Ma 1c Practical - Solutions to Homework Set 5

All exercises are from the Vector Calculus text, Marsden and Tromba (Fifth Edition)

1. **Exercise 4.1.14:** *Show that, at a local maximum or minimum of the quantity* $\|\mathbf{r}(t)\|$, $\mathbf{r}'(t)$ *is perpendicular to* $\mathbf{r}(t)$.

   Solution: Notice that at the time t where a local maximum or minimum for $\|\mathbf{r}(t)\|$ occurs, a local maximum or minimum for $\|\mathbf{r}(t)\|^2 = \mathbf{r}(t) \cdot \mathbf{r}(t)$ also occurs. And at those particular t's, the first derivative of $\|\mathbf{r}(t)\|^2$ is equal to zero. Therefore

   $$0 = (\mathbf{r}(t) \cdot \mathbf{r}(t))' = \mathbf{r}'(t) \cdot \mathbf{r}(t) + \mathbf{r}(t) \cdot \mathbf{r}'(t) = 2\mathbf{r}'(t) \cdot \mathbf{r}(t),$$

   which means that $\mathbf{r}'(t)$ is perpendicular to $\mathbf{r}(t)$. $\quad\diamond$

2. **Exercise 4.2.4:** *Find the arc length of the curve*

   $$c(t) = \left(t + 1, \frac{2\sqrt{2}}{3}t^{3/2} + 7, \frac{1}{2}t^2\right)$$

   on the interval $1 \leq t \leq 2$.

   Solution: The arc length is

   $$L(c) = \int_1^2 \|\mathbf{c}'(t)\| dt = \int_1^2 \sqrt{1 + 2t + t^2} dt = \int_1^2 (t + 1) dt = \frac{5}{2}. \quad \diamond$$

3. **Exercise 4.2.18:** *In special relativity, the proper time of a path* $\gamma : [a, b] \to \mathbb{R}^4$ *with* $\gamma(\lambda) = (x(\lambda), y(\lambda), z(\lambda), t(\lambda))$ *is defined to be the quantity*

   $$\frac{1}{c}\int_a^b \sqrt{-[x'(\lambda)]^2 - [y'(\lambda)]^2 - [z'(\lambda)]^2 + c^2[t'(\lambda)]^2} d\lambda,$$

   where c *is the velocity of light, a constant. Referring to Figure 1, show that, using self-explanatory notation,*

   $$\text{proper time (AB)} + \text{proper time (BC)} < \text{proper time (AC)}.$$

   (*This inequality is a special case of what is known as the twin paradox.*)

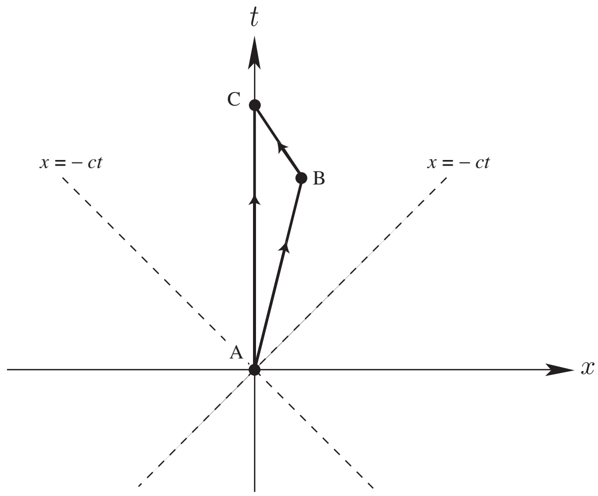

Figure 1: The relativistic triangle inequality.

Solution: We proceed in three steps. First we parametrize the paths.

i Let $\mathrm{A} = (0,0,0,0), \mathrm{B} = (x_{\mathrm{B}}, 0, 0, t_{\mathrm{B}}), \mathrm{C} = (0,0,0,t_{\mathrm{C}})$. Let c_1, c_2, c_3 be the paths from A to B, B to C, A to C, respectively. Then

$$\begin{aligned} c_1(\lambda) &= (1-\lambda)(0,0,0,0) + \lambda(x_{\mathrm{B}},0,0,t_{\mathrm{B}}) \\ c_2(\lambda) &= (1-\lambda)(x_{\mathrm{B}},0,0,t_{\mathrm{B}}) + \lambda(0,0,0,t_{\mathrm{C}}) \\ c_3(\lambda) &= (1-\lambda)(0,0,0,0) + \lambda(0,0,0,t_{\mathrm{C}}) \end{aligned}$$

ii Denote the proper time of AB, BC, AC by T_{AB}, etc., then

$$T_{\mathrm{AB}} = \frac{1}{c}\int_0^1 \sqrt{-x_{\mathrm{B}}^2 + c^2 t_{\mathrm{B}}^2}\, d\lambda = \frac{1}{c}\sqrt{-x_{\mathrm{B}}^2 + c^2 t_{\mathrm{B}}^2}.$$

Similarly, we have

$$\begin{aligned} T_{\mathrm{BC}} &= \frac{1}{c}\sqrt{-x_{\mathrm{B}}^2 + c^2(t_{\mathrm{C}} - t_{\mathrm{B}})^2} \\ T_{\mathrm{AC}} &= \frac{1}{c}\sqrt{c^2 t_{\mathrm{C}}^2} = \frac{1}{c}(ct_{\mathrm{C}}). \end{aligned}$$

iii It suffices to show that

$$\sqrt{-x_{\mathrm{B}}^2 + c^2 t_{\mathrm{B}}^2} + \sqrt{-x_{\mathrm{B}}^2 + c^2(t_{\mathrm{C}} - t_{\mathrm{B}})^2} < ct_{\mathrm{C}}.$$

But the above is true if and only if

$$\sqrt{-x_{\mathrm{B}}^2 + c^2(t_{\mathrm{C}} - t_{\mathrm{B}})^2} < ct_{\mathrm{C}} - \sqrt{-x_{\mathrm{B}}^2 + c^2 t_{\mathrm{B}}^2}$$

if and only if

$$-x_{\mathrm{B}}^2 + c^2(t_{\mathrm{C}} - t_{\mathrm{B}})^2 < c^2 t_{\mathrm{C}}^2 - x_{\mathrm{B}}^2 + c^2 t_{\mathrm{B}}^2 - 2ct_{\mathrm{C}}\sqrt{-x_{\mathrm{B}}^2 + c^2 t_{\mathrm{B}}^2}$$

if and only if

$$ct_{\mathrm{B}} > \sqrt{-x_{\mathrm{B}}^2 + c^2 t_{\mathrm{B}}^2}.$$

The last inequality is true as $x_B^2 > 0$. Thus the proof is complete. $\Diamond$

4. **Exercise 4.3.14:** *Show that the curve*

$$\mathbf{c}(t) = (t^2, 2t-1, \sqrt{t}), t > 0$$

is a flow line of the velocity vector field

$$\mathbf{F}(x, y, z) = (y+1, 2, 1/2z).$$

Solution: We must verify that $\mathbf{c}'(t) = \mathbf{F}(\mathbf{c}(t))$. The left side is $(2t, 2, 1/(2\sqrt{t}))$ while the right side is $\mathbf{F}(t^2, 2t-1, \sqrt{t}) = (2t, 2, 1/(2\sqrt{t}))$. Thus $c(t)$ is a flow line of $\mathbf{F}$. $\Diamond$

5. **Exercise 4.4.16:** *Find the curl of the vector field*

$$\mathbf{F}(x, y, z) = \frac{yz\mathbf{i} - xz\mathbf{j} + xy\mathbf{k}}{x^2 + y^2 + z^2}.$$

Solution: Let $r = \sqrt{x^2 + y^2 + z^2}$. We take the formal cross product of the ∇ operator with the given vector field to calculate the curl:

$$\begin{aligned}
\nabla \times \mathbf{F} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \dfrac{yz}{r^2} & \dfrac{-xz}{r^2} & \dfrac{xy}{r^2} \end{vmatrix} \\
&= \left[\frac{\partial}{\partial y}\left(\frac{xy}{r^2}\right) + \frac{\partial}{\partial z}\left(\frac{xz}{r^2}\right)\right]\mathbf{i} - \left[\frac{\partial}{\partial x}\left(\frac{xy}{r^2}\right) - \frac{\partial}{\partial z}\left(\frac{yz}{r^2}\right)\right]\mathbf{j} \\
&\quad + \left[\frac{\partial}{\partial x}\left(\frac{-xz}{r^2}\right) - \frac{\partial}{\partial y}\left(\frac{yz}{r^2}\right)\right]\mathbf{k} \\
&= \left[\frac{xr^2 - 2xy^2}{r^4} + \frac{xr^2 - 2xz^2}{r^4}\right]\mathbf{i} - \left[\frac{yr^2 - 2yx^2}{r^4} - \frac{yr^2 - 2yz^2}{r^4}\right]\mathbf{j} \\
&\quad - \left[\frac{zr^2 - 2zx^2}{r^4} + \frac{zr^2 - 2zy^2}{r^4}\right]\mathbf{k} \\
&= \frac{2}{r^4}\left(x^3\mathbf{i} + y(x^2 - z^2)\mathbf{j} - z^3\mathbf{k}\right).
\end{aligned}$$

Therefore,

$$\nabla \times \mathbf{F} = \left(\frac{2x^3}{(x^2+y^2+z^2)^2}, \frac{2y(x^2-z^2)}{(x^2+y^2+z^2)^2}, \frac{-2z^3}{(x^2+y^2+z^2)^2}\right). \quad \Diamond$$

6. **Exercise 4.4.26:** *Show that* $\mathbf{F} = (x^2 + y^2)\mathbf{i} - 2xy\mathbf{j}$ *is not a gradient field.*

   Solution: **Method 1 -** Suppose $\mathbf{F}$ is a gradient field of some C^2 function U, *i.e.,* $\mathbf{F} = \nabla U$. This means

$$\begin{aligned} \frac{\partial U}{\partial x} &= x^2 + y^2 \\ \frac{\partial U}{\partial y} &= -2xy. \end{aligned}$$

   But then we would have

$$\frac{\partial^2 U}{\partial y \partial x} = 2y \neq -2y = \frac{\partial^2 U}{\partial x \partial y}.$$

   This contradiction proves that $\mathbf{F}$ is not a gradient field.

   Method 2 - If $\mathbf{F}$ were a gradient field we would have $\nabla \times \mathbf{F} = \mathbf{0}$. However,

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 + y^2 & -2xy & 0 \end{vmatrix} = (-2y - 2y)\mathbf{k} = -4y\mathbf{k} \neq 0.$$

   Hence $\mathbf{F}$ is not a gradient. $\Diamond$

7. **Exercise 5.1.4:** *Using Cavalieri's principle, compute the volume of the structure shown in Figure* 5.1.11 *of the textbook; each section is a rectangle of length* 5 *and width* 3.

   Solution: By Cavalieri's principle the volume of the solid in Figure 5.1.11 is the same as that of a rectangular parallelepiped of dimensions $3 \times 5 \times 7$ or $(3)(5)(7) = 105$. $\Diamond$

8. **Exercise 5.2.6:** *Compute the volume of the solid bounded by the surface* $z = \sin y$, *the planes* $x = 1, x = 0, y = 0$ *and* $y = \pi/2$ *and the* xy *plane.*

   Solution: Since $\sin y \geq 0$, for $0 \leq y \leq \pi/2$, this volume is given by

$$\int_0^1 \int_0^{\pi/2} \sin y dy\, dx = \int_0^1 [-\cos y]_0^{\pi/2}\, dx = \int_0^1 dx = 1. \quad \Diamond$$

9. **Exercise 5.3.2(a):** *Evaluate the following integral and sketch the region of integration*

$$\int_{-3}^{2}\int_{0}^{y^2}(x^2+y)dx\,dy.$$

Solution: The region of integration is shown in Figure 2.

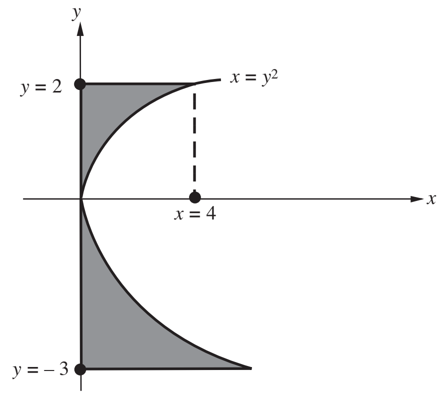

Figure 2: An x-simple region of integration.

This region is x-simple but not y-simple, since it is bounded on the left and right by graphs, but not top and bottom (unless we broke it into two pieces, one above the x-axis and one below. The integral is

$$\int_{-3}^{2}\left(\frac{x^3}{3}+xy\Big|_{x=0}^{x=y^2}\right)dy=\int_{-3}^{2}\left(\frac{y^6}{3}+y^3\right)dy=\frac{y^7}{21}+\frac{y^4}{4}\Big|_{-3}^{2}$$
$$=\frac{2^7}{21}+\frac{3^7}{21}+\frac{2^4}{4}-\frac{3^4}{4}=\frac{2^7+3^7}{21}-\frac{65}{4}.\quad\Diamond$$

10. **Exercise 5.4.2(a):** *Find*

$$\int_{-1}^{1}\int_{|y|}^{1}(x+y)^2 dx\, dy.$$

Solution: Changing the order of integration (see Figure 3) we see that this iterated integral is equal to

$$\begin{aligned}\int_0^1\int_{-x}^{x}(x+y)^2 dy\, dx &= \frac{1}{3}\int_0^1 \left[(x+y)^3\right]_{-x}^{x} dx \\ &= \frac{8}{3}\int_0^1 x^3 dx = \frac{2}{3}. \quad \diamondsuit\end{aligned}$$

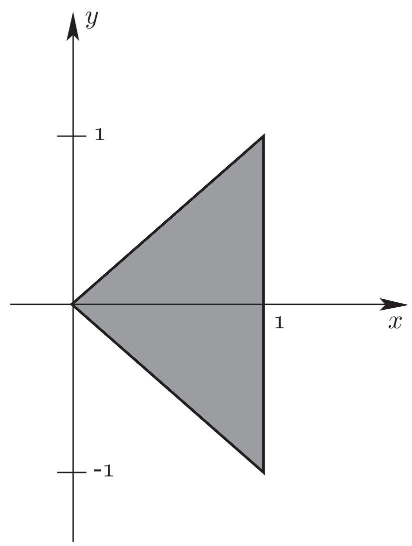

Figure 3: Region of integration for Exercise 5.4.2(a).