Ma 1c Practical - Solutions to Homework Set 6

All exercises are from the Vector Calculus text, Marsden and Tromba (Fifth Edition)

Exercise 5.5.4 *Evaluate*

$$\iiint_B ze^{x+y}\,dx\,dy\,dz$$

where $B = [0,1]\times[0,1]\times[0,1]$.

Solution.

$$\begin{aligned}\iiint_B ze^{x+y}\,dx\,dy\,dz &= \int_0^1 dx\int_0^1 dy\left(\int_0^1 ze^x\cdot e^y dz\right)\\ &= \int_0^1 e^x dx\int_0^1 e^y dy\int_0^1 zdz\\ &= \left(\frac{1}{2}z^2\right)\Big|_0^1 (e-1)^2 = \frac{(e-1)^2}{2}.\quad\diamondsuit\end{aligned}$$

Exercise 5.5.15 *Evaluate*

$$\iiint_W (x^2+y^2+z^2)dx\,dy\,dz,$$

where W *is the region bounded by* $x+y+z=a$ *(where* $a>0$*),* $x=0, y=0,$ *and* $z=0$.

Solution. The set W is shown in the figure.

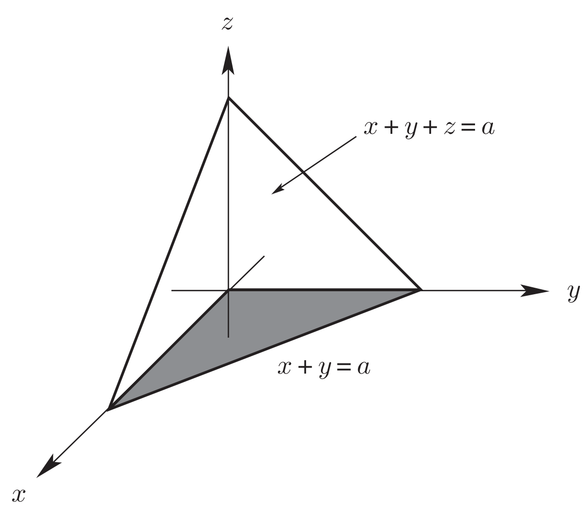

Figure 1: The region of integration for Exercise 5.5.15.

Note that W is defined by the inequalities

$$x\geq 0,\quad y\geq 0,\quad z\geq 0,\quad \text{and}\quad 0\leq x+y+z\leq a.$$

Thus,

$$
\begin{aligned}
&\iiint_W (x^2+y^2+z^2)dx\,dy\,dz \\
&\quad = \int_0^a dx \int_0^{a-x} dy \int_0^{a-x-y} dz(x^2+y^2+z^2) \\
&\quad = \int_0^a dx \left(\int_0^{a-x} \left[(x^2+y^2)(a-x-y) + \frac{1}{3}(a-x-y)^3 \right] dy \right) \\
&\quad = \int_0^a \left[x^2(a-x)^2 + \frac{1}{12}(a-x)^4 - \frac{1}{2}x^2(a-x)^2 + \frac{1}{12}(a-x)^4 \right] dx \\
&\quad = \frac{1}{2}\int_0^a (a^2x^2 - 2ax^3 + x^4)dx + \frac{1}{60}(x-a)^5\Big|_0^a + \frac{1}{60}(x-a)^5\Big|_0^a \\
&\quad = \frac{1}{6}a^5 - \frac{1}{4}a^5 + \frac{1}{10}a^5 + \frac{1}{60}a^5 + \frac{1}{60}a^5 = \frac{1}{20}a^5. \quad \Diamond
\end{aligned}
$$

Exercise 6.1.6 *Let D^* be the parallelogram with vertices*

$$(-1,3), \quad (0,0), \quad (2,-1) \quad \textit{and} \quad (1,2)$$

and D be the rectangle $D = [0,1]\times[0,1]$. Find a transformation T such that D is the image set of D^* under T.

Solution. We will seek a linear mapping T with $T(D^*) = D$. We look for $T(u,v) = (x,y)$ where

$$x = au + bv \quad \text{and} \quad y = cu + dv.$$

We require vertices to be mapped to vertices in the same clockwise order and observe that we already have $T(0,0) = (0,0)$. Thus, we suppose $T(1,2) = (1,1), T(-1,3) = (1,0)$ and $T(2,-1) = (0,1)$. This gives us three sets of equations

$$
\begin{aligned}
1 &= a + 2b \quad \text{and} \quad 1 = c + 2d \\
1 &= -a + 3b \quad \text{and} \quad 0 = -c + 3d \\
0 &= 2a - b \quad \text{and} \quad 1 = 2c - d.
\end{aligned}
$$

Using the last line, $b = 2a$ and so from the first equation, we find $a = 1/5, b = 2/5$ and similarly from the second line, $c = 3d$ and so from one of the other two equations for c, d, we get $c = 3/5, d = 1/5$. Therefore, $T(u,v) = (u+2v, 3u+v)/5$. $\Diamond$

Exercise 6.2.8 *Calculate*

$$\iint_R \frac{dx\,dy}{x+y},$$

where R is the region bounded by $x = 0, y = 0, x+y = 1,$ and $x+y = 4$ by using the mapping $T(u,v) = (u-uv, uv)$.

Solution. The region D, bounded by $x = 0, y = 0, x + y = 1$ and $x + y = 4$, is a quadrilateral. If $x = u - uv$ and $y = uv$, then $u = x + y$, and $v = y/(x + y)$. Therefore, away from the v axis ($u = 0$), the mapping is one to one. (Note that any portion of the v axis is mapped to $(0, 0)$.) The pre-image of the interval $y = 0, 1 \leq x \leq 4$ is the interval $1 \leq u \leq 4, v = 0$. The pre-image of the line $x+y = 4$, where $0 \leq x \leq 4$ and $0 \leq y \leq 4$, is the line described by $u = 4, 0 \leq v \leq 1$. The pre-image of the line $x+y = 1, 0 \leq x \leq 1, 0 \leq y \leq 1$ is the line $u = 1, 0 \leq v \leq 1$ and finally, the pre-image of the interval $x = 0, 1 \leq y \leq 4$, is the line $v = 1, 1 \leq u \leq 4$. Thus, D^* is the rectangle $[1, 4] \times [0, 1]$ in the uv plane, and $T : D^* \to D$ is one to one. The Jacobian determinant is given by

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} 1 - v & -u \\ v & u \end{vmatrix} = u.$$

Therefore,

$$\iint_R \frac{dx\, dy}{x + y} = \iint_{D^*} \frac{u}{u} du\, dv = \int_1^4 \int_0^1 dv\, du = 3. \quad \Diamond$$

Exercise 6.3.5 *A sculptured gold plate D is defined by $0 \leq x \leq 2\pi$ and $0 \leq y \leq \pi$ (centimeters) and has mass density $\delta(x, y) = y^2 \sin^2 4x + 2$ (grams per cm^2). If gold sells for \$7 per gram, how much is the gold in the plate worth?*

Solution. The mass of the gold plate is given by

$$\begin{aligned} \iint_D (y^2 \sin^2 4x + 2) dx\, dy &= \int_0^\pi \int_0^{2\pi} (y^2 \sin^2 4x + 2) dx\, dy \\ &= \int_0^\pi y^2\, dy \int_0^{2\pi} \sin^2 4x\, dx + 2\pi \cdot 2\pi \\ &= \frac{\pi^3}{3} \cdot \frac{1}{2} 2\pi + 4\pi^2 = \frac{\pi^4}{3} + 4\pi^2 \quad \text{(grams)}. \end{aligned}$$

Thus, the price of the plate is $7 \cdot (4\pi^2 + \frac{\pi^4}{3}) \approx \$503.64$. $\quad \Diamond$

Exercise 7.1.4(a) *Evaluate the path integral of $f(x, y, z) = x \cos z$ along the path $\mathbf{c} : t \mapsto t\mathbf{i} + t^2\mathbf{j}, t \in [0, 1]$.*

Solution. The path integral is

$$\begin{aligned} \int_{\mathbf{c}} f\, ds &= \int_0^1 (t \cos 0) \sqrt{1 + 4t^2} dt \\ &= \int_0^1 t(1 + 4t^2)^{1/2}\, dt \end{aligned}$$

This is readily integrated using the substitution $u = 1 + 4t^2$ and we get

$$\frac{1}{12} \left[\left(1 + 4t^2\right)^{3/2} \right] \Big|_0^1 = \frac{1}{12} \left(5^{3/2} - 1\right). \quad \Diamond$$

Exercise 7.2.2 *Evaluate each of the following integrals:*

(a) $\displaystyle\int_{\mathbf{c}} x\,dy - y\,dx, \quad \mathbf{c}(t) = (\cos t, \sin t), \quad 0 \le t \le 2\pi$

(b) $\displaystyle\int_{\mathbf{c}} x\,dx + y\,dy, \quad \mathbf{c}(t) = (\cos \pi t, \sin \pi t), \quad 0 \le t \le 2$

Solution. (a) By definition,

$$\begin{aligned}\int_{\mathbf{c}} x\,dy - y\,dx &= \int_0^{2\pi} [\cos t(\cos t\,dt) - (\sin t)(-\sin t\,dt)] \\ &= \int_0^{2\pi} \left[\cos^2 t + \sin^2 t\right] dt = \int_0^{2\pi} dt = 2\pi.\end{aligned}$$

(b) Here,

$$\begin{aligned}\int_{\mathbf{c}} x\,dx + y\,dy &= \int_0^2 (\cos \pi t)(-\pi \sin \pi t)dt + \int_0^2 (\sin \pi t)(\pi \cos \pi t)dt \\ &= \left[\frac{\cos^2 \pi t}{2}\right]\Bigg|_0^2 + \left[\frac{\sin^2 \pi t}{2}\right]\Bigg|_0^2 = \frac{1}{2}\left[\cos^2 \pi t + \sin^2 \pi t\right]\Big|_0^2 \\ &= \frac{1}{2}\,[1]|_0^2 = 0.\end{aligned}$$

Exercise 7.2.18 *A cyclist rides up a mountain along the path shown in* Figure 7.2.16 *in the text. She makes one complete revolution around the mountain in reaching the top, while her vertical rate of climb is constant. Throughout the trip, she exerts a force described by the vector field*

$$\mathbf{F}(x, y, z) = y\mathbf{i} + x\mathbf{j} + \mathbf{k}.$$

What is the work done by the cyclist in travelling from A *to* B*? What is unrealistic about this model of a cyclist?*

Solution. Notice that $\mathbf{F} = \nabla f$, where

$$f(x, y, z) = xy + z.$$

Also, $\mathrm{A} = (\sqrt{2\pi}, 0, 0)$ and $\mathrm{B} = (0, 0, 2\pi)$. Thus the work is

$$f(\mathrm{B}) - f(\mathrm{A}) = 2\pi.$$

The last question is of course subject to interpretation—it does challenge the students to think about modeling. One possible answer is this. The path followed is

$$\mathbf{c}(t) = \left(\sqrt{(2\pi - t)}\cos t, \sqrt{(2\pi - t)}\sin t, t\right)$$

for $0 \le t \le 2\pi$. Near $t = 2\pi$, the speed $\|\mathbf{c}'(t)\|$ approaches infinity. This is clearly not possible for a real bicycle rider. $\Diamond$

Exercise 7.3.2 *Find an equation for the tangent plane to the surface* $x = u^2 - v^2, y = u + v, z = u^2 + 4v$ *at* $\left(-\frac{1}{4}, \frac{1}{2}, 2\right)$.

Solution. For this surface,

$$\begin{aligned} \mathbf{T}_u &= (2u)\mathbf{i} + \mathbf{j} + (2u)\mathbf{k} \\ \mathbf{T}_v &= (-2v)\mathbf{i} + \mathbf{j} + 4\mathbf{k}. \end{aligned}$$

To calculate these vectors explicitly at $\left(-\frac{1}{4}, \frac{1}{2}, 2\right)$ we must find the point(s) in the uv plane corresponding to the point $\left(-\frac{1}{4}, \frac{1}{2}, 2\right)$ on the surface. To find (u, v) we must solve

$$\begin{aligned} u^2 - v^2 &= -\frac{1}{4} \\ u + v &= \frac{1}{2} \\ u^2 + 4v &= 2. \end{aligned}$$

Subtracting the first equation from the last we obtain

$$v^2 + 4v = \frac{9}{4}$$

i.e.,

$$4v^2 + 16v - 9 = 0.$$

By the quadratic formula the two possible solutions for v are:

$$\begin{aligned} \frac{-16 \pm \sqrt{16 \cdot 16 + 16 \cdot 9}}{8} &= \frac{-16 \pm 20}{8} \\ &= \frac{1}{2} \quad \text{or} \quad -4\frac{1}{2}. \end{aligned}$$

The solution for u corresponding to $v = \frac{1}{2}$ is $u = 0$. One readily checks that the image of the point $\left(0, \frac{1}{2}\right)$ is, in fact the point $\left(-\frac{1}{4}, \frac{1}{2}, 2\right)$ on the surface. The other possible value of v does not yield a value of u that solves all three equations. A normal to the surface is given by

$$\begin{aligned} \mathbf{T}_u \times \mathbf{T}_v &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2u & 1 & 2u \\ -2v & 1 & 4 \end{vmatrix} \\ &= (4 - 2u)\mathbf{i} + (-8u - 4uv)\mathbf{j} + (2u + 2v)\mathbf{k}. \end{aligned}$$

At $\left(0, \frac{1}{2}\right) = (u, v)$,

$$\mathbf{T}_u \times \mathbf{T}_v = 4\mathbf{i} + \mathbf{k} = \mathbf{n}.$$

Since the equation of a tangent plane at the point (x_0, y_0, z_0), where the normal vector $\mathbf{n}$ to the surface is known, is given by

$$\mathbf{n} \cdot (x - x_0, y - y_0, z - z_0) = 0$$

the equation of the desired tangent plane is

$$4\left(x + \frac{1}{4}\right) + (z - 2) = 0$$

i.e.,

$$4x + z - 1 = 0. \quad \Diamond$$

Exercise 7.3.6 *Find an expression for a unit vector normal to the surface*

$$x = 3\cos\theta\sin\phi, \quad y = 2\sin\theta\sin\phi, \quad z = \cos\phi$$

for θ *in* $[0, 2\pi]$ *and* ϕ *in* $[0, \pi]$.

Solution. Here,

$$\mathbf{T}_\theta = (-3\sin\theta\sin\phi, 2\cos\theta\sin\phi, 0)$$

and

$$\mathbf{T}_\phi = (3\cos\theta\cos\phi, 2\sin\theta\cos\phi, -\sin\phi).$$

Thus,

$$\mathbf{T}_\theta \times \mathbf{T}_\phi = (-2\cos\theta\sin^2\phi, -3\sin\theta\sin^2\phi, -6\sin\phi\cos\phi)$$

and

$$\|\mathbf{T}_\theta \times \mathbf{T}_\phi\| = \sin\phi(5\sin^2\theta\sin^2\phi + 32\cos^2\phi + 4)^{1/2}.$$

Hence a unit normal vector is

$$\begin{aligned}\mathbf{n} &= \frac{\mathbf{T}_\theta \times \mathbf{T}_\phi}{\|\mathbf{T}_\theta \times \mathbf{T}_\phi\|} = \frac{1}{\sin\phi\sqrt{5\sin^2\theta\sin^2\phi + 32\cos^2\phi + 4}} \\ &\quad \times(-2\cos\theta\sin^2\phi, -3\sin\theta\sin^2\phi, -6\sin\phi\cos\phi).\end{aligned}$$

Since

$$\frac{x^2}{9} + \frac{y^2}{4} + z^2 = 1,$$

the surface is an ellipsoid. $\Diamond$