Homework 1 Solutions

Math 1c Practical, 2008

All questions are from the Linear Algebra text, O'Nan and Enderton

Question 1: 6.4.2 Apply Gram-Schmidt orthogonalization to the following sequence of vectors in $\mathbb{R}^3$: $\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 8 \\ 1 \\ -6 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

Solution Apply the process on page 365, with $x_1 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, x_2 = \begin{bmatrix} 8 \\ 1 \\ -6 \end{bmatrix}, x_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

Step 1 produces an orthogonal basis:

$v_1 = x_1 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$.

$$v_2 = x_2 - \frac{(x_2,v_1)}{(v_1,v_1)} v_1 = \begin{bmatrix} 8 \\ 1 \\ -6 \end{bmatrix} - \frac{\left(\begin{bmatrix} 8 \\ 1 \\ -6 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}\right)}{\left(\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}\right)} \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 8 \\ 1 \\ -6 \end{bmatrix} - \frac{10}{5} \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 6 \\ -3 \\ -6 \end{bmatrix}.$$

$$v_3 = x_3 - \frac{(x_3,v_1)}{(v_1,v_1)} v_1 - \frac{(x_3,v_2)}{(v_2,v_2)} v_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} - \frac{\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}\right)}{\left(\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}\right)} \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} - \frac{\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 6 \\ -3 \\ -6 \end{bmatrix}\right)}{\left(\begin{bmatrix} 6 \\ -3 \\ -6 \end{bmatrix}, \begin{bmatrix} 6 \\ -3 \\ -6 \end{bmatrix}\right)} \begin{bmatrix} 6 \\ -3 \\ -6 \end{bmatrix} =$$

$$\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} - \frac{0}{5} \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} - \frac{-6}{81} \begin{bmatrix} 6 \\ -3 \\ -6 \end{bmatrix} = \frac{1}{9} \begin{bmatrix} 4 \\ -2 \\ 5 \end{bmatrix}.$$

Step 2 produces an orthonormal basis by replacing each vector with a vector of norm 1:

Replace v_1 with $\frac{v_1}{|v_1|} = \frac{\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}}{\left\| \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \right\|} = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$.

Replace v_2 with $\frac{v_2}{|v_2|} = \frac{\begin{bmatrix} 6 \\ -3 \\ -6 \end{bmatrix}}{\left\| \begin{bmatrix} 6 \\ -3 \\ -6 \end{bmatrix} \right\|} = \frac{1}{9} \begin{bmatrix} 6 \\ -3 \\ -6 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 2 \\ -1 \\ -2 \end{bmatrix}$.

Replace v_3 with $\frac{v_3}{|v_3|} = \frac{\frac{1}{9}\begin{bmatrix} 4 \\ -2 \\ 5 \end{bmatrix}}{\left| \frac{1}{9}\begin{bmatrix} 4 \\ -2 \\ 5 \end{bmatrix} \right|} = \frac{1}{3\sqrt{5}} \begin{bmatrix} 4 \\ -2 \\ 5 \end{bmatrix}$.

So the final solution is $v_1 = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, v_2 = \frac{1}{3} \begin{bmatrix} 2 \\ -1 \\ -2 \end{bmatrix}, v_3 = \frac{1}{3\sqrt{5}} \begin{bmatrix} 4 \\ -2 \\ 5 \end{bmatrix}$.

Question 2: 6.4.10 Let $A = \begin{bmatrix} 1 & -2 & -4 \\ 2 & -5 & -3 \\ 3 & -7 & -7 \end{bmatrix}$ and $x = \begin{bmatrix} 20 \\ -16 \\ 14 \end{bmatrix}$ and express x as the sum of a vector in the row space of A and a vector in the nullspace of A.

Solution We proceed as in Example 5 on page 369. We must find vectors $x_n \in$ nullspace(A) and $x_r \in$ rowspace(A) such that $x = x_n + x_r$. We find the reduced row echelon form of A, which is $\begin{bmatrix} 1 & 0 & -14 \\ 0 & 1 & -5 \\ 0 & 0 & 0 \end{bmatrix}$. Since nullspace$(A)$=$sp \left\{ \begin{bmatrix} 14 \\ 5 \\ 1 \end{bmatrix} \right\}$ is 1-dimensional, let's project onto the nullspace of A. Clearly $\begin{bmatrix} 14 \\ 5 \\ 1 \end{bmatrix}$ is an orthogonal basis for nullspace(A). So we can use the Second Projection Theorem to find the projection of x onto nullspace(A): $x_n = \frac{\left(\begin{bmatrix} 20 \\ -16 \\ 14 \end{bmatrix}, \begin{bmatrix} 14 \\ 5 \\ 1 \end{bmatrix} \right)}{\left(\begin{bmatrix} 14 \\ 5 \\ 1 \end{bmatrix}, \begin{bmatrix} 14 \\ 5 \\ 1 \end{bmatrix} \right)} \begin{bmatrix} 14 \\ 5 \\ 1 \end{bmatrix} =$

$\frac{107}{111}\begin{bmatrix} 14 \\ 5 \\ 1 \end{bmatrix}$. Then x_r is whatever is left over: $x_r = x - x_n = \begin{bmatrix} 20 \\ -16 \\ 14 \end{bmatrix} - \frac{107}{111}\begin{bmatrix} 14 \\ 5 \\ 1 \end{bmatrix} = \frac{1}{111}\begin{bmatrix} 722 \\ -2311 \\ 1447 \end{bmatrix}$.

Question 3: 6.4.13 For the space $\mathbb{R}^4$, let $w_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, w_2 = \begin{bmatrix} 3 \\ 3 \\ -1 \\ -1 \end{bmatrix}$, $y = \begin{bmatrix} 6 \\ 0 \\ 2 \\ 0 \end{bmatrix}$ and let $W = sp\{w_1, w_2\}$. (a) Find a basis for W consisting of two orthogonal vectors. (b) express y as the sum of a vector in W and a vector in $W^\perp$.

Solution (a) Apply step 1 of Gram-Schmidt:

$v_1 = w_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$.

$$v_2 = w_2 - \frac{(w_2, v_1)}{(v_1, v_1)} v_1 = \begin{bmatrix} 3 \\ 3 \\ -1 \\ -1 \end{bmatrix} - \frac{\left(\begin{bmatrix} 3 \\ 3 \\ -1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}\right)}{\left(\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}\right)} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ -2 \\ -2 \end{bmatrix}.$$

This gives us an orthogonal basis $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ -2 \\ -2 \end{bmatrix}$ for W.

(b) We must find vectors $w \in W$ and $w' \in W^\perp$ such that $y = w + w'$. Using our orthogonal basis from (a) and the Second Projection Theorem, we get

$$w = \frac{\left(\begin{bmatrix} 6 \\ 0 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}\right)}{\left(\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}\right)} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + \frac{\left(\begin{bmatrix} 6 \\ 0 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ -2 \\ -2 \end{bmatrix}\right)}{\left(\begin{bmatrix} 2 \\ 2 \\ -2 \\ -2 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ -2 \\ -2 \end{bmatrix}\right)} \begin{bmatrix} 2 \\ 2 \\ -2 \\ -2 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ 1 \\ 1 \end{bmatrix}.$$ Then w'

is whatever is left over: $w' = y - w = \begin{bmatrix} 6 \\ 0 \\ 2 \\ 0 \end{bmatrix} - \begin{bmatrix} 3 \\ 3 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ -3 \\ 1 \\ -1 \end{bmatrix}$.

Question 4: 6.5.4 Let u and v be orthogonal vectors. If $u + v$ and $u - v$ are orthogonal, show that $|u| = |v|$.

Solution $u + v$ and $u - v$ are orthogonal $\Rightarrow 0 = (u + v, u - v) = (u, u - v) + (v, u - v) = (u, u) - (u, v) + (v, u) - (v, v) = (u, u) - (v, v)$ since u and v are orthogonal. Hence $(u, u) = (v, v) \Rightarrow \sqrt{(u, u)} = \sqrt{(v, v)} \Rightarrow |u| = |v|$.

Question 5: 6.5.5 Let T be a linear operator on $\mathbb{R}^2$. Suppose that T has the following property: whenever a and b are orthogonal, then $T(a)$ and $T(b)$ are orthogonal. Show that T is a scalar multiple of an isometry. [Hint: let $u = T(i)$ and $v = T(j)$. Use the preceding exercise to show that $|u| = |v|$.]

Solution Recall in $\mathbb{R}^2$, $i = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $j = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Clearly i and j are orthogonal vectors, and the pair $i + j = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $i - j = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ is orthogonal too. Hence by hypothesis, $T(i)$ and $T(j)$ is an orthogonal pair, as is $T(i + j)$ and $T(i - j)$. So by question 6.5.4 above, $|T(i)| = |T(j)|$.
Now, T is a scalar multiple of the linear operator S, where $S(x) = \frac{T(x)}{|T(i)|}$. By the first representation theorem, and using the fact $|T(i)| = |T(j)|$, we get that $A_S = \begin{bmatrix} \frac{T(i)}{|T(i)|} & \frac{T(j)}{|T(j)|} \end{bmatrix}$, where A_S is the 2×2 matrix representing S. Since $T(i) \perp T(j)$, we calculate that $A_S^{tr} A = I_2$. By definition A_S is an orthogonal matrix, and so by Theorem 2 on page 376, S is an isometry. So T is a scalar multiple of the isometry S.

Question 6: 6.5.11 Calculate the orthogonal matrix associated with a rotation of $\mathbb{R}^3$ of θ degrees about the z axis.

Solution Let R_θ be the linear operator that is a rotation of $\mathbb{R}^3$ by θ degrees about the z axis. Check that $R_\theta\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} \cos\theta \\ \sin\theta \\ 0 \end{bmatrix}$, $R_\theta\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} -\sin\theta \\ \cos\theta \\ 0 \end{bmatrix}$, $R_\theta\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$. Hence by the First Representation Theorem, the matrix associated with R_θ is $\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$.
To see that this matrix (call it A) is orthogonal, either calculate that $A^T A = I_3$, or use Theorem 2 on page 376 and the fact that R_θ is an isometry.

Question 7: 6.5.13 Show that any unitary 2×2 matrix of determinant 1 is of the form $\begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix}$ where $|a|^2 + |b|^2 = 1$.

Solution Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, with $a, b, c, d \in \mathbb{C}$, be an arbitrary unitary 2×2 matrix of determinant 1. Then $A^*A = I_2 \Rightarrow \begin{bmatrix} \bar{a} & \bar{c} \\ \bar{b} & \bar{d} \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, giving us the four equations
(1) $|a|^2 + |c|^2 = 1$
(2) $a\bar{b} + c\bar{d} = 0$
(3) $\bar{a}b + \bar{c}d = 0$
(4) $|b|^2 + |d|^2 = 1$
Since det(A)=1, we get a fifth equation (5) $ad - bc = 1$.
Case 1: $a = 0$. Then equation (1) implies $|c|^2 = 1$, so in polar form $c = e^{i\theta}$. Plugging $a = 0$ and $c = e^{i\theta}$ into equation (2) implies $d = 0$, and into equation (5) implies $b = -e^{-i\theta} = -\bar{c}$, or $c = -\bar{b}$. Hence equation (1) implies $|a|^2 + |b|^2 = 1$ and $A = \begin{bmatrix} 0 & b \\ -\bar{b} & 0 \end{bmatrix}$ is of the correct form.
Case 2: $a \neq 0$. Then we can divide by $\bar{a}$, so (3) gives us $b = \frac{-\bar{c}d}{\bar{a}}$. Plug this into (5) to get $ad + \frac{\bar{c}d}{\bar{a}}c = 1$. Multiply both sides by $\bar{a}$ to get $|a|^2 d + |c|^2 d = \bar{a}$. Equation (1) then implies $d = \bar{a}$. Equation (2) then implies $c = -\bar{b}$. Hence equation (1) imples $|a|^2 + |b|^2 = 1$ and $A = \begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix}$ is of the correct form.

Question 8: 7.1.2(b) Determine the eigenvalues and corresponding eigenspaces of the matrix $\begin{bmatrix} -5 & 3 & 0 \\ -6 & 4 & 2 \\ 2 & -1 & 1 \end{bmatrix}$.

Solution Call the above matrix A. By Theorem 2(b) on page 386, the eigenvalues of A are the solutions λ to $det(\lambda I - A) = 0$. We solve: $det(\lambda I - A) = det\left(\begin{bmatrix} \lambda+5 & -3 & 0 \\ 6 & \lambda-4 & -2 \\ -2 & 1 & \lambda-1 \end{bmatrix}\right) = \lambda^3 - \lambda = \lambda(\lambda^2 - 1) = \lambda(\lambda - 1)(\lambda + 1)$. So the matrix A has three eigenvalues: $\lambda = -1, 0, 1$.
The eigenspace corresponding to an eigenvalue λ is $ker(\lambda I - A)$; see page 382.
The eigenspace corresponding to $\lambda = -1$ is $ker(-I - A) = sp\left\{\begin{bmatrix} -3 \\ -4 \\ 1 \end{bmatrix}\right\}$ since the reduced row echelon form of $-I - A$ is $\begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & 0 \end{bmatrix}$.
The eigenspace corresponding to $\lambda = 0$ is $ker(-A) = sp\left\{\begin{bmatrix} -3 \\ -5 \\ 1 \end{bmatrix}\right\}$ since the

reduced row echelon form of $-A$ is $\begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 5 \\ 0 & 0 & 0 \end{bmatrix}$.

The eigenspace corresponding to $\lambda = 1$ is $ker(I - A) = sp \left\{ \begin{bmatrix} \frac{1}{2} \\ 1 \\ 0 \end{bmatrix} \right\}$ since the reduced row echelon form of $I - A$ is $\begin{bmatrix} 1 & -\frac{1}{2} & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$.

Question 9: 7.1.6(c) For the matrix $A = \begin{bmatrix} 3 & 2 & -2 \\ 4 & 1 & -2 \\ 8 & 4 & -5 \end{bmatrix}$, find a matrix B such that $B^{-1}AB$ is diagonal.

Solution We proceed as in Example 3, page 385. We find the eigenvalues of A by solving $0 = det(\lambda I - A) = det \left(\begin{bmatrix} \lambda - 3 & -2 & 2 \\ -4 & \lambda - 1 & 2 \\ -8 & -4 & \lambda + 5 \end{bmatrix} \right) = \lambda^3 + \lambda^2 - \lambda - 1$. To factor this cubic polynomial, we try to find a root: we plug in small integers (0, ± 1, ± 2, etc) until we find that 1 is a root. This means that $\lambda - 1$ is a factor. Performing long division, we find the factorization $\lambda^3 + \lambda^2 - \lambda - 1 = (\lambda - 1)(\lambda^2 + 2\lambda + 1) = (\lambda - 1)(\lambda + 1)^2$. So the eigenvalues of A are $\lambda = -1, 1$. The eigenspace corresponding to $\lambda = -1$ is $ker(-I - A) = sp \left\{ \begin{bmatrix} -\frac{1}{2} \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} \frac{1}{2} \\ 0 \\ 1 \end{bmatrix} \right\}$ since the reduced row echelon form of $-I - A$ is $\begin{bmatrix} 1 & \frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. The eigenspace corresponding to $\lambda = 1$ is $ker(I - A) = sp \left\{ \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 1 \end{bmatrix} \right\}$ since the reduced row echelon form of $I - A$ is $\begin{bmatrix} 1 & 0 & -\frac{1}{2} \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 0 \end{bmatrix}$. The eigenvectors $\begin{bmatrix} -\frac{1}{2} \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} \frac{1}{2} \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 1 \end{bmatrix}$ are linearly independent, and hence form a basis for $\mathbb{R}^3$. So the change of coordinate matrix $B = \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 1 & 0 & \frac{1}{2} \\ 0 & 1 & 1 \end{bmatrix}$ satisfies $B^{-1}AB = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, a diagonal matrix whose diagonal entries are the corresponding eigenvalues.

Question 10: 7.1.8(c) Find a general formula for the nth power of the matrix $\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$.

Solution This problem would be easy if we were given a diagonal matrix, as

$\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}^n = \begin{bmatrix} a^n & 0 \\ 0 & b^n \end{bmatrix}$. This motivates us check if the matrix $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ is at least diagonalizable.

We find the eigenvalues of A by solving $0 = det(\lambda I - A) = det\left(\begin{bmatrix} \lambda-2 & -1 \\ -1 & \lambda-2 \end{bmatrix}\right) = \lambda^2 - 4\lambda + 3 = (\lambda-1)(\lambda-3)$. So the eigenvalues of A are $\lambda = 1, 3$. The eigenspace corresponding to $\lambda = 1$ is $ker(I - A) = sp\left\{\begin{bmatrix} -1 \\ 1 \end{bmatrix}\right\}$ since the reduced row echelon form of $I - A$ is $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$. The eigenspace corresponding to $\lambda = 3$ is $ker(I - A) = sp\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}$ since the reduced row echelon form of $3I - A$ is $\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$. The eigenvectors $\begin{bmatrix} -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ are linearly independent, and hence form a basis for $\mathbb{R}^2$. So the change of coordinate matrix $B = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$ satisfies $B^{-1}AB = C$, where $C = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$ is the diagonal matrix whose diagonal entries are the corresponding eigenvalues. Equivalently, $BCB^{-1} = A$. Now, we can solve the problem: $A^n = (BCB^{-1})^n = (BCB^{-1})(BCB^{-1})...(BCB^{-1}) = BC^nB^{-1}$ since $B^{-1}B = I$. Hence, $A^n = BC^nB^{-1} = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}^n\begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1^n & 0 \\ 0 & 3^n \end{bmatrix}\begin{bmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 3^n+1 & 3^n-1 \\ 3^n-1 & 3^n+1 \end{bmatrix}$. This is a general formula for the nth power of the matrix A.