## HOMEWORK 4 SOLUTIONS

All questions are from Vector Calculus, by Marsden and Tromba

**Question 1: 3.1.16** *Let $w = f(x, y)$ be a function of two variables, and let*

$$x = u + v, y = u - v.$$

*Show that*

$$\frac{\partial^2 w}{\partial u \partial v} = \frac{\partial^2 w}{\partial x^2} - \frac{\partial^2 w}{\partial y^2}.$$

**Solution.** By the chain rule,

$$\frac{\partial w}{\partial v} = \frac{\partial w}{\partial x} \cdot \frac{\partial x}{\partial v} + \frac{\partial w}{\partial y} \cdot \frac{\partial y}{\partial v} = w_x - w_y.$$

Thus,

$$\begin{aligned} \frac{\partial^2 w}{\partial u \partial v} &= \frac{\partial}{\partial u}\left(\frac{\partial w}{\partial v}\right) = \frac{\partial}{\partial u}(w_x - w_y) = \frac{\partial}{\partial u} w_x - \frac{\partial}{\partial u} w_y \\ &= \frac{\partial w_x}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial w_x}{\partial y} \cdot \frac{\partial y}{\partial u} - \left(\frac{\partial w_y}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial w_y}{\partial y} \cdot \frac{\partial y}{\partial u}\right) \\ &= w_{xx} + w_{xy} - (w_{yx} + w_{yy}) = w_{xx} - w_{yy} \end{aligned}$$

*i.e.,*

$$\frac{\partial^2 w}{\partial u \partial v} = \frac{\partial^2 w}{\partial x^2} - \frac{\partial^2 w}{\partial y^2}.$$

**Question 2: 3.1.22**

(a) **:** *Show that the function*

$$g(x, t) = 2 + e^{-t} \sin x$$

*satisfies the heat equation: $g_t = g_{xx}$. [Here $g(x, t)$ represents the temperature in a metal rod at position $x$ and time $t$.]*

(b) **:** *Sketch the graph of $g$ for $t \geq 0$. (Hint: Look at sections by the planes $t = 0, t = 1,$ and $t = 2$.)*

(c) **:** *What happens to $g(x, t)$ as $t \to \infty$? Interpret this limit in terms of the behavior of heat in the rod.*

**Solution.**

(a) **:** Since $g(x, y) = 2 + e^{-t} \sin x$, then $g_t = -e^{-t} \sin x, g_x = e^{-t} \cos x$, and $g_{xx} = -e^{-t} \sin x$. Therefore, $g_t = g_{xx}$.

(b) **:** The graph of $g$ is shown in Figure 1.

*Date*: Math 1c Practical, 2008.

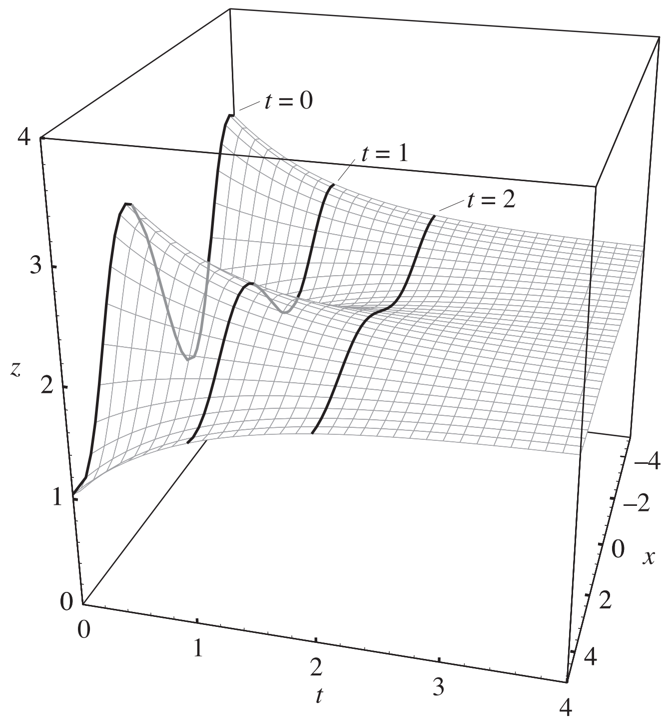


FIGURE 1. The graph of $g$ at $t = 0, 1$, and 2.

(c) : Note that

$$\lim_{t\to\infty} g(x,t) = \lim_{t\to\infty} (2 + e^{-t}\sin x) = 2$$

This means that the temperature in the rod at position $x$ tends to be a constant ($= 2$) as the time $t$ is large enough. $\Diamond$

**Question 3: 3.2.2** *Determine the second-order Taylor formula for*

$$f(x,y) = \frac{1}{x^2+y^2+1} \quad \textit{about} \quad x_0 = 0, y_0 = 0.$$

**Solution.** We first compute the partial derivatives up through second order:

$$\begin{aligned}
f_x &= \frac{-2x}{(1+x^2+y^2)^2}, \quad f_y = \frac{-2y}{(1+x^2+y^2)^2} \\
f_{xy} &= \frac{8xy}{(1+x^2+y^2)^3}, \quad f_{yx} = \frac{8xy}{(1+x^2+y^2)^3} \\
f_{xx} &= \frac{-2}{(1+x^2+y^2)^2} + \frac{8x^2}{(1+x^2+y^2)^3} \\
f_{yy} &= \frac{-2}{(1+x^2+y^2)^2} + \frac{8y^2}{(1+x^2+y^2)^3}.
\end{aligned}$$

Next, we evaluate these derivatives at $(0,0)$, obtaining

$$f_x(0,0) = f_y(0,0) = 0,$$
$$f_{xy}(0,0) = f_{yx}(0,0) = 0$$

and

$$f_{xx}(0,0) = f_{yy}(0,0) = -2.$$

Therefore, the second order Taylor formula is

$$f(\mathbf{h}) = -h_1^2 - h_2^2 + R_2(\mathbf{0}, \mathbf{h}),$$

where $\mathbf{h} = (h_1, h_2)$ and where

$$\frac{R_2(\mathbf{0}, \mathbf{h})}{\|\mathbf{h}\|} \to 0 \quad \text{as} \quad \|\mathbf{h}\| \to 0.$$

**Question 4: 3.2.6** *Determine the second-order Taylor formula for the function*

$$f(x, y) = e^{(x-1)^2} \cos y$$

*expanded about the point* $x_0 = 1, y_0 = 0$.

**Solution.** The ingredients needed in the second-order Taylor formula are computed as follows:

$$\begin{aligned} f_x &= 2(x-1)e^{(x-1)^2} \cos y \\ f_y &= -e^{(x-1)^2} \sin y \\ f_{xx} &= 2e^{(x-1)^2} \cos y + 4(x-1)^2 e^{(x-1)^2} \cos y \\ f_{xy} &= -2(x-1)e^{(x-1)^2} \sin y = f_{yx} \\ f_{yy} &= -e^{(x-1)^2} \cos y. \end{aligned}$$

Evaluating the function and these derivatives at the point $(1,0)$ gives

$$\begin{aligned} f(1,0) &= 1 \\ f_x(1,0) &= f_y(1,0) = 0 \\ f_{xx}(1,0) &= 2 \\ f_{xy}(1,0) &= f_{yx}(1,0) = 0 \quad \text{and} \\ f_{yy}(1,0) &= -1. \end{aligned}$$

Consequently, the second order Taylor formula is

$$f(\mathbf{h}) = 1 + h_1^2 - \frac{1}{2}h_2^2 + R_2((1,0), \mathbf{h}),$$

where $\mathbf{h} = (h_1, h_2)$ and where

$$\frac{R_2((1,0), \mathbf{h})}{\|\mathbf{h}\|} \to 0 \quad \text{as} \quad \|\mathbf{h}\| \to 0.$$

**Question 5: 3.3.7** *Find the critical points for the function*

$$f(x, y) = 3x^2 + 2xy + 2x + y^2 + y + 4.$$

*and then determine whether they are local maxima, local minima, or saddle points.*

**Solution.** Here,

$$\frac{\partial f}{\partial x} = 6x + 2y + 2, \quad \frac{\partial f}{\partial y} = 2x + 2y + 1.$$

We have

$$\frac{\partial f}{\partial x} = 0, \quad \frac{\partial f}{\partial y} = 0$$

when $x = y = -1/4$. Therefore, the only critical point is $(-1/4, -1/4)$. Now, $\frac{\partial^2 f}{\partial x^2}(-1/4, -1/4) = 6$, $\frac{\partial^2 f}{\partial y^2}(-1/4, -1/4) = 2$, and $\frac{\partial^2 f}{\partial x \partial y}(-1/4, -1/4) = 2$, which yields $D = 6.2 - 2 = 10 > 0$. Therefore $(-1/4, -1/4)$ is a local minimum.

**Question 6: 3.3.17** *Find the local maxima and minima for* $z = (x^2+3y^2)e^{1-x^2-y^2}$.

**Solution.** We first locate the critical points of $f(x, y) = (x^2 + 3y^2)e^{1-x^2-y^2}$.
$\nabla f(x, y) = e^{1-x^2-y^2}(2x(1 - 3y^2 - x^2)\mathbf{i} + 2y(3 - x^2 - 3y^2)\mathbf{j})$ Thus, $\nabla f(x, y) = 0$ if and only if $(x, y) = (0, 0), (0, \pm 1)$, or $(\pm 1, 0)$. To determine whether they are maxima or minima, we need to calculate the second partial derivatives.
$\frac{\partial^2 f}{\partial x^2} = (1 + 2x^4 - 3y^2 + x^2(6y^2 - 5))e^{1-x^2-y^2}$
$\frac{\partial^2 f}{\partial y^2} = (3 - 15y^2 + 6y^4 + x^2(2y^2 - 1))e^{1-x^2-y^2}$, and
$\frac{\partial^2 f}{\partial x \partial y} = 4(3y^2 + x^2 - 4)e^{1-x^2-y^2}$.
Therefore, $\frac{\partial^2 f}{\partial x^2}(0, 0) = 2e$, $\frac{\partial^2 f}{\partial y^2}(0, 0) = 6e$, and $\frac{\partial^2 f}{\partial x \partial y}(0, 0) = 0$, which yields $D = (2e)(6e) = 12e^2 > 0$, and $(0, 0)$ is a local minimum.
$\frac{\partial^2 f}{\partial x^2}(0, \pm 1) = -4$, $\frac{\partial^2 f}{\partial y^2}(0, \pm 1) = -12$, and $\frac{\partial^2 f}{\partial x \partial y}(0, \pm 1) = 0$, which yields $D = (-4)(-12) = 24 > 0$, and $(0, \pm 1)$ are local maxima.
$\frac{\partial^2 f}{\partial x^2}(\pm 1, 0) = -4$, $\frac{\partial^2 f}{\partial y^2}(\pm 1, 0) = 4$, and $\frac{\partial^2 f}{\partial x \partial y}(0, \pm 1) = 0$, which yields $D = (-4)(4) = -16 < 0$, and $(\pm 1, 0)$ are saddle points.

**Question 7: 3.3.25** *Write the number* 120 *as a sum of three numbers so that the sum of the products taken two at a time is a maximum.*

**Solution.** Let the three numbers be $x, y, z$. Thus,

$$x + y + z = 120, \quad z = 120 - x - y.$$

We want to find the maximum value for

$$\begin{aligned} S(x, y) &= xy + yz + xz = xy + (x + y)(120 - x - y) \\ &= -x^2 - xy - y^2 + 120x + 120y. \end{aligned}$$

We differentiate to get

$$\frac{\partial S}{\partial x} = -2x - y + 120, \quad \frac{\partial S}{\partial y} = -x - 2y + 120.$$

These vanish when $x = y = 40$, then $z = 120 - (x + y) = 40$. Therefore, when $x = y = z = 40$ is the only critical point. The condition $0 \leq x \leq 120, 0 \leq y \leq 120, 0 \leq z \leq 120$ describes a cube in $\mathbb{R}^3$ and on the boundary of the cube (either $x = 0, x = 120, y = 0, y = 120, z = 0, z = 120$), $S$ is zero. Therefore the maximum of $S$ occurs on the interior of this cube, *i.e.,* at a local maximum. Since $x = 40, y = 40, z = 40$ is the only critical point, it must be a maximum.

**Question 8: 3.4.2** *Find the extrema of $f(x,y) = x - y$ subject to the constraint $x^2 - y^2 = 2$.*

**Solution.** By the method of Lagrange multipliers, we write the constraint as $g = 0$, where $g(x,y) = x^2 - y^2 - 2$ and then write the Lagrange multiplier equations as $\nabla f = \lambda \nabla g$. Thus, we get

$$\begin{aligned} 1 &= \lambda \cdot 2x \\ 1 &= \lambda \cdot 2y \\ x^2 - y^2 - 2 &= 0. \end{aligned}$$

First of all, the first two equations imply that $x \neq 0$ and $y \neq 0$. Hence we can eliminate $\lambda$, giving $x = y$. From the last equation this would imply that $2 = 0$. Hence there are no extrema.

**Question 9: 3.4.22** *Let P be a point on a surface $S$ in $\mathbb{R}^3$ defined by the equation $f(x,y,z) = 1$, where $f$ is of class $C^1$. Suppose that P is a point where the distance from the origin to $S$ is maximized. Show that the vector emanating from the origin and ending at P is perpendicular to $S$.*

**Solution.** We want to maximize the function $g(x,y,z) = x^2 + y^2 + z^2$ subject to the constraint $f(x,y,z) = 1$. Suppose this maximum occurs at $\mathrm{P} = (x_0, y_0, z_0)$, then by the method of Lagrange multipliers we have the equations

$$\begin{aligned} 2x_0 &= \lambda \left\{\nabla f(x_0,y_0,z_0)\right\}_1 \\ 2y_0 &= \lambda \left\{\nabla f(x_0,y_0,z_0)\right\}_2 \\ 2z_0 &= \lambda \left\{\nabla f(x_0,y_0,z_0)\right\}_3 \end{aligned}$$

where $\left\{\nabla f(x_0,y_0,z_0)\right\}_i$ denotes the $i$th component of $\nabla f(x_0,y_0,z_0), 1 \leq i \leq 3$. If $\mathbf{v} = (x_0,y_0,z_0)$ is the vector from the origin ending at P, then these equations say that $\mathbf{v} = \left(\frac{\lambda}{2}\right) \cdot \nabla f(x_0,y_0,z_0)$. But $\nabla f(x_0,y_0,z_0)$ is perpendicular to $S$ at P, and since $\mathbf{v}$ is a scalar multiple of $\nabla f(x_0,y_0,z_0)$ it is also perpendicular to $S$ at P.

**Question 10: 3.4.28** *A company's production function is $Q(x,y) = xy$. The cost of production is $C(x,y) = 2x + 3y$. If this company can spend $C(x,y) = 10$, what is the maximum quantity that can be produced?*

**Solution.** We want to maximize $Q$ subject to the constraint $C(x,y) = 10$. Since both $x, y \geq 0$, this imposes the condition that $0 \leq x \leq 5, 0 \leq y \leq 10/3$. Thus, we wish to maximize $Q$ on the line segment $2x + 3y = 10, x \geq 0, y \geq 0$. If the maximum occurs at an interior point $(x_0, y_0)$ of this segment, then $\nabla Q(x_0,y_0) = \lambda \nabla C(x_0,y_0)$; that is,

$$\begin{aligned} y_0 &= 2\lambda \\ x_0 &= 3\lambda \\ 2x_0 + 3y_0 &= 10. \end{aligned}$$

Thus $6\lambda + 6\lambda = 10, \lambda = 5/6, y_0 = 5/3, x_0 = 5/2, Q(x_0,y_0) = 25/6$. The value of $Q$ at the endpoints of this segment are $Q(0, \frac{10}{3}) = 0 = Q(5,0)$. Consequently the maximum occurs at $(5/2, 5/3)$ and the maximum value of $Q$ is $25/6$.