Ma 1c Practical - Solutions to Homework Set 7

All exercises are from the Vector Calculus text, Marsden and Tromba (Fifth Edition)

Exercise 7.4.6. *Find the area of the portion of the unit sphere that is cut out by the cone*

$$z \geq \sqrt{x^2 + y^2}.$$

Solution. The intersection of the unit sphere and the cone $z = \sqrt{x^2 + y^2}$ is found by solving the equations

$$x^2 + y^2 + z^2 = 1 \quad \text{and} \quad x^2 + y^2 - z^2 = 0$$

(with $z \geq 0$), which is easily done by subtracting these two equations. This gives the circle described by $z = 1/\sqrt{2}$ and $x^2 + y^2 = 1/2$, as in Figure 1. We are to find the area of the surface above this circle.

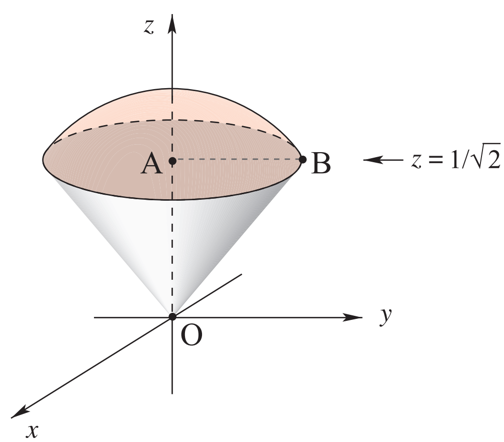

FIGURE 1. Find the area of the "ice cream" part of this surface.

Notice that the triangle AOB has two sides of length $1/\sqrt{2}$, and hypotenuse of length 1, so the vertex angle AOB is $\pi/4$. Using this geometry and spherical coordinates, we find that a parametrization is

$$\begin{aligned} x &= \sin\phi\cos\theta \\ y &= \sin\phi\sin\theta \\ z &= \cos\phi, \end{aligned}$$

for $0 \leq \theta \leq 2\pi, 0 \leq \phi \leq \frac{\pi}{4}$. We find that

$$\begin{aligned} \text{Area} &= \int_0^{2\pi} \int_0^{\frac{\pi}{4}} \sin\phi \, d\phi \, d\theta \\ &= \left(1 - \frac{\sqrt{2}}{2}\right) 2\pi \\ &= (2 - \sqrt{2})\pi. \quad \Diamond \end{aligned}$$

Exercise 7.5.2. Evaluate

$$\iint_S xyz\, dS$$

where S is the triangle with vertices $(1,0,0),(0,2,0)$ and $(0,1,1)$.

Solution. The triangle is contained in a plane whose equation is of the form $ax + by + cz + d = 0$. Since $(1,0,0)$ lies on it, $a + d = 0$, so $a = -d$. Since $(0,2,0)$ is on it, $b = -\frac{1}{2}d$. Since $(0,1,1)$ is on it, $b + c = -d$, so $c = -d + \frac{1}{2}d = -\frac{1}{2}d$. Letting $d = -2$, we get $2x + y + z - 2 = 0$ *i.e.,* the equation of the plane is given by

$$2x + y + z = 2.$$

See Figure 7.5.2.

A normal vector is obtained from the coefficients as $(2,1,1)$, so a unit normal is

$$\mathbf{n} = \frac{1}{\sqrt{6}}(2\mathbf{i} + \mathbf{j} + \mathbf{k}).$$

The domain D in the xy plane is the triangle with vertices $(1,0),(0,2)$ and $(0,1)$, as in Figure 2.

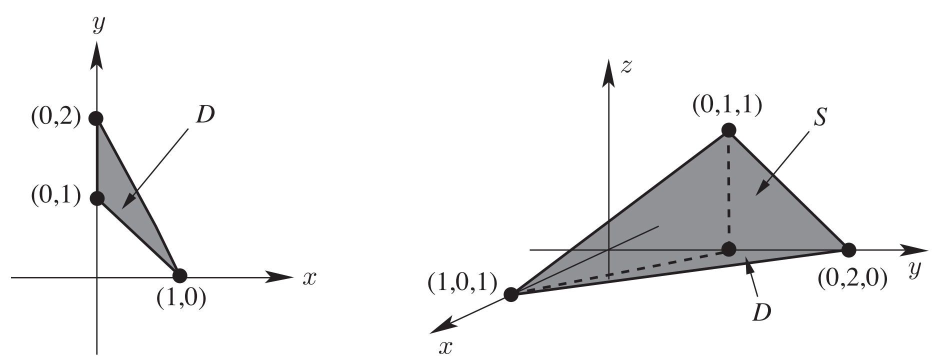

FIGURE 2. The domain D in the xy-plane for plane in Exercise 2 regarded as a graph: $z = 2 - 2x - y$.

Now

$$dS = \frac{dx\,dy}{\mathbf{n}\cdot\mathbf{k}} = \sqrt{6}\,dx\,dy,$$

and so

$$\begin{aligned} \iint_S f\,dS &= \iint_D xy(2 - 2x - y)\sqrt{6}\,dx\,dy \\ &= \sqrt{6}\int_0^1 \int_{1-x}^{2(1-x)} [2(x - x^2)y - xy^2]dy\,dx \end{aligned}$$

Carrying out the y-integration gives

$$\begin{aligned}\iint_S f\,dS &= \sqrt{6}\int_0^1 \left(2(x-x^2)\frac{y^2}{2} - \frac{xy^3}{3}\right)\Bigg|_{1-x}^{2(1-x)} dx \\ &= \sqrt{6}\int_0^1 \Bigg[2x(1-x)\left(\frac{[2(1-x)]^2}{2} - \frac{[1-x]^2}{2}\right) \\ &\quad -\frac{x}{3}([2(1-x)]^3 - (1-x)^3)\Bigg]\,dx \\ &= \sqrt{6}\int_0^1 \frac{2}{3}x(1-x)^3 dx = \sqrt{6}\int_0^1 \frac{2}{3}\cdot\frac{1}{4}(1-x)^4 dx = \frac{\sqrt{6}}{30},\end{aligned}$$

where the last steps were done using integration by parts. ◊

Exercise 7.6.3. *Let S be the closed surface that consists of the hemisphere $x^2 + y^2 + z^2 = 1, z \geq 0$, and its base $x^2 + y^2 \leq 1, z = 0$. Let $\mathbf{E}$ be the electric field defined by $\mathbf{E}(x, y, z) = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}$. Find the electric flux across S.*

Solution. Write $S = H \cup D$ where H is the upper hemisphere and D is the disk. Hence

$$\iint_S \mathbf{E}\cdot d\mathbf{S} = \iint_H \mathbf{E}\cdot d\mathbf{S} + \iint_D \mathbf{E}\cdot d\mathbf{S}.$$

(i) Let $x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ be the unit normal $\mathbf{n}$ pointing outward from H. Then

$$\begin{aligned}\iint_H \mathbf{E}\cdot d\mathbf{S} &= \iint_H \mathbf{E}\cdot\mathbf{n}\,dS = \iint_H (2x, 2y, 2z)\cdot(x, y, z)dS \\ &= 2\iint_H (x^2+y^2+z^2)dS = 2\iint_H dS = 4\pi.\end{aligned}$$

(ii) The unit normal is $-\mathbf{k}$ and $z = 0$ on D. Hence,

$$\iint_D \mathbf{E}\cdot d\mathbf{S} = \iint_D \mathbf{E}\cdot\mathbf{n}\,dS = \iint_D (2x, 2y, 2z)\cdot(0, 0, -1)dS = 0.$$

Therefore,

$$\iint_S \mathbf{E}\cdot d\mathbf{S} = 4\pi. \quad \Diamond$$

Exercise 7.6.15. *Let the velocity field of a fluid be given by $\mathbf{v} = \mathbf{i} + x\mathbf{j} + z\mathbf{k}$ in meters/second. How many cubic meters of fluid per second are crossing the surface $x^2 + y^2 + z^2 = 1, z \geq 0$? (Distances are in meters.)*

Solution. Here, $\mathbf{v}\cdot d\mathbf{S} = \mathbf{v}\cdot\mathbf{n}\,dS$ and $\mathbf{n} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, so

$$\mathbf{v}\cdot\mathbf{n} = x + xy + z^2.$$

By symmetry, the integrals of x and of xy vanish. Thus, the flux is

$$\iint_S \mathbf{v}\cdot d\mathbf{S} = \iint_S z^2 dS.$$

Using spherical coordinates, $z = \cos\phi$, so we get

$$\int_{\phi=0}^{\pi/2}\int_{\theta=0}^{2\pi} \cos^2\phi\sin\phi\,d\theta\,d\phi = -2\pi\,\frac{\cos^3\phi}{3}\Bigg|_0^{\pi/2} = \frac{2\pi}{3}. \quad \Diamond$$

Exercise 8.1.3(d). *Verify Green's theorem for the disk D with center $(0,0)$ and radius R for $P = 2y, Q = x$.*

Solution. Green's theorem

$$\int_{\partial D} P dx + Q dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx\, dy$$

becomes

$$\int_{\partial D} 2y\, dx + x\, dy = \iint_D (1-2) dx\, dy = -\iint_D dx\, dy$$

The right side is $-\pi R^2$ while the left side is, since $x = R\cos\theta$ and $y = R\sin\theta$,

$$\int_0^{2\pi} (2R\sin\theta)(-R\sin\theta)d\theta + (R\cos\theta)(R\cos\theta)d\theta$$

$$= -2R^2 \int_0^{2\pi} \sin^2\theta d\theta + R^2 \int_0^{2\pi} \cos^2\theta d\theta.$$

Using the fact that $\sin^2\theta$ and $\cos^2\theta$ have averages $\frac{1}{2}$, namely

$$\frac{1}{2\pi} \int_0^{2\pi} \sin^2\theta d\theta = \frac{1}{2}$$

(this is one way of remembering the formula for the integrals of $\sin^2\theta$ and $\cos^2\theta$ on $[0, 2\pi]$ and $[0, \pi]$), we get $-2R^2 \cdot \pi + R^2 \cdot \pi = -\pi R^2$. Thus, Green's theorem checks. $\diamondsuit$

Exercise 8.2.10. *Find the surface integral* $\iint_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S}$, *where S is the ellipsoid $x^2 + y^2 + 2z^2 = 10$ and $\mathbf{F} = (\sin xy)\mathbf{i} + e^x\mathbf{j} - yz\mathbf{k}$.*

Solution. Notice that the ellipsoid S is a closed surface and has no boundary. Therefore, by Stokes' theorem,

$$\iint_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S} = \int_{\partial S} \mathbf{F} \cdot d\mathbf{S} = 0. \quad \diamondsuit$$

Note: The same conclusion also follows from the divergence theorem since $\operatorname{div}\operatorname{curl}\mathbf{F} = 0$.

Exercise 8.2.23. *Let $\mathbf{F} = x^2\mathbf{i} + (2xy + x)\mathbf{j} + z\mathbf{k}$. Let C be the circle $x^2 + y^2 = 1$ in the plane $z = 0$ oriented counterclockwise and S the disk $x^2 + y^2 \le 1$ oriented with the normal vector $\mathbf{k}$. Determine:*

(a) *The integral of $\mathbf{F}$ over S.*
(b) *The circulation of $\mathbf{F}$ around C.*
(c) *Find the integral of $\nabla \times \mathbf{F}$ over S. Verify Stokes' theorem directly in this case.*

Solution.

(a) Notice that $F = (x^2, 2xy + x, 0)$ on S. Hence

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_S (x^2, 2xy + x, 0) \cdot (0,0,1)\, dS = 0.$$

(b) Let $c(t) = (\cos t, \sin t, 0)$ be the parameterization of C. Then

$$\begin{aligned}\int_C \mathbf{F} \cdot d\mathbf{S} &= \int_0^{2\pi} (\cos^2 t, 2\cos t \sin t + \cos t, 0) \cdot (-\sin t, \cos t, 0) dt \\ &= \int_0^{2\pi} (\cos^2 t \sin t + \cos^2 t) dt = \pi.\end{aligned}$$

(c) Routine computation shows that $\nabla \times \mathbf{F} = (0, 0, 2y + 1)$. Hence

$$\begin{aligned}\iint_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S} &= \int_0^1 \int_0^{2\pi} (0, 0, 2r \sin\theta + 1) \cdot (0, 0, 1) r \, d\theta \, dr \\ &= \int_0^1 \int_0^{2\pi} (2r \sin\theta + 1) r \, d\theta \, dr = \pi.\end{aligned}$$

Combining the results in (b) and (c), Stokes' theorem is verified. ◊

Exercise 8.3.14. *Determine which of the following vector fields* $\mathbf{F}$ *in the plane is the gradient of a scalar function* f*. If such an* f *exists, find it.*

(a) $\mathbf{F}(x, y) = (\cos xy - xy \sin xy)\mathbf{i} - (x^2 \sin xy)\mathbf{j}$

(b) $\mathbf{F}(x, y) = (x\sqrt{x^2y^2 + 1})\mathbf{i} + (y\sqrt{x^2y^2 + 1})\mathbf{j}$

(c) $\mathbf{F}(x, y) = (2x \cos y + \cos y)\mathbf{i} - (x^2 \sin y + x \sin y)\mathbf{j}$.

Solution. In this problem, we apply the cross-derivative test. For example, for problem (a),

$$\frac{\partial F_1}{\partial y} - \frac{\partial F_2}{\partial x} = (x \sin xy - x \sin xy - x^2 y \cos xy) - (-2x \sin xy - x^2 y \cos xy) = 0,$$

so $\mathbf{F}$ is indeed the gradient of some function on the plane. To find such a function, we seek f satisfying

$$\frac{\partial f}{\partial y} = F_2 = x^2 \sin xy,$$

for example, $f(x, y) = x \cos xy$. Note that f is unique only up to an additive that could be a function of x. However, we don't need to add it in this case as this function is checked to have gradient the given vector field.

Part (b) and (c) proceed similarly. One sees that (b) is not a gradient field, while (c) is a gradient. For part (c), $f(x, y) = x^2 \cos y + x \cos y$ is a function whose gradient is the given field.

Exercise 8.4.10. *Evaluate the surface integral* $\iint_S \mathbf{F} \cdot \mathbf{n} \, dS$*, where*

$$\mathbf{F}(x, y, z) = \mathbf{i} + \mathbf{j} + z(x^2 + y^2)^2 \mathbf{k}$$

and S *is the surface of the cylinder* $x^2 + y^2 \le 1, 0 \le z \le 1$*, including the sides and both lids.*

Solution. Use Gauss' divergence theorem in space:

$$\iint_S \mathbf{F} \cdot \mathbf{n} \, dS = \iiint_W (\operatorname{div} \mathbf{F}) dx \, dy \, dz$$

Here,

$$\operatorname{div} \mathbf{F} = \frac{\partial}{\partial x}(1) + \frac{\partial}{\partial y}(1) + \frac{\partial}{\partial z} z(x^2 + y^2)^2 = (x^2 + y^2)^2.$$

The region W is a cylinder, so it is the easiest to evaluate the integral in cylindrical coordinates:

$$\int_0^1 \int_0^{2\pi} \int_0^1 r \cdot (r^2)^2 \, dr \, d\theta \, dz = \frac{2\pi}{6} = \frac{\pi}{3}. \quad \diamondsuit$$

Exercise 8.4.14. *Fix k vectors $\mathbf{v}_1, \dots, \mathbf{v}_k$ in space and numbers ("charges") $q_1, \dots, q_k$. Define*

$$\phi(x, y, z) = \sum_{i=1}^{k} \frac{q_i}{4\pi \|\mathbf{r} - \mathbf{v}_i\|},$$

where $\mathbf{r} = (x, y, z)$. Show that for a closed surface S and $\mathbf{e} = -\nabla\phi$,

$$\iint_S \mathbf{e} \cdot d\mathbf{S} = Q,$$

where $Q = q_1 + \cdots + q_k$ is the total charge inside S. Assume that none of the charges are on S.

Solution. Surround each charge at vector $\mathbf{v}_i$ by a small ball B_i in such a way that the B_i are mutually disjoint and do not intersect S. Assume that $B_1, \dots, B_n$, (where $n \leq k$) are those balls contained within S. Then since div $\mathbf{e} = 0$, and as in Theorem 10,

$$\iint_S \mathbf{e} \cdot d\mathbf{S} = \sum_{i=1}^{n} \iint_{\partial B_i} \mathbf{e} \cdot d\mathbf{S}$$

where ∂B_i is given the outward orientation. But again, as in Theorem 10,

$$\iint_{\partial B_i} \mathbf{e} \cdot d\mathbf{S} = q_i.$$

Thus,

$$\iint_S \mathbf{e} \cdot d\mathbf{S} = \sum_{i=1}^{n} q_i = Q,$$

the total charge inside S. $\diamondsuit$