Ma 1c Practical - Solutions to Homework Set 5

All exercises are from the Vector Calculus text, Marsden and Tromba (Fifth Edition)

1. **Exercise 4.1.14:** Show that, at a local maximum or minimum of the quantity \( \| r(t) \| \), \( r'(t) \) is perpendicular to \( r(t) \).

**Solution:** Notice that at the time \( t \) where a local maximum or minimum for \( \| r(t) \| \) occurs, a local maximum or minimum for \( \| r(t) \|^2 = r(t) \cdot r(t) \) also occurs. And at those particular \( t \)'s, the first derivative of \( \| r(t) \|^2 \) is equal to zero. Therefore

\[
0 = (r(t) \cdot r(t))' = r'(t) \cdot r(t) + r(t) \cdot r'(t) = 2r'(t) \cdot r(t),
\]

which means that \( r'(t) \) is perpendicular to \( r(t) \). ♦

2. **Exercise 4.2.4:** Find the arc length of the curve 

\[
c(t) = \left( t + 1, \frac{2\sqrt{2}}{3} t^{3/2} + 7, \frac{1}{2} t^2 \right)
\]
on the interval \( 1 \leq t \leq 2 \).

**Solution:** The arc length is

\[
L(c) = \int_1^2 \| c'(t) \| dt = \int_1^2 \sqrt{1 + 2t + t^2} dt = \int_1^2 (t + 1) dt = \frac{5}{2}. ♦
\]

3. **Exercise 4.2.18:** In special relativity, the proper time of a path \( \gamma : [a, b] \to \mathbb{R}^4 \) with \( \gamma(\lambda) = (x(\lambda), y(\lambda), z(\lambda), t(\lambda)) \) is defined to be the quantity

\[
\frac{1}{c} \int_a^b \sqrt{-[x'(\lambda)]^2 - [y'(\lambda)]^2 - [z'(\lambda)]^2 + c^2[t'(\lambda)]^2} d\lambda,
\]

where \( c \) is the velocity of light, a constant. Referring to Figure 1, show that, using self-explanatory notation,

proper time (AB) + proper time (BC) < proper time (AC).

(This inequality is a special case of what is known as the twin paradox.)
Solution: We proceed in three steps. First we parametrize the paths.

\( i \) Let \( A = (0,0,0,0), B = (x_B,0,0,t_B), C = (0,0,0,t_C) \). Let \( c_1, c_2, c_3 \) be the paths from \( A \) to \( B \), \( B \) to \( C \), \( A \) to \( C \), respectively. Then

\[
\begin{align*}
    c_1(\lambda) &= (1-\lambda)(0,0,0,0) + \lambda(x_B,0,0,t_B) \\
    c_2(\lambda) &= (1-\lambda)(x_B,0,0,t_B) + \lambda(0,0,0,t_C) \\
    c_3(\lambda) &= (1-\lambda)(0,0,0,0) + \lambda(0,0,0,t_C)
\end{align*}
\]

\( ii \) Denote the proper time of \( AB, BC, AC \) by \( T_{AB} \), etc., then

\[
T_{AB} = \frac{1}{c} \int_0^1 \sqrt{-x_B^2 + c^2 t_B^2} \, d\lambda = \frac{1}{c} \sqrt{-x_B^2 + c^2 t_B^2}.
\]

Similarly, we have

\[
\begin{align*}
    T_{BC} &= \frac{1}{c} \sqrt{-x_B^2 + c^2 (t_C - t_B)^2} \\
    T_{AC} &= \frac{1}{c} \sqrt{c^2 t_C^2} = \frac{1}{c} (ct_C).
\end{align*}
\]

\( iii \) It suffices to show that

\[
\sqrt{-x_B^2 + c^2 t_B^2} + \sqrt{-x_B^2 + c^2 (t_C - t_B)^2} < ct_C.
\]

But the above is true if and only if

\[
\sqrt{-x_B^2 + c^2 (t_C - t_B)^2} < ct_C - \sqrt{-x_B^2 + c^2 t_B^2}
\]

if and only if

\[
-x_B^2 + c^2 (t_C - t_B)^2 < c^2 t_C^2 - x_B^2 + c^2 t_B^2 - 2ct_C \sqrt{-x_B^2 + c^2 t_B^2}
\]
if and only if
\[ c t_B > \sqrt{-x_B^2 + c^2 t_B^2}. \]

The last inequality is true as \( x_B^2 > 0 \). Thus the proof is complete. \( \diamond \)

4. **Exercise 4.3.14:** Show that the curve
\[ c(t) = (t^2, 2t - 1, \sqrt{t}), \ t > 0 \]

is a flow line of the velocity vector field
\[ F(x, y, z) = (y + 1, 2, 1/2z). \]

**Solution:** We must verify that \( c'(t) = F(c(t)) \). The left side is \((2t, 2, 1/(2\sqrt{t}))\) while the right side is \( F(t^2, 2t - 1, \sqrt{t}) = (2t, 2, 1/(2\sqrt{t})) \). Thus \( c(t) \) is a flow line of \( F \). \( \diamond \)

5. **Exercise 4.4.16:** Find the curl of the vector field
\[ F(x, y, z) = \frac{yz}{r^2}i - \frac{xz}{r^2}j + \frac{xy}{r^2}k. \]

**Solution:** Let \( r = \sqrt{x^2 + y^2 + z^2} \). We take the formal cross product of the \( \nabla \) operator with the given vector field to calculate the curl:

\[
\nabla \times F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{yz}{r^2} & -\frac{xz}{r^2} & \frac{xy}{r^2} \end{vmatrix} \\
= \left[ \frac{\partial}{\partial y} \left( \frac{xy}{r^2} \right) + \frac{\partial}{\partial z} \left( \frac{xz}{r^2} \right) \right] i - \left[ \frac{\partial}{\partial x} \left( \frac{xy}{r^2} \right) - \frac{\partial}{\partial z} \left( \frac{yz}{r^2} \right) \right] j \\
+ \left[ \frac{\partial}{\partial x} \left( -\frac{xz}{r^2} \right) - \frac{\partial}{\partial y} \left( \frac{yz}{r^2} \right) \right] k \\
= \left[ \frac{xr^2 - 2xy^2}{r^4} + \frac{xr^2 - 2xz^2}{r^4} \right] i - \left[ \frac{yr^2 - 2yx^2}{r^4} - \frac{yr^2 - 2yz^2}{r^4} \right] j \\
- \left[ \frac{zr^2 - 2zx^2}{r^4} + \frac{zr^2 - 2zy^2}{r^4} \right] k \\
= \frac{2}{r^4} \left( x^3 i + y(x^2 - z^2)j - z^3 k \right). \\
\]

Therefore,
\[
\nabla \times F = \left( \frac{2x^3}{(x^2 + y^2 + z^2)^2}, \frac{2y(x^2 - z^2)}{(x^2 + y^2 + z^2)^2}, \frac{-2z^3}{(x^2 + y^2 + z^2)^2} \right). \ \diamond
6. **Exercise 4.4.26:** *Show that \( \mathbf{F} = (x^2 + y^2)\mathbf{i} - 2xy\mathbf{j} \) is not a gradient field.*

**Solution:** **Method 1** - Suppose \( \mathbf{F} \) is a gradient field of some \( C^2 \) function \( U \), i.e., \( \mathbf{F} = \nabla U \). This means

\[
\frac{\partial U}{\partial x} = x^2 + y^2 \\
\frac{\partial U}{\partial y} = -2xy.
\]

But then we would have

\[
\frac{\partial^2 U}{\partial y \partial x} = 2y \neq -2y = \frac{\partial^2 U}{\partial x \partial y}.
\]

This contradiction proves that \( \mathbf{F} \) is not a gradient field.

**Method 2** - If \( \mathbf{F} \) were a gradient field we would have \( \nabla \times \mathbf{F} = \mathbf{0} \). However,

\[
\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 + y^2 & -2xy & 0 \end{vmatrix} = (-2y - 2y)\mathbf{k} = -4y\mathbf{k} \neq 0.
\]

Hence \( \mathbf{F} \) is not a gradient. \( \diamond \)

7. **Exercise 5.1.4:** *Using Cavalieri’s principle, compute the volume of the structure shown in Figure 5.1.11 of the textbook; each section is a rectangle of length 5 and width 3.*

**Solution:** By Cavalieri’s principle the volume of the solid in Figure 5.1.11 is the same as that of a rectangular parallelepiped of dimensions \( 3 \times 5 \times 7 \) or \( (3)(5)(7) = 105 \).

\( \diamond \)

8. **Exercise 5.2.6:** *Compute the volume of the solid bounded by the surface \( z = \sin y \), the planes \( x = 1, x = 0, y = 0 \) and \( y = \pi/2 \) and the xy plane.*

**Solution:** Since \( \sin y \geq 0 \), for \( 0 \leq y \leq \pi/2 \), this volume is given by

\[
\int_0^1 \int_0^{\pi/2} \sin y dy \, dx = \int_0^1 [-\cos y]_0^{\pi/2} \, dx = \int_0^1 dx = 1. \diamond
\]
9. **Exercise 5.3.2(a):** Evaluate the following integral and sketch the region of integration

\[
\int_{-3}^{2} \int_{0}^{y^2} (x^2 + y) \, dx \, dy.
\]

**Solution:** The region of integration is shown in Figure 2.

\begin{figure}
\centering
\includegraphics[width=0.5\textwidth]{image.png}
\caption{An \(x\)-simple region of integration.}
\end{figure}

This region is \(x\)-simple but not \(y\)-simple, since it is bounded on the left and right by graphs, but not top and bottom (unless we broke it into two pieces, one above the \(x\)-axis and one below. The integral is

\[
\int_{-3}^{2} \left( \frac{x^3}{3} + xy \bigg|_{x=0}^{x=y^2} \right) \, dy = \int_{-3}^{2} \left( \frac{y^6}{3} + y^3 \right) \, dy = \left. \frac{y^7}{21} + \frac{y^4}{4} \right|_{-3}^{2}^2
\]

\[
= \frac{2^7}{21} + \frac{3^7}{21} + \frac{2^4}{4} - \frac{3^4}{4} = \frac{2^7 + 3^7}{21} - \frac{65}{4}.
\]
10. **Exercise 5.4.2(a):** Find
\[
\int_{-1}^{1} \int_{|y|}^{1} (x + y)^2 \, dx \, dy.
\]

**Solution:** Changing the order of integration (see Figure 3) we see that this iterated integral is equal to
\[
\int_{0}^{1} \int_{-x}^{x} (x + y)^2 \, dy \, dx = \frac{1}{3} \int_{0}^{1} \left[ (x + y)^3 \right]_{-x}^{x} \, dx
\]
\[
= \frac{8}{3} \int_{0}^{1} x^3 \, dx = \frac{2}{3}. \quad \diamondsuit
\]

Figure 3: Region of integration for Exercise 5.4.2(a).