \mathbb{R}^n \to \mathbb{R}$ be C^1. Consider the problem of extremizing f on a level set of g, say $g(\mathbf{x}) = c$. If $\mathbf{x}_0$ is such an extremum and if $\nabla g(\mathbf{x}_0) \neq \mathbf{0}$ then the ***Lagrange multiplier equations*** hold:

$$\nabla f(\mathbf{x}_0) = \lambda \nabla g(\mathbf{x}_0)$$

   for a constant λ, the ***multiplier***.

2. The idea of the proof is to use the fact that f has a critical point along any curve in the level set through $\mathbf{x}_0$, which shows, via the chain rule, that $\nabla f(\mathbf{x}_0)$ is perpendicular to that level set; but $\nabla g(\mathbf{x}_0)$ is also perpendicular, so these two vectors are parallel.

3. The Lagrange multiplier method produces *candidates* for extrema; one must make sure there is an extremum and then f can be evaluated at the candidates to choose the maximum or minimum as desired.

4. If there are k constraints

$$g_1 = c_1, \cdots, g_k = c_k,$$

   for C^1 functions $g(x_1, \ldots, x_n), \ldots g_k(x_1, \ldots, x_n)$ and constants $c_1, \ldots, c_k$, then the Lagrange multiplier equations become

$$\nabla f(\mathbf{x}_0) = \lambda_1 \nabla g(\mathbf{x}_0) + \cdots + \lambda_k \nabla g(\mathbf{x}_0).$$

5. The Lagrange multiplier method is an effective tool for finding the extrema of $f|\partial U$ in the strategy for finding global extrema described in the last section.

6. **Second Derivative Test with Constraints.** Let $\mathbf{x}_0$ satisfy the conditions of the Lagrange multiplier theorem (in point 1.) Let $h = f - \lambda g$ and $|\bar{H}|$ be the ***bordered Hessian determinant***:

$$|\bar{H}| = \begin{vmatrix} 0 & -\dfrac{\partial g}{\partial x} & -\dfrac{\partial g}{\partial y} \\ -\dfrac{\partial g}{\partial x} & \dfrac{\partial^2 h}{\partial x^2} & \dfrac{\partial^2 h}{\partial x \partial y} \\ -\dfrac{\partial g}{\partial y} & \dfrac{\partial^2 h}{\partial x \partial y} & \dfrac{\partial^2 h}{\partial y^2} \end{vmatrix}$$

evaluated at $\mathbf{x}_0$.

If $|\bar{H}| > 0$, then $\mathbf{x}_0$ is a local maximum of f subject to the constraint $g = c$ and if $|\bar{H}| < 0$, it is a local minimum.

3.5 The Implicit Function Theorem

Key Points in this Section.

1. **One-Variable Version.** If $f : (a, b) \to \mathbb{R}$ is C^1 and if $f'(x_0) \neq 0$, then locally near x_0, f has a C^1 inverse function $x = f^{-1}(y)$. If $f'(x) > 0$ on all of (a, b) and is continuous on $[a, b]$, then f has an inverse defined on $[f(a), f(b)]$. This result is used in one-variable calculus to define, for example, the log function as the inverse of $f(x) = e^x$ and $\sin^{-1}$ as the inverse of $f(x) = \sin x$.

2. **Special n-variable Version.** If $F : \mathbb{R}^{n+1} \to \mathbb{R}$ is C^1 and at a point $(\mathbf{x}_0, z) \in \mathbb{R}^{n+1}$, $F(\mathbf{x}_0, z) = 0$ and $\frac{\partial F}{\partial z}(\mathbf{x}_0, z_0) \neq 0$, then locally near $(\mathbf{x}_0, z_0)$ there is a unique solution $z = g(\mathbf{x})$ of the equation $F(\mathbf{x}, z) = 0$. We say that $F(\mathbf{x}, z) = 0$ ***implicitly defines*** z as a function of $\mathbf{x} = (x_1, \ldots, x_n)$.

3. The partial derivatives are computed by ***implicit differentiation***:
$$\frac{\partial F}{\partial x_i} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial x_i} = 0,$$
so
$$\frac{\partial z}{\partial x_i} = -\frac{\partial F/\partial x_i}{\partial F/\partial z}$$

4. The special implicit function theorem guarantees that if $\nabla g(\mathbf{x}_0) \neq \mathbf{0}$, then the level set $g = c$ is a smooth surface near $\mathbf{x}_0$, a fact needed in the proof of the Lagrange multiplier theorem.

5. The general implicit function theorem deals with solving m equations
$$\begin{array}{rcl} F_1(x_1, \ldots, x_n, z_1, \ldots, z_m) & = & 0 \\ \vdots & & \vdots \\ F_m(x_1, \ldots, x_n, z_1, \ldots, z_m) & = & 0 \end{array}$$
for m unknowns $\mathbf{z} = (z_1, \ldots, z_m)$. If
$$\begin{vmatrix} \dfrac{\partial F_1}{\partial z_1} & \cdots & \dfrac{\partial F_1}{\partial z_m} \\ \vdots & & \vdots \\ \dfrac{\partial F_m}{\partial z_1} & \cdots & \dfrac{\partial F_m}{\partial z_m} \end{vmatrix} \neq 0$$
at $(\mathbf{x}_0, \mathbf{z}_0)$, then these equations define $(z_1, \ldots, z_m)$ as functions of $(x_1, \ldots, x_n)$. The partial derivatives $\partial z_i/\partial x_j$ may again be computed by using implicit differentiation.

6. The ***Inverse Function Theorem***, which is a special case of the general implicit function theorem, states that a system

$$\begin{aligned} f_1(x_1,\ldots,x_n) &= y_1 \\ \vdots \quad & \quad \vdots \\ f_n(x_1,\ldots,x_n) &= y_n \end{aligned}$$

where $f = (f_1, \ldots, f_n)$ is a C^1 mapping, can be solved for the x_i's as functions of $(y_1, \ldots, y_n)$ near a given point $\mathbf{x}_0$, $\mathbf{y}_0 = f(\mathbf{x}_0)$ provided the ***Jacobian determinant***

$$\left.\frac{\partial(f_1,\ldots,f_n)}{\partial(x_1,\ldots,x_n)}\right|_{\mathbf{x}=\mathbf{x}_0} = J(f)(\mathbf{x}_0) = \begin{vmatrix} \dfrac{\partial f_1}{\partial x_1} & \cdots & \dfrac{\partial f_1}{\partial x_n} \\ \vdots & & \vdots \\ \dfrac{\partial f_n}{\partial x_1} & \cdots & \dfrac{\partial f_n}{\partial x_n} \end{vmatrix}$$

(where partials are evaluated at $\mathbf{x}_0$) is non-zero. Again the partial derivatives $\partial x_i/\partial y_j$ can be determined by implicit differentiation.

4

Vector Valued Functions

4.1 Acceleration and Newton's Second Law

Key Points in this Section.

1. Two of the more important rules for differentiating paths are

   (a) ***Dot Product Rule:***

   $$\frac{d}{dt}[\mathbf{b}(t) \cdot \mathbf{c}(t)] = \mathbf{b}'(t) \cdot \mathbf{c}(t) + \mathbf{b}(t) \cdot \mathbf{c}'(t)$$

   (b) ***Cross Product Rule:***

   $$\frac{d}{dt}[\mathbf{b}(t) \times \mathbf{c}(t)] = \mathbf{b}'(t) \times \mathbf{c}(t) + \mathbf{b}(t) \times \mathbf{c}'(t).$$

2. The ***acceleration*** of a path is $\mathbf{a}(t) = \mathbf{c}''(t)$.

3. A C^1 path is ***regular*** at t_0 when $\mathbf{c}'(t_0) \neq \mathbf{0}$. Non-intersecting regular paths have images that look smooth.

4. If $\mathbf{F}$ is a force field acting on a particle of mass m, then the particle follows a path satisfying ***Newton's Second Law:*** $\mathbf{F}(\mathbf{c}(t)) = m\mathbf{a}(t)$, or $\mathbf{F} = m\mathbf{a}$ for short.

5. ***Newton's Law of Gravity:***

   $$\mathbf{F}(\mathbf{r}) = -\frac{GmM}{r^3}\mathbf{r}.$$

6. ***Kepler's Law.*** For a particle moving in a circular orbit under Newton's law of gravity, the square of the period is proportional to the cube of the radius.

4.2 Arc Length

Key Points in this Section.

1. The length of a C^1 path $\mathbf{c}(t)$, $a \le t \le b$, is

$$L(\mathbf{c}) = \int_a^b \|\mathbf{c}'(t)\| dt.$$

2. If the path is only piecewise C^1, then the length is the sum of the lengths of the pieces.

3. If $\mathbf{c}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$, the ***vector arc length differential***, also called the ***infinitesimal displacement***, is

$$d\mathbf{s} = dx\mathbf{i} + dy\mathbf{j} + dz\mathbf{k} = \left(\frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j} + \frac{dz}{dt}\mathbf{k}\right) dt = \mathbf{c}'(t)dt$$

and its length, called the (scalar) arc length differential, is

$$ds = \sqrt{dx^2 + dy^2 + dz^2} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt = \|\mathbf{c}'(t)\| dt.$$

4. The ***arc length function*** of a path is

$$s(t) = \int_a^t \|\mathbf{c}'(\tau)\| d\tau.$$

5. The formula for arc length may be justified by either Riemann sums, thinking of a path as being made up of many little, nearly straight segments, or by thinking of a moving particle and using

$$\text{distance} = \int \text{speed}.$$

4.3 Vector Fields

Key Points in this Section.

1. A ***vector field in*** $\mathbb{R}^3$ assigns a vector to each point in space. Similarly, a vector field in $\mathbb{R}^2$ assigns a vector to each point in the plane.

2. A vector field is a ***gradient vector field*** if it equals the gradient of some function.

3. The gravitational vector field
$$\mathbf{F} = -\frac{mMG}{r^3}\mathbf{r}$$
is a gradient. In fact, $\mathbf{F} = -\nabla V$, where
$$V = -\frac{mMG}{r}.$$

4. A particle moving according to Newton's second law $\mathbf{F} = m\mathbf{a}$ in a gradient field, say$\mathbf{F} = -\nabla V$ conserves energy; that is,
$$E = \frac{1}{2}m\|\mathbf{r}'(t)\|^2 + V(\mathbf{r}(t))$$
is constant in time.

5. Not all vector fields are gradient fields.

6. A ***flow line*** of a vector field $\mathbf{F}$ is a path $\mathbf{c}(t)$ satisfying
$$\mathbf{c}'(t) = \mathbf{F}(\mathbf{c}(t)).$$

4.4 Divergence and Curl

Key Points in this Section.

1. The ***del operator*** is

$$\nabla = \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{z}\frac{\partial}{\partial k}.$$

2. The gradient of a function may be thought of as ∇ ***operating*** on that function.

3. The ***divergence*** of a vector field $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ is

$$\text{div } \mathbf{F} = \nabla \cdot \mathbf{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

(omit R for planar vector fields). The divergence may be thought of as the dot product of ∇ and F.

4. **Expansion and the Divergence.** The divergence measures the rate at which $\mathbf{F}$ expands (if $\nabla \cdot \mathbf{F} > 0$) or contracts (if $\nabla \cdot \mathbf{F} < 0$) volumes, or areas in the case of planar vector fields.

5. The ***curl*** of $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ is

$$\begin{aligned}\text{curl}\,\mathbf{F} = \nabla \times \mathbf{F} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} \\ &= \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\mathbf{i} - \left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right)\mathbf{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathbf{k}\end{aligned}$$

and may be thought of as the cross product of ∇ and $\mathbf{F}$.

If $\mathbf{F} = P\mathbf{i} + Q\mathbf{j}$ is two dimensional, only the last term is present and it gives the scalar function,

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y},$$

which is called the ***scalar curl***.

6. **Rotations and the Curl.** The vector field describing rigid rotational motion of a body about a fixed axis has curl equal in magnitude to twice the angular velocity and points along the axis of rotation (using the right hand rule).

7. **Vector Identities.** There are many basic identities involving div, grad and curl, such as

   (a) $\nabla \times \nabla f = 0$ (c.f. $\mathbf{v} \times \mathbf{v} = 0$)
   (b) $\nabla \cdot (\nabla \times \mathbf{F}) = 0$) (c.f. $\mathbf{v} \cdot (\mathbf{v} \times \mathbf{w}) = 0$)
   (c) $\text{div}\ (f\mathbf{F}) = f\ \text{div}\ \mathbf{F} + (\nabla f) \cdot \mathbf{F}$
   (d) $\text{curl}\ (f\mathbf{F}) = f\ \text{curl}\ \mathbf{F} + \nabla f \times \mathbf{F}$
   (e) $\nabla(r^n) = nr^{n-2}\mathbf{r}$
   (f) $\nabla^2(1/r) = 0$ (for $r \neq 0$).

   Here,
   $$\nabla^2 f = \nabla \cdot \nabla f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$
   is the ***Laplacian*** of f.

5
Double and Triple Integrals

5.1 Introduction

Key Points in this Section.

1. If $R = [a, b] \times [c, d]$ is a rectangle in the plane and $f : R \to \mathbb{R}$ is a non-negative function, then

$$\iint_R f(x, y)dA = \iint_R f(x, y)dx\,dy$$

is the volume of the region under the graph of f and above the rectangle R. This is an 'informal' definition in that it assumes one knows about volumes. A 'rigorous' definition is given in §5.2.

2. **Cavalieri's Principle.** Suppose that one is given the following data (see Figure 5.1.1):
   (a) A solid S,
   (b) An x-axis in space,
   (c) Planes P_x perpendicular to the x-axis cutting S in regions R_x with areas $A(x)$ for x ranging between $x = a$ and $x = b$.
   (d) Then the volume of S is

$$V = \int_a^b A(x)dx.$$

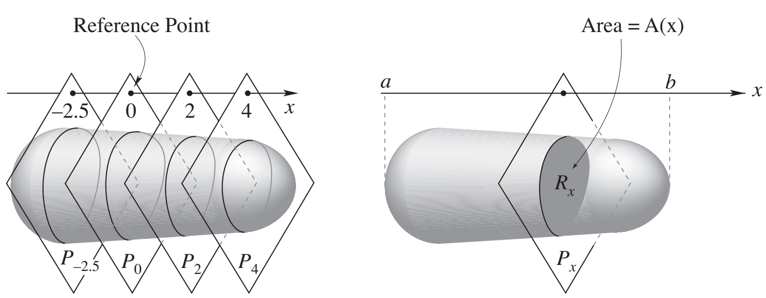

FIGURE 5.1.1. The data used in Cavalieri's principle: Volume = $\int_a^b A(x)dx$

3. **Iterated Integrals.** Using slices along the x and y-axes, together with the interpretation of the one-variable integral as an area, Cavalieri's principle leads to the double integral written as iterated integrals:

$$\iint_R f(x, y)dA = \int_a^b \left[\int_c^d f(x, y)dy\right] dx = \int_c^d \left[\int_a^b f(x, y)dx\right] dy.$$

5.2 The Double Integral over a Rectangle

Key Points in this Section.

1. A ***Riemann sum*** for a function f defined on a rectangle $R = [a, b] \times [c, d]$ has the form

$$S_n = \sum_{j,k=0}^{n-1} f(\mathbf{c}_{jk}) \Delta x \Delta y,$$

where R is divided into n^2 equal sub-rectangles obtained by dividing $[a, b]$ and $[c, d]$ into n equal parts, and where $\mathbf{c}_{jk}$ is a point chosen in the jk^{th} sub-rectangle, $0 \le j,k \le n-1$, of width Δx and height Δy.

2. **Definition of the Integral.** If $\lim_{n\to\infty} S_n = S$ exists and is independent of the choice of $\mathbf{c}_{jk}$, f is called ***integrable*** over R and the limit is denoted

$$\iint_R f(x, y)dA, \text{ or } \iint_R f(x, y)dxdy, \text{ or } \iint_R f dA.$$

3. Continuous functions as well as functions that are bounded and that are continuous except along a finite union of graphs of functions (of either x or y) are integrable.

4. The integral is linear in its argument and is additive with respect to the region. It also satisfies

$$\left| \iint_R f dA \right| \le \iint_R |f|\, dA$$

5. For $f \ge 0$, the rigorous definition in point 2 justifies interpreting $\iint_R f dA$ as the volume of the region under the graph of f and over R, as well as giving a theoretical foundation for the definition of the volume of a region.

6. ***Fubini's Theorem*** states that for f continuous, the reduction to iterated integrals holds:

$$\iint_R f(x, y)dA = \int_a^b \left[\int_c^d f(x, y)dy \right] dx = \int_c^d \left[\int_a^b f(x, y)dx \right] dy.$$

A similar result holds for bounded functions with discontinuities along a finite number of graphs provided the iterated integrals exist.

5.3 The Double Integral Over More General Regions

Key Points in this Section.

1. **Elementary Regions.** A y-***simple region*** is one that lies between two continuous curves $y = \phi_1(x)$ and $y = \phi_2(x)$, where $\phi_1(x) \leq \phi_2(x)$ and $a \leq x \leq b$. Similarly, x-***simple regions*** are those lying between two continuous curves $x = \psi_1(y)$ and $x = \psi_2(y)$, where $\psi_1(y) \leq \psi_2(y)$ and $c \leq y \leq d$. An ***elementary region*** is one that is either y-simple or is x-simple. If it is *both*, then it is called ***simple***.

2. The integral of a function f over an elementary region D is obtained by extending f to f^*, the function defined to be f on D and zero outside D but inside a containing rectangle R. The integral of f over D is defined by
$$\iint_D f dA = \iint_R f^* dA.$$

3. For a y-simple region
$$\iint_D f dA = \int_a^b \int_{\phi_1(x)}^{\phi_2(x)} f(x, y) dy dx$$
and for an x-simple region
$$\iint_D f dA = \int_c^d \int_{\psi_1(y)}^{\psi_2(y)} f(x, y) dx dy.$$

5.4 Changing the Order of Integration

Key Points in this Section.

1. If D is a ***simple*** region, that is, it is both x-simple and y-simple, then

$$\int_a^b \int_{\phi_1(x)}^{\phi_2(x)} f(x, y) dy dx = \int_c^d \int_{\psi_1(y)}^{\psi_2(y)} f(x, y) dx dy.$$

   Sometimes one of these orders is simpler to evaluate than the other.

2. If $m \leq f(x, y) \leq M$ on an elementary region D, then the ***mean value inequality*** holds:

$$m \text{ Area}(D) \leq \int\int_D f dA \leq M \text{ Area}(D).$$

3. If f is continuous and D is an elementary region (that is, it is either x-simple or y-simple), then the ***mean value equality*** holds:

$$\int\int_D f(x, y) dA = f(x_0, y_0) \text{ Area}(D).$$

   for some point (x_0, y_0) in D.

5.5 The Triple Integral

Key Points in this Section.

1. **Definition of the Integral.** If f is a bounded function defined on a box $B = [a,b] \times [c,d] \times [p,q]$ in $\mathbb{R}^3$, the triple integral, denoted

$$\iiint_B f\,dV, \quad \iiint_B f(x,y,z)dV, \text{ or } \iiint_B f(x,y,z)dx\,dy\,dz$$

is defined as a limit of Riemann sums analogous to that for double integrals; if the limit exists, f is called ***integrable***.

2. **Reduction to Iterated Integrals.** When f is integrable and an iterated integral exists, one has equality; for example,

$$\iiint_B f\,dV = \int_a^b \left\{ \int_p^q \left[\int_c^d f(x,y,z)dy \right] dz \right\} dx$$

3. **Elementary Regions.** An example of an ***elementary region*** W in $\mathbb{R}^3$ is one defined by inequalities $a \le x \le b$, $\phi_1(x) \le y \le \phi_2(x)$ (an elementary region in the plane) and $\gamma_1(x,y) \le z \le \gamma_2(x,y)$.

4. The ***integral*** $\iiint_W f\,dV$ of a function f defined on an elementary region W is obtained, as for double integrals, by extending f to be zero outside W but inside a box B containing W.

5. For the elementary region W described in point 3,

$$\iiint_W f\,dV = \int_a^b \int_{\phi_1(x)}^{\phi_2(x)} \int_{\gamma_1(x,y)}^{\gamma_2(x,y)} f(x,y,z)dz\,dy\,dx.$$

6. For regions that can be described as elementary regions in more than one way, one can, as with double integrals, change the order of integration.

6

The Change of Variables Formula and Applications

6.1 The Geometry of Maps from $\mathbb{R}^2$ to $\mathbb{R}^2$

Key Points in this Section.

1. A ***mapping*** T of a region D^* in $\mathbb{R}^2$ to $\mathbb{R}^2$ associates to each point (u, v) in D^* a point $(x, y) = T(u, v)$. The set of all such (x, y) is the ***image*** domain $D = T(D^*)$.

2. If T is ***linear***; that is if $T(u, v) = A\left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right]$, where A is a 2×2 matrix (and identifying points (u, v) with column vectors $\left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right]$), then T maps parallelograms to parallelograms, mapping the sides and vertices of the first, to those of the second.

3. A map T is called ***one-to-one*** if different points (that is, $(u, v) \neq (u', v')$) get sent to different points (that is $T(u, v) \neq T(u', v')$).

4. If T is linear, determined by a 2×2 matrix A, then T is one-to-one when $\det A \neq 0$.

5. When D is the image of T; that is, $D = T(D^*)$, we say T maps D^* ***onto*** D.

6.2 The Change of Variables Theorem

Key Points in this Section.

1. The ***Jacobian determinant*** of a C^1 mapping $T : D^* \subset \mathbb{R}^2 \to \mathbb{R}$; $T(u,v) = (x(u,v), y(u,v))$ is defined by

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix}.$$

2. The ***singe variable change of variables formula***, which is an integrated version of the chain rule, states that for $u \mapsto x(u)$ a C^1 mapping and $f(x)$ continuous,

$$\int_{x(a)}^{x(b)} f(x)\,dx = \int_a^b f(x(u))\frac{dx}{du}du$$

3. The ***two-variable change of variables formula*** states that for a C^1 map $\tau : D^* \to D$ that is one-to-one and onto D, and an integrable function $f : D \to \mathbb{R}$,

$$\iint_D f(x,y)\,dx\,dy = \iint_{D^*} f(x(u,v), y(u,v)) \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv.$$

4. The key idea in the proof is to put together these facts
   (a) the double integral is a limit of Riemann sums
   (b) the mapping T is nearly equal to its linear approximation on each term in the Riemann sum
   (c) the absolute value of the determinant of a linear map is the factor by which the map distorts area.

5. For polar coordinates $(r,\theta) \mapsto (x,y)$, where $x = r\cos\theta$ and $y = r\sin\theta$, the change of variables formula reads

$$\iint_D f(x,y)\,dx\,dy = \iint_{D^*} f(r\cos\theta, r\sin\theta) r\,dr\,d\theta$$

   and we write the relation between the area elements as

$$dx\,dy = r\,dr\,d\theta$$

6. **Guassian Integral.** An interesting combination of reduction to iterated integrals and a change of variables to polar coordinates applied to the integral $\iint_{\mathbb{R}^2} e^{-x^2-y^2}\,dx\,dy$ shows that

$$\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}.$$

7. The ***triple integral change of variables formula*** states that for a C^1 one-to-one map $T : W^* \to W$ that is onto W (except possibly on a finite union of curves), and an integrable function $f : W \to \mathbb{R}$,

$$\begin{aligned}&\iiint_W f(x,y,z)\,dx\,dy\,dz\\ &= \iiint_{W^*} f(x(u,v,w),y(u,v,w),z(u,v,w))\left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| du\,dv\,dw,\end{aligned}$$

where $T(u,v,w) = (x(u,v,w),y(u,v,w),z(u,v,w))$ and where the ***Jacobian determinant***

$$\frac{\partial(x,y,z)}{\partial(u,v,w)}$$

is the determinant of $\mathbf{D}T$, the matrix of partial derivatives of T.

8. **Cylindrical Coordinates.** For $x = r\cos\theta$, $y = r\sin\theta$, $z = z$,

$$\iiint_W f(x,y,z)\,dx\,dy\,dz = \iiint_{W^*} f(r\cos\theta, r\sin\theta, z)\,r\,dr\,d\theta\,dz$$

and the volume elements are related by

$$dx\,dy\,dz = r\,dr\,d\theta\,dz$$

9. **Spherical Coordinates.** For $x = \rho\sin\phi\cos\theta$, $y = \rho\sin\phi\sin\theta$, $z = \rho\cos\phi$,

$$\begin{aligned}&\iiint_W f(x,y,z)\,dx\,dy\,dz\\ &= \iiint_{W^*} f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)\,\rho^2\sin\phi\,d\rho\,d\theta\,d\phi\end{aligned}$$

and the volume elements are related by

$$dx\,dy\,dz = \rho^2\sin\phi\,d\rho\,d\theta\,d\phi.$$

6.3 Applications of Double and Triple Integrals

Key Points in this Section.

1. The ***average value*** of a function $f : [a, b] \to \mathbb{R}$ is

$$[f]_{\text{av}} = \frac{1}{b-a} \int_a^b f(x)dx,$$

of $f : D \subset \mathbb{R}^2 \to \mathbb{R}$ is

$$[f]_{\text{av}} = \frac{1}{\text{Area}(D)} \iint_D f(x,y)\,dx\,dy$$

where $\text{Area}(D) = \iint_D dx\,dy$ and of $f : W \subset \mathbb{R}^3 \to \mathbb{R}$ is

$$[f]_{\text{av}} = \frac{1}{\text{Volume}(W)} \iiint_W f(x,y,z)\,dx\,dy\,dz,$$

where $\text{Volume}(W) = \iiint_W dx\,dy\,dz$.

2. The ***center of mass*** of a distribution of masses $m_1, \ldots, m_n$ at points $x_1, \ldots, x_n$ on $\mathbb{R}$ is

$$\bar{x} = \frac{1}{m_1 + \cdots + m_n} \left(x_1 m_1 + \cdots + x_n m_n\right),$$

of material with a mass density $\delta(x)$ on $[a, b]$ is

$$\bar{x} = \frac{1}{\int_a^b \delta(x)dx} \int_a^b x\delta(x)dx,$$

and of material with mass density $\delta(x, y)$ on $D \subset \mathbb{R}^2$ is $(\bar{x}, \bar{y})$, where

$$\bar{x} = \frac{1}{\iint_D \delta(x,y)dxdy} \iint_D x\delta(x,y)dxdy$$
$$\bar{y} = \frac{1}{\iint_D \delta(x,y)dxdy} \iint_D y\delta(x,y)dxdy,$$

and of a distribution of material with mass density $\delta(x, y, z)$ on a region $W \subset \mathbb{R}^3$ is $(\bar{x}, \bar{y}, \bar{z})$, where

$$\bar{x} = \frac{1}{\iiint_W \delta(x,y,z)dxdydz} \iiint_W x\delta(x,y,z)dxdydz$$

with similar formulas for $\bar{y}$ and $\bar{z}$. In each of these formulas, the denominator is the ***total mass***.

3. The ***moments of inertia*** of a solid body occupying a region $W \subset \mathbb{R}^3$ with mass density $\delta(x, y, z)$ about the x,y,and z-axes are

$$I_x = \iiint_W (y^2 + z^2)\delta(x, y, z)dxdydz,$$
$$I_y = \iiint_W (x^2 + z^2)\delta(x, y, z)dxdydz,$$
$$I_z = \iiint_W (x^2 + y^2)\delta(x, y, z)dxdydz.$$

4. The gravitational potential of a particle with mass m due to matter occupying a region W with mass density $\delta(x, y, z)$ at a point (X, Y, Z) outside the body is

$$V(X, Y, Z) = -Gm \iiint_W \frac{\delta(x, y, z)dxdydz}{\sqrt{(x - X)^2 + (y - Y)^2 + (z - Z)^2}}$$

6.4 Improper Integrals

Key Points in this Section.

1. Improper integrals occur when either (a) the function being integrated is unbounded in an elementary region D or (b) the region itself is unbounded. In case (a), if $f : D \to \mathbb{R}$ is unbounded at parts of the boundary of D, then we find a sequence of smaller regions, say $D_{\eta,\delta}$ obtained by "backing off" by an amount η from the sides and δ from the top and bottom. Then we define

$$\iint_D f\,dA = \lim_{(\eta,\delta)\to(0,0)} \iint_{D_{\eta,\delta}} f\,dA$$

if the limit exists. For y-simple regions,

$$\iint_{D_{\eta,\delta}} f\,dA = \int_{a+\eta}^{b-\eta} \int_{\phi_1(x)+\delta}^{\phi_2(x)-\delta} f(x,y)\,dy\,dx.$$

In case (b) one similarly finds a family of bounded regions expanding to the given region and again takes the limit of the integrals over the bounded regions.

2. **Fubini's Theorem.** If f is a function, satisfying $f \geq 0$, continuous except possibly on the boundary of a y-simple region D, and if the iterated (improper) integral

$$\int_a^b \int_{\phi_1(x)}^{\phi_2(x)} f(x,y)\,dy\,dx$$

exists, then f itself is integrable and $\iint_D f\,dA$ equals the iterated integral. Here, for each x,

$$g(x) = \int_{\phi_1(x)}^{\phi_2(x)} f(x,y)\,dy = \lim_{\alpha\to 0^+} \int_{\phi_1(x)+\alpha}^{\phi_2(x)-\alpha} f(x,y)\,dy,$$

and $\int_a^b g(x)\,dx = \lim_{\beta\to 0^+} \int_{a+\beta}^{b-\beta} g(x)\,dx$, as in one variable calculus. There is a similar statement for x-simple regions.

The subtlety here is that for positive functions, two *single limits* can be replaced by one *double limit*. Exercise 18 shows that positivity of f is essential, or this result is not true.

7
Integrals over Curves and Surfaces

7.1 The Path Integral

Key Points in this Section.

1. **Definition.** The ***path integral*** of a scalar function f in $\mathbb{R}^3$ along a path $\mathbf{c}(t)$, where $a \leq t \leq b$, is defined by
$$\int_{\mathbf{c}} f\, ds = \int_a^b f(x(t), y(t), z(t)) \|\mathbf{c}'(t)\|\, dt.$$
2. The ***scalar element of arc length*** is
$$ds = \|\mathbf{c}'(t)\|\, dt.$$
3. There is a similar definition for path integrals in the plane (just leave out the z-dependence).
4. If f has the interpretation of the mass density along a wire, then the path integral is the total mass of the wire.
5. If the curve is in the xy-plane and f is interpreted as the height of a fence along the curve, then the path integral is the area of (one side of) this fence.
6. **Arc Length a Special Case.** If $f = 1$ (is identically one), then the definition of the path integral reduces to that for the arc length of the path.

7.2 Line Integrals

Key Points in this Section.

1. **Definition.** The ***line integral*** of a given continuous vector field $\mathbf{F}$ (defined in the plane or in space) along a path $\mathbf{c}(t)$, where $a \leq t \leq b$, is defined by
$$\int_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{s} = \int_a^b \mathbf{F}(\mathbf{c}(t)) \cdot \mathbf{c}'(t)\, dt.$$

2. The ***vector line element*** is
$$d\mathbf{s} = \mathbf{c}'(t)\, dt.$$

3. **Interpretation as Work.** If $\mathbf{F}$ represents a force field, then the line integral of $\mathbf{F}$ along $\mathbf{c}$ is the ***work done*** by the force field in moving a particle subject to this force field, along the path. (Another interpretation in terms of circulation, when $\mathbf{F}$ represents the velocity field of a fluid, is given in Chapter 8).

4. **Line Integral of a Gradient.** If $\mathbf{F} = \nabla f$, then an analog of the *Fundamental Theorem of Calculus* holds
$$\int_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{s} = f(\mathbf{c}(b)) - f(\mathbf{c}(a)).$$
In fact, this result follows directly from the single variable Fundamental Theorem of Calculus since, by the Chain Rule,
$$\frac{d}{dt} f(\mathbf{c}(t)) = \mathbf{F}(\mathbf{c}(t)) \cdot \mathbf{c}'(t).$$

5. Line integrals are independent of orientation preserving reparametrizations and path integrals are independent of *any* reparametrization. This is proved using the single-variable change of variables formula.

6. Because of the independence of parametrization one can define the line integral of a vector field along a geometric curve C, denoted
$$\int_C \mathbf{F} \cdot d\mathbf{s} \quad \text{or} \quad \int_C \mathbf{F} \cdot d\mathbf{r}$$
as long as an orientation along the curve is specified. To actually evaluate such an integral, any parametrization may be chosen, or some other method (such as the fundamental theorem in item 3) is used.

7.3 Parametrized Surfaces

Key Points in this Section.

1. To be able to deal with surfaces such as the sphere, one needs to move beyond graphs to more general objects, such as parametrized surfaces.

2. A ***parametrized surface*** is a map

$$\mathbf{\Phi} : D \to \mathbb{R}^3$$

written as

$$\mathbf{\Phi}(u,v) = (x(u,v), y(u,v), z(u,v)).$$

3. The actual surface S is the image of the map $\mathbf{\Phi}$.

4. Tangent vectors to the surface are given by

$$\mathbf{T}_u = \frac{\partial \mathbf{\Phi}}{\partial u} = \frac{\partial x}{\partial u}\mathbf{i} + \frac{\partial y}{\partial u}\mathbf{j} + \frac{\partial z}{\partial u}\mathbf{k}.$$

and

$$\mathbf{T}_v = \frac{\partial \mathbf{\Phi}}{\partial v} = \frac{\partial x}{\partial v}\mathbf{i} + \frac{\partial y}{\partial v}\mathbf{j} + \frac{\partial z}{\partial v}\mathbf{k}.$$

with a normal vector being given by

$$\mathbf{n} = \mathbf{T}_u \times \mathbf{T}_v.$$

5. A surface is called ***regular*** if $\mathbf{T}_u \times \mathbf{T}_v \neq \mathbf{0}$. This nonzero normal vector is useful for finding the equation of the tangent plane to the surface. The tangent plane at a point (x_0, y_0, z_0) on the surface is given by

$$(x - x_0, y - y_0, z - z_0) \cdot \mathbf{n} = 0,$$

where the normal vector $\mathbf{n}$ is evaluated at the point $(x_0, y_0, z_0) = \mathbf{\Phi}(u_0, v_0)$.

7.4 Area of a Surface

Key Points in this Section.

1. **Area of a parametrized surface**:

$$\begin{aligned} A(S) &= \iint_D \|\mathbf{T}_u \times \mathbf{T}_v\| \, du \, dv \\ &= \iint_D \sqrt{\left[\frac{\partial(y,z)}{\partial(u,v)}\right]^2 + \left[\frac{\partial(x,y)}{\partial(u,v)}\right]^2 + \left[\frac{\partial(x,z)}{\partial(u,v)}\right]^2} \, du \, dv \end{aligned}$$

2. The ***scalar surface area element*** is the integrand:

$$\begin{aligned} dS &= \|\mathbf{T}_u \times \mathbf{T}_v\| \, du \, dv \\ &= \sqrt{\left[\frac{\partial(y,z)}{\partial(u,v)}\right]^2 + \left[\frac{\partial(x,y)}{\partial(u,v)}\right]^2 + \left[\frac{\partial(x,z)}{\partial(u,v)}\right]^2} \, du \, dv \end{aligned}$$

3. The formula for the area element is motivated by the fact that on a small patch, the surface is approximated by the parallelogram with sides $\mathbf{T}_u \, du$ and $\mathbf{T}_v \, dv$ and the fact that the area of a parallelogram with sides $\mathbf{a}$ and $\mathbf{b}$ is given by $\|\mathbf{a} \times \mathbf{b}\|$.

4. **Sphere.** $x^2 + y^2 + z^2 = R^2$, the scalar surface element is given by:

$$dS = R^2 \sin\phi \, d\phi \, d\theta$$

5. **Graph.** $z = g(x, y)$ (where $(x, y) \in D \subset \mathbb{R}^2$ can be parametrized by

$$x = u, \quad y = v, \quad z = g(u, v).$$

6. **Surface area of a graph.**

$$A(S) = \iint_D \left(\sqrt{\left[\frac{\partial g}{\partial x}\right]^2 + \left[\frac{\partial g}{\partial y}\right]^2 + 1} \right) du \, dv$$

7. **Surfaces of Revolution**.

   (a) Revolve $y = f(x)$, where $a \le x \le b$, about the x-axis:

$$A(S) = 2\pi \int_a^b \left(|f(x)| \sqrt{1 + (f'(x))^2} \right) dx$$

   (b) Revolve $y = f(x)$, where $a \le x \le b$, about the y-axis:

$$A(S) = 2\pi \int_a^b \left(|x| \sqrt{1 + (f'(x))^2} \right) dx$$

8. The formulas in points 6 and 7 are derived from the general area formula in point 1 for a parametrized surface by parametrizing the circles making up the surface using sines and cosines.

7.5 Integrals of Scalar Functions over Surfaces

Key Points in this Section.

1. **Definition of Scalar Surface Integral.**

$$\iint_S f\,dS = \iint_D f(x(u,v), y(u,v), z(u,v))\|\mathbf{T}_u \times \mathbf{T}_v\|\,du\,dv$$

2. **Graph.** For $z = g(x, y)$ with $\mathbf{\Phi}(u, v) = (u, v, g(u, v))$,

$$\mathbf{T}_u = \mathbf{i} + \frac{\partial g}{\partial u}\mathbf{k}; \qquad \mathbf{T}_v = \mathbf{j} + \frac{\partial g}{\partial v}\mathbf{k}$$

and

$$\mathbf{T}_u \times \mathbf{T}_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & \frac{\partial g}{\partial u} \\ 0 & 1 & \frac{\partial g}{\partial v} \end{vmatrix} = -\frac{\partial g}{\partial u}\mathbf{i} - \frac{\partial g}{\partial v}\mathbf{j} + \mathbf{k}$$

3. **Scalar Surface Element Formulas.**

   (a) **Parametrized Surface.**

$$dS = \|\mathbf{T}_u \times \mathbf{T}_v\|\ du\,dv$$

   (b) **Graph.**

$$dS = \frac{dxdy}{\cos\theta} = \frac{dx\,dy}{\mathbf{n}\cdot\mathbf{k}} = \left(\sqrt{\left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial y}\right)^2 + 1}\right)\,dx\,dy$$

   where $\cos\theta = \mathbf{n}\cdot\mathbf{k}$, and $\mathbf{n}$ is the upward pointing unit normal vector to the surface. See Figure 7.5.1.

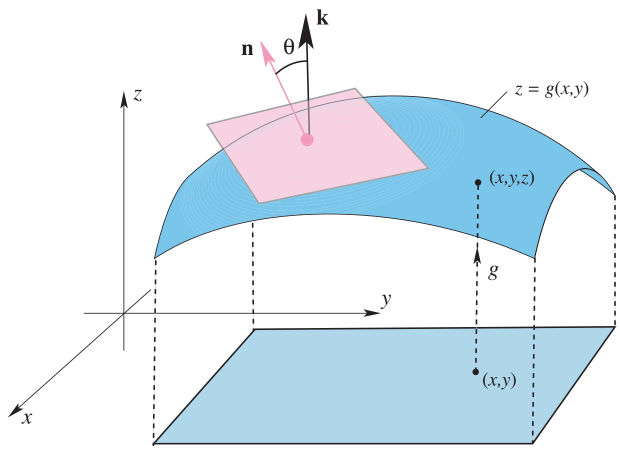

FIGURE 7.5.1. The area element on a graph is $dS = \frac{dx dy}{\cos\theta} = \frac{dx\, dy}{\mathbf{n}\cdot\mathbf{k}}$.

(c) **Sphere** $x^2 + y^2 + z^2 = R^2$:

$$dS = R^2 \sin\phi \, d\phi \, d\theta$$

4. Surface integrals are independent of the parametrization of the surface chosen (this is discussed in the next section).

5. **Interpretation.** The total mass of a surface with a surface mass density m (mass per unit area) is given by

$$M(S) = \iint_S m(x, y, z) dS.$$

7.6 Surface Integrals of Vector Functions

Key Points in this Section.

1. **Definition.** The formula for the surface integral of a vector field $\mathbf{F}$ over a parametrized surface is given by:

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_D \mathbf{F} \cdot (\mathbf{T}_u \times \mathbf{T}_v)\, du\, dv$$

2. The $d\mathbf{S}$ and dS notation helps one remember the formulas for integrals of scalar and vector functions on surfaces.

3. **Surface Area Elements—Parametrized Surface.**

$$d\mathbf{S} = \mathbf{T}_u \times \mathbf{T}_v\, du\, dv, \quad dS = \|\mathbf{T}_u \times \mathbf{T}_v\|\, du\, dv$$

   or, in other notation,

$$d\mathbf{S} = \mathbf{\Phi}_u \times \mathbf{\Phi}_v\, du\, dv, \quad dS = \|\mathbf{\Phi}_u \times \mathbf{\Phi}_v\|\, du\, dv.$$

4. **Vector vs Scalar Surface Element**. Since the unit normal is $\mathbf{n} = (\mathbf{T}_u \times \mathbf{T}_v)/\|\mathbf{T}_u \times \mathbf{T}_v\|$, it follows from the preceding points that

$$d\mathbf{S} = \mathbf{n}\, dS.$$

5. **Vector Surface Element for a Sphere of Radius** R:

$$d\mathbf{S} = (x\mathbf{i} + y\mathbf{j} + z\mathbf{k})R \sin\phi \ \ d\phi\, d\theta = \mathbf{r} R \sin\phi \ \ d\phi\, d\theta$$

6. **Geometric Surface**. This is similar to the geometric curve idea met in line integrals. To integrate over a geometric surface, we need an orientation, or handedness. This is done by specifying a direction for the unit normal.

7. **Möbius Band.** Many students are fascinated by the fact that the Möbius band cannot be oriented. A classroom demonstration of this may be useful.

8. **Graphs.** If S is a graph $z = g(x, y)$, the default orientation is the *upward* normal. In the case of graphs, many students will want to memorize the formula

$$d\mathbf{S} = \left(-\frac{\partial g}{\partial x}\mathbf{i} - \frac{\partial g}{\partial y}\mathbf{j} + \mathbf{k}\right) dx\, dy,$$

   which is just $\mathbf{\Phi}_x \times \mathbf{\Phi}_y\, dx\, dy$ where $\mathbf{\Phi}(x, y) = (x, y, g(x, y))$.

9. **Independence of Parametrization.** As long as the orientation is respected, the surface integral over a geometric surface is well defined, independent of the parametrization. That is, for two parametrizations $\mathbf{\Phi}_1$ and $\mathbf{\Phi}_2$, describing the same geometric surface (including the orientation), then

$$\iint_{\mathbf{\Phi}_1} \mathbf{F} \cdot d\mathbf{S} = \iint_{\mathbf{\Phi}_2} \mathbf{F} \cdot d\mathbf{S}.$$

   Their common value is denoted

$$\iint_S \mathbf{F} \cdot d\mathbf{S}.$$

10. **Normal Component.** Since $d\mathbf{S} = \mathbf{n}\, dS$, we find that

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_S (\mathbf{F} \cdot \mathbf{n})\, dS,$$

   that is, the surface integral of the vector function $\mathbf{F}$ is equal to the scalar integral of the normal component of $\mathbf{F}$.

11. **Physical Interpretation.** If $\mathbf{F}$ represents the velocity field of a fluid, then the surface integral

$$\iint_S \mathbf{F} \cdot d\mathbf{S}.$$

   represents the *rate of flow* of fluid across the surface. For example, one can talk about an imaginary surface across a creek, where the flow rate might be measured in cubic meters per second. For other vector fields, the surface integral is called the ***flux***. Figure 7.6.1 indicates why the flux is the integral of the normal component.

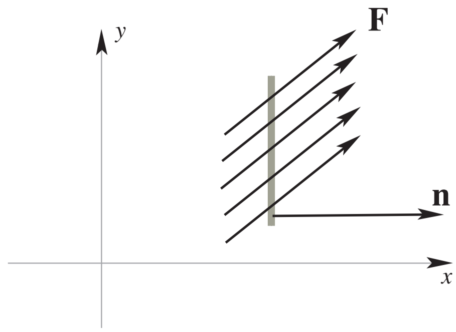

FIGURE 7.6.1. The flux across a surface (a line in two dimensions) is the integral of the normal component of the vector field.

12. **Gauss' Law.** This says (in appropriate units) that

$$\iint_S \mathbf{E} \cdot d\mathbf{S} = Q,$$

where $\mathbf{E}$ is the electric field caused by a charge distribution and Q is the total charge enclosed by the surface S.

13. **Coulomb's law**. If the charge is symmetrically placed, S is chosen to be a sphere, and one assumes (as is reasonable) that the electric field is $\mathbf{E} = E\mathbf{n}$, then one finds that

$$E = \frac{Q}{4\pi R^2}$$

and in particular, for a point charge, one gets ***Coulomb's law*** stating that the above gives a formula for the field of a point charge.

7.7 Applications: Differential Geometry, Physics, Forms of Life

Key Points in this Section.

1. The theory of curvature for surfaces is one of the most exciting chapters in the history of mathematics, in part because it is a core idea in Einstein's General Theory of Relativity.

2. The ***Gauss curvature*** $K(p)$ of a surface S at a point P is given by

$$K(p) = \frac{ln - m^2}{W}$$

and the ***mean curvature*** $H(p)$ at P is given by

$$H(p) = \frac{Gl + En - 2Fm}{2W},$$

where, if S is parameterized by the mapping $\mathbf{\Phi}$,

$$\begin{aligned} l &= \mathbf{N} \cdot \mathbf{\Phi}_{uu} \\ m &= \mathbf{N} \cdot \mathbf{\Phi}_{uv} \\ n &= \mathbf{N} \cdot \mathbf{\Phi}_{vv} \end{aligned}$$

and

$$\mathbf{N} = \frac{\mathbf{T}_u \times \mathbf{T}_v}{\sqrt{W}}, \qquad W = \|\mathbf{T}_u \times \mathbf{T}_v\|^2 = EG - F$$

and

$$E = \|\mathbf{\Phi}_u\|^2, \qquad F = \mathbf{\Phi}_u \cdot \mathbf{\Phi}_v, \qquad G = \|\mathbf{\Phi}_v\|^2.$$

8

The Integral Theorems of Vector Analysis

8.1 Green's Theorem

Key Points in this Section.

1. **Statement of Green's Theorem.** For a simple region D with bounding curve $C = \partial D$ and two C^1 functions P and Q on D, we have
$$\int_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

2. **Orientation.** The orientation is chosen so that as you proceed along the boundary curve in the positive direction, the region is on your left. For simple regions this means that you go around the regions *counter-clockwise*; if there are holes inside the region, those boundaries get traversed *clockwise*.

3. **Strategy of the Proof.** For a y-simple region, one proves by reduction to iterated integrals, the Fundamental Theorem of Calculus and the definition of the line integral that
$$\int_C P\,dx = -\iint_D \left(\frac{\partial P}{\partial y}\right) dx\,dy$$
Similarly, for a x-simple region, we have
$$\int_C Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x}\right) dx\,dy$$
One gets Green's theorem for simple regions by simply adding these two results.

4. **More General Regions.** One gets Green's theorem for more general regions by breaking up a given region into simple ones as in Figure 8.1.5 of the Text. Here is another example of how to break up a region.

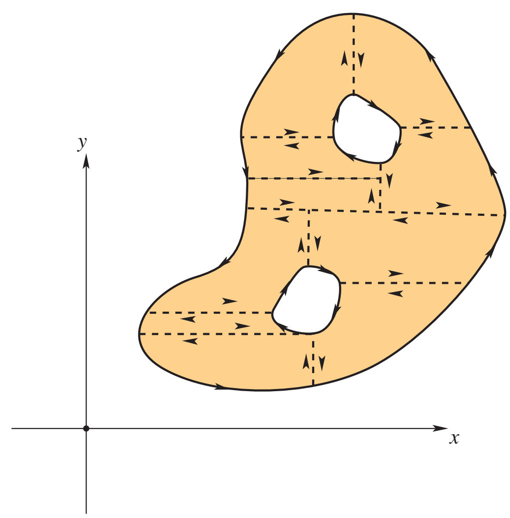

FIGURE 8.1.1. How to break a two-holed region up into simple regions.

5. **Area.** As a special case of Green's theorem, one finds that the area of a region is

$$A = \frac{1}{2}\int_{\partial D} x\,dy - y\,dx$$

6. **Vector form of Green's theorem.** If $\mathbf{F}$ is a vector field in the plane, then

$$\int_{\partial D} \mathbf{F}\cdot d\mathbf{s} = \iint_D (\nabla\times\mathbf{F})\cdot\mathbf{k}\,dx\,dy.$$

This is proved by simply writing $\mathbf{F} = P\mathbf{i} + Q\mathbf{j}$ and applying Green's theorem and noting that

$$\nabla\times\mathbf{F} = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathbf{k}.$$

7. **Divergence theorem in the plane.** This result says that

$$\int_{\partial D} \mathbf{F}\cdot\mathbf{n}\,ds = \iint_D (\operatorname{div}\mathbf{F})\,dx\,dy.$$

where $\mathbf{n}$ is the outward normal to the boundary. This is proved by again writing $\mathbf{F} = P\mathbf{i} + Q\mathbf{j}$ and noting that the unit outward normal is given by

$$\mathbf{n} = \frac{y'\mathbf{i} - x'\mathbf{j}}{\sqrt{(x')^2 + (y')^2}}$$

using

$$ds = \sqrt{(x')^2 + (y')^2}\, dt,$$

substituting into the left side to get

$$\int_{\partial D} P\, dy - Q\, dx,$$

and then using Green's theorem.

8.2 Stokes' Theorem

Key Points in this Section.

1. **Statement of Stoke's Theorem.** Let S be the oriented surface defined by the graph of a C^2 function $z = f(x, y)$, where $(x, y) \in D$, a region in the plane to which Green's theorem applies, and let F be a C^1 vector field on a region containing the surface. If ∂S denotes the oriented boundary curve of S, then

$$\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = \iint_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S} = \int_{\partial S} \mathbf{F} \cdot d\mathbf{s}.$$

2. The main idea in the proof of this result is to reduce the problem to Green's theorem over the region D by everywhere substituting z in terms of x and y.

3. The same statement holds for parametrized surfaces as well, and the main idea of the proof is the same; this time one reduces it to Green's theorem by substituting for x, y and z their expressions in terms of the surface parameters, u and v.

4. The default orientation for graphs is that the surface is oriented by the upward pointing normal vector; that is, by

$$\mathbf{n} = -\frac{\partial g}{\partial y}\mathbf{i} - \frac{\partial g}{\partial x}\mathbf{j} + \mathbf{k}$$

(note that this vector need not be a *unit vector*). One traverses the boundary in the same way as one traverses the boundary in the domain D as in Green's theorem.

5. For a parametrized surface, if one's head is pointing in the direction of the chosen normal vector (which determines the orientation of the surface), and if one walks along the boundary curve ∂S in the correct oriented direction, then the surface is on your left. (If the surface is on your right, then you are going in the wrong direction and you must change direction or change the orientation of the surface).

6. Stokes' Theorem together with the mean value theorem gives the interpretation of the curl of a vector field $\mathbf{F}$ as the circulation per unit area. That is, if we choose a point P and a *unit* vector $\mathbf{n}$ at this point, then

$$(\operatorname{curl} \mathbf{F}(\mathrm{P})) \cdot \mathbf{n} = \lim_{\rho \to 0} \frac{1}{A(S_\rho)} \int_{\partial S_\rho} \mathbf{F} \cdot d\mathbf{s}$$

where S_ρ is a disk of radius ρ in the plane perpendicular to $\mathbf{n}$ and centered at the point P and $A(S_\rho) = \pi\rho^2$ is its area (shapes other than disks can be used just as well).

7. The interpretation of the curl as circulation per unit area is useful in deriving formulas for the curl in cylindrical and spherical coordinates.

8.3 Conservative Fields

Key Points in this Section.

1. The main result in this section states that the following statements concerning a vector field **F** defined and C^1 on all of $\mathbb{R}^3$ are equivalent:

   (a) The integral of **F** around any closed loop is zero

   (b) The integral of **F** from one point to another is independent of the path taken between those points

   (c) **F** is a gradient field

   (d) $\nabla \times \mathbf{F} = \mathbf{0}$.

2. A similar result holds in the plane (where the curl is interpreted as the scalar curl)

3. Stokes' theorem is used to show that if **F** is curl free, then its integral around a closed loop is zero.

4. If **F** is not defined at a finite number of points in $\mathbb{R}^3$, then the same result is true. This does not necessarily hold in the plane. (A counter example is given in Exercise 12).

5. Special Case: In the plane, a vector field $\mathbf{F} = P\mathbf{i} + Q\mathbf{j}$ defined and C^1 everywhere, is a gradient if and only if

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}.$$

8.4 Gauss' Theorem

Key Points in this Section.

1. If S is a closed surface enclosing a region W, we adopted the convention that $S = \partial W$ is given the ***outward orientation***, with outward unit normal denoted by $\mathbf{n}(x, y, z)$ at each point (x, y, z) of S. If we denote the surface with the opposite (inward) orientation by ∂W_{op}, then the associated unit normal direction for this orientation is $-\mathbf{n}$. Thus,

$$\iint_{\partial W} \mathbf{F} \cdot d\mathbf{S} = \iint_S (\mathbf{F} \cdot \mathbf{n}) dS = -\iint_S [\mathbf{F} \cdot (-\mathbf{n})] dS = -\iint_{\partial W_{\text{op}}} \mathbf{F} \cdot d\mathbf{S}.$$

2. ***Gauss' Divergence Theorem*** states that for a (symmetric, elementary) region W with boundary ∂W oriented by the outward pointing unit normal and if $\mathbf{F}$ is a smooth vector field defined on W, then

$$\iiint_W (\nabla \cdot \mathbf{F}) dV = \iint_{\partial W} \mathbf{F} \cdot d\mathbf{S}.$$

3. The ***key idea of the proof*** is to proceed in these steps:

   (a) Write $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ so that

   $$\nabla \cdot \mathbf{F} = \partial P/\partial x + \partial Q/\partial y + \partial R/\partial z.$$

   (b) Establish the separate identities

   $$\iiint_W \frac{\partial P}{\partial x} dV = \iint_{\partial W} P\mathbf{i} \cdot d\mathbf{S}$$
   $$\iiint_W \frac{\partial Q}{\partial y} dV = \iint_{\partial W} Q\mathbf{j} \cdot d\mathbf{S}$$
   $$\iiint_W \frac{\partial R}{\partial z} dV = \iint_{\partial W} R\mathbf{k} \cdot d\mathbf{S},$$

   which is parallel to what was done in the proof of Green's theorem.

   (c) Adding these identities gives the divergence theorem

   (d) To establish the above identities, proceed in a manner similar to Green's theorem, namely reduce the triple integral to a double + single integral and apply the fundamental theorem of calculus to the single integral.

(e) For the third identity (the one involving R), for instance, write the region as that between the graphs of two functions $z = f_2(x, y)$ and $z = f_1(x, y)$ over a region D in the xy-plane. Then,

$$\iiint_W \frac{\partial R}{\partial z} dV = \iint_D \left[\int_{z=f_1(x,y)}^{z=f_2(x,y)} \frac{\partial R}{\partial z}\, dz \right] dx\, dy$$
$$= \iint_D \left[R(x, y, f_2(x, y)) - R(x, y, f_1(x, y)) \right] dx\, dy.$$

(f) Write out the boundary integral using the formulas for the surface element of the bounding graphs:

$$d\mathbf{S} = \left(-\frac{\partial f_2}{\partial x}\mathbf{i} - \frac{\partial f_2}{\partial y}\mathbf{j} + \mathbf{k} \right) dx\, dy,$$

and

$$d\mathbf{S} = \left(-\frac{\partial f_1}{\partial x}\mathbf{i} - \frac{\partial f_1}{\partial y}\mathbf{j} - \mathbf{k} \right) dx\, dy,$$

(g) Note that on the upper surface

$$R\mathbf{k} \cdot d\mathbf{S} = R(x, y, f_2(x, y))\, dx\, dy$$

while on the lower surface,

$$R\mathbf{k} \cdot d\mathbf{S} = -R(x, y, f_1(x, y))\, dx\, dy$$

(h) There is no contribution to the surface integral from the sides of the region as $R\mathbf{k}$ and $d\mathbf{S}$ are orthogonal. Comparing this with the preceding formula for the triple integral of $\partial R/\partial z$ gives the result.

4. As with Green's and Stokes' Theorems, the result is seen to be valid on a more general region, by breaking it up into a union of symmetric elementary regions.

5. From the divergence theorem and the mean value theorem, it follows that

$$\nabla \cdot \mathbf{F}(P) = \lim_{\rho \to 0} \frac{1}{V(W_\rho)} \iint_{\partial W_\rho} \mathbf{F} \cdot d\mathbf{S}$$

where W_ρ is a family of regions that approaches the point P as ρ tends to zero. This makes precise the idea (already discussed in Chapter 4) that the divergence is the net outward ***flux per unit volume.***

6. A vector field $\mathbf{F}$ is called ***divergence free*** or ***incompressible*** when $\nabla \cdot \mathbf{F} = 0$. By the divergence theorem this is equivalent to the property that the flux of $\mathbf{F}$ out of any surface is zero. This agrees with the earlier intuition about the divergence as the rate of change of volume under motion along flow lines.

7. ***Gauss' Law*** states that for a region W containing the origin,

$$\iint_{\partial W} \frac{\mathbf{r} \cdot d\mathbf{S}}{r^3} = 4\pi$$

(the integral is zero if the region does not contain the origin). This is a good example where students must be a little careful with places where the integrand is not defined. One uses the divergence theorem to write

$$\iint_{\partial W} \frac{\mathbf{r} \cdot d\mathbf{S}}{r^3} = \iiint_W \nabla \cdot \frac{\mathbf{r}}{r^3}\, dV$$

but for $r \neq 0$, $\nabla \cdot (\mathbf{r}/r^3) = 0$. Thus, one can deform the region to that of a small sphere surrounding the origin and for the sphere one evaluates the integral easily to be 4π.

8.5 Applications: Physics, Engineering & Differential Equations

Key Points in this Section.

1. The ***law of conservation of mass*** for a vector field $\mathbf{V}$ and a function ρ, is the condition

$$\frac{d}{dt}\iiint_W \rho\,dV = -\iint_{\partial W} \mathbf{J}\cdot\mathbf{n}\,dA$$

where $\mathbf{J} = \rho\mathbf{V}$ and where W is an arbitrary region in $\mathbb{R}^3$.

2. The divergence theorem shows that conservation of mass is equivalent to the ***continuity equation***

$$\operatorname{div}\mathbf{J} + \frac{\partial\rho}{\partial t} = 0.$$

3. The ***material derivative*** of a function f with respect to a vector field $\mathbf{F}$ is

$$\frac{Df}{Dt} = \frac{\partial f}{\partial t} + \nabla f\cdot\mathbf{F}.$$

4. If $\phi(x,t)$ is the ***flow*** of the vector field $\mathbf{F}$, (that is, $\phi(x,0) = x$ and the map $t \mapsto \phi(x,t)$ for each fixed x is a flow line of F), and J is the Jacobian determinant of the flow map $x \mapsto \phi(x,t)$, then

$$\frac{\partial J}{\partial t} = J\operatorname{div}\mathbf{F}$$

and the ***transport theorem*** holds for any function f of (x,y,z,t):

$$\frac{d}{dt}\iiint_{W_t} f\,dV = \iiint_{W_t}\left(\frac{Df}{Dt} + f\operatorname{div}\mathbf{F}\right)dV$$

where W_t is the image of a region W in $\mathbb{R}^3$ under the flow map.

5. ***Euler's equation for a perfect fluid*** is

$$\rho\left(\frac{\partial\mathbf{V}}{\partial t} + \mathbf{V}\cdot\nabla\mathbf{V}\right) = -\nabla p$$

where $\mathbf{V}$ is the fluid velocity field, ρ is the fluid density and p is the pressure.

6. Conservation of energy applied to heat energy gives the ***heat equation***:

$$\frac{\partial T}{\partial t} = k\nabla^2 T,$$

where T is the temperature and k is the material heat conductivity.

7. ***Maxwell's equations*** for an electric field $\mathbf{E}$ and a magnetic field $\mathbf{H}$ state that

$$\begin{aligned} \operatorname{div} \mathbf{E} &= \rho \\ \operatorname{div} \mathbf{H} &= 0 \\ \operatorname{curl} \mathbf{E} + \frac{\partial \mathbf{H}}{\partial t} &= 0 \\ \operatorname{curl} \mathbf{H} - \frac{\partial \mathbf{E}}{\partial t} &= J, \end{aligned}$$

where ρ is the charge density and J is the current.

8. Stokes' and Gauss' theorems are the key to understanding the integral versions of these equations. For example, the integral version of the last of Maxwell's equations is ***Faraday's law***, which was studied in §8.2 (see Example 5).

8.6 Differential Forms

Key Points in this Section.

1. 0-forms are real valued functions
2. 1-forms have the expression
$$\omega = P\,dx + Q\,dy + R\,dz.$$
3. 2-forms have the expression
$$\eta = F dxdy + G dydz + H dzdx,$$
4. 3-forms have the expression
$$\nu = f(x, y, z)dxdydz.$$
5. The integral of a 1-form corresponds to a line integral, of a 2-form to a surface integral and of a 3-form to a volume integral.
6. The basic operations on forms involve the wedge operation, written $\omega \wedge \eta$ and the d operation, written $d\alpha$.
7. The d operation includes the gradient, divergence and curl into one operation.
8. The general Stokes' theorem reads
$$\int_{\partial S} \omega = \int_S d\omega,$$
where S can be
    (a) a curve (one dimensional, and correspondingly, ω is a 0-form and $d\omega$ is a 1-form),
    (b) a surface in the plane or space, (two dimensional, and correspondingly, ω is a 1-form and $d\omega$ is a 2-form), or
    (c) a solid region in space (three dimensional, and correspondingly, ω is a 2-form and $d\omega$ is a 3-form).
9. These three cases correspond to the Fundamental Theorem of Calculus, to Stokes' Theorem (or Green's Theorem if the surface is in the plane), and to Gauss' Theorem.