# HOMEWORK 2 SOLUTIONS

All questions are from the Linear Algebra text, O'Nan and Enderton, and Vector Calculus, Marsden and Tromba.

**Question 1: 7.2.1c** Show that the following matrix is diagonalizable, and find the diagonal matrix to which it is similar: $\begin{bmatrix} -2 & -4 & -5 \\ 1 & 3 & 1 \\ 2 & 2 & 5 \end{bmatrix}$

**Solution** Call the above matrix $A$. We will show that $A$ is diagonalizable by showing that it has distinct eigenvalues. The characteristic polynomial for $A$ is $\det(\lambda I - A) = \det \begin{bmatrix} \lambda+2 & 4 & 5 \\ -1 & \lambda-3 & -1 \\ -2 & -2 & \lambda-5 \end{bmatrix} = \lambda^3 - 6\lambda^2 + 11\lambda - 6 = (\lambda-1)(\lambda-2)(\lambda-3)$. So the matrix $A$ has three eigenvalues: $\lambda = 1, 2, 3$, which are distinct. By Theorem 2(b) on Page 397, $A$ is diagonalizable. Furthermore, $A$ is similar to the diagonal matrix with the eigenvalues of $A$ as its entries, namely the matrix $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$.

**Question 2: 7.2.2** Let $T$ be the linear operator on $P_2$ defined by the equation:$T(f) = f + (1+x)f'$

(a) Calculate the eigenvalues of $T$.

(b) Give a diagonal matrix representing $T$.

**Solution** Let $\{1, x, x^2\}$ be a basis for $P_2$. Write $f(x) = a_2x^2 + a_1x + a_0$. If $\lambda$ is an eigenvalue of $T$, $T(f(x)) = f(x) + (1+x)f'(x) = \lambda f(x)$, and $a_2x^2 + a_1x + a_0 + (1+x)(2a_2x + a1) = 3a_2x^2 + (2a_1 + 2a_2)x + a_0 + a_1 = \lambda a_2x^2 + \lambda a_1x + \lambda a_0$. We compare the coefficients of both sides. If $a_2 = a_1 = 0$, $\lambda = 1$. Here the set of eigenvectors is *span* $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$. If $a_2 = 0$, and $a_1 \neq 0$, $\lambda = 2$. It follows that $a_1 = a_0$, and the set of eigenvectors is *span* $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ Finally, if $a_2 \neq 0$, $\lambda = 3$. In this case, $2a_2 = a_1$, and $a_1 = 2a_0$. Therefore the set of eigenvectors is *span* $\begin{bmatrix} 1/2 \\ 1 \\ 1/2 \end{bmatrix}$. In conclusion, the three eigenvalues are 1, 2, and 3, and the diagonal matrix representing $T$ is $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$.

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**Question 3: 7.3.1f** For the following matrix $A$ determine an orthogonal matrix $U$ such that $U^{-1}AU$ is diagonal: $\begin{bmatrix} -3 & -6 & 0 \\ -6 & 0 & 6 \\ 0 & 6 & 3 \end{bmatrix}$

**Solution** Since $A$ is a real symmetric matrix, it is orthogonally similar to a real diagonal matrix. We can determine $U$ by finding a orthonormal basis of the eigenvectors of $A$. The characteristic polynomial for $A$ is $\det(\lambda I - A) = \det \begin{bmatrix} \lambda+3 & 6 & 0 \\ 6 & \lambda & -6 \\ 0 & -6 & \lambda-3 \end{bmatrix} = \lambda^3 - 81\lambda = \lambda(\lambda+9)(\lambda-9)$. So the matrix $A$ has three eigenvalues: $\lambda = 0, 9, -9$. For $\lambda = 0$, the eigenspaces can be found by solving the equations
$-3x - 6y = 0$
$-6x + 6z = 0$
$6y + 3z = 0.$

The set of solutions is *span* $\begin{bmatrix} -2 \\ 1 \\ -2 \end{bmatrix}$. For $\lambda = 9$, we solve the equations
$12x + 6y = 0$
$6x + 9y - 6z = 0$
$6y - 6z = 0.$

The set of solutions is *span* $\begin{bmatrix} 1 \\ -2 \\ -2 \end{bmatrix}$. For $\lambda = -9$, we solve the equations
$6x - 6y = 0$
$6x - 9y - 6z = 0$
$6y + 12z = 0$

The set of solutions is *span* $\begin{bmatrix} -2 \\ -2 \\ 1 \end{bmatrix}$. After normalization, we get eigenvectors $\begin{bmatrix} -2/3 \\ 1/3 \\ -2/3 \end{bmatrix}$, $\begin{bmatrix} 1/3 \\ -2/3 \\ -2/3 \end{bmatrix}$, and $\begin{bmatrix} -2/3 \\ -2/3 \\ 1/3 \end{bmatrix}$. Since the eigenvectors correspond to distinct eigenvectors, they're pairwise orthogonal. Take $U = \frac{1}{3}\begin{bmatrix} -2 & 1 & -2 \\ 1 & -2 & -2 \\ -2 & -2 & 1 \end{bmatrix}$, then U is the desired orthogonal matrix.

**Question 4: 7.3.5** To what diagonal matrix is the matrix $\begin{bmatrix} a & c \\ c & b \end{bmatrix}$ similar?

**Solution** Call the above matrix $A$. Again, we see that $A$ is real symmetric. The characteristic polynomial for $A$ is $\det(\lambda I - A) = \det \begin{bmatrix} \lambda - a & -c \\ -c & \lambda b \end{bmatrix} = \lambda^2 - (a+b)\lambda + (ab - c^2)$. So the matrix $A$ has two eigenvalues: $\lambda = \frac{a+b+\sqrt{(a-b)^2+4c^2}}{2}, \frac{a+b-\sqrt{(a-b)^2+4c^2}}{2}$. They're the same iff $a = b$, and $c = 0$, in which case $A$ is itself diagonal. Otherwise, the eigenvalues are distinct, and $A$ is similar to the diagonal matrix $\begin{bmatrix} \frac{a+b+\sqrt{(a-b)^2+4c^2}}{2} & 0 \\ 0 & \frac{a+b-\sqrt{(a-b)^2+4c^2}}{2} \end{bmatrix}$

**Question 5: 7.3.7** Let $A$ be an $n \times n$ real skew-symmetric matrix. Show that $I + A$ is invertible and $(I - A)(I + A)^{-1}$ is orthogonal

**Solution** We first show that $A$ does not have $-1$ as an eigenvalue. Suppose $\lambda$

is a real eigenvalue with $v$ an (real) eigenvector. Then
$Av = \lambda v$
$v^T A^T = \lambda v^T$
$v^T A^T Av = \lambda^2 v^T v$
Since $A$ is skew-symmetric, $A^T = -A$, and
$-v^T A^2 v = \lambda^2 v^T v$
Now $A^2 v = AAv = \lambda Av = \lambda^2 v$, Therefore $-v^T \lambda^2 v = \lambda^2 v^T v$. Since $v \neq 0$, $v^T v \neq 0$, and $\lambda = 0$. Therefore $\lambda \neq -1$, and $\det(-I - A) \neq 0$. $I + A = -(-I - A)$is invertible. Similarly, since 1 is not an eigenvalue of $A$, $\det(I - A) \neq 0$, and $I - A$ is invertible. Since $A$ and $I$ commute, $I + A$ commutes with $I - A$, and hence $I + A$ also commutes with $(I - A)^{-1}$. Let $B = (I - A)(I + A)^{-1}$; then $B^{-1} = (I + A)(I - A)^{-1} = (I - A)^{-1}(I + A)$. Also, for any two matrices $M$ and $N$, $(MN)^T = N^T M^T$, and $(M + N)^T = M^T + N^T$. If $M$ is invertible then $(M^T)^{-1} = (M^{-1})^T$. Thus as $A^T = -A$,

$B^T = ((I - A)(I + A)^{-1})^T = ((I + A)^{-1})^T (I - A)^T = ((I + A)^T)^{-1}(I - A)^T = (I + A^T)^{-1}(I - A)^T = (I - A)^{-1}(I + A)$,

so $B^{-1} = B^T$ and hence B is orthogonal.

**Question 6: P90.20** Show that two planes given by the equations $Ax+By+Cz+D_1 = 0$ and $Ax + By + Cz + D_2 = 0$ are parrallel, and that the distance between them is $\frac{|D_1 - D_2|}{\sqrt{A^2+B^2+C^2}}$

**Solution** $A\mathbf{i} + B\mathbf{j} + C\mathbf{k}$ is a normal vector for both planes. Therefore the two

planes are parallel. The distance between two parallel planes is the same as the distance between a point on one of the planes and the other plane. Let $P_0 = (x_0, y_0, z_0)$ be a point on the plane given by $Ax + By + Cz + D_2 = 0$. Then the distance from $P_0$ to the plane $Ax + By + Cz + D_1 = 0$ is $\frac{|Ax_0+By_0+Cz_0+D_1|}{\sqrt{A^2+B^2+C^2}} = \frac{|-D_2+D_1|}{\sqrt{A^2+B^2+C^2}}$.

**Question 7: P90.21** (a)Prove that the area of the triangle in the plane with vertices $(x_1, y_1)$, $(x_2, y_2)$, $(x_3, y_3)$ is the absolute value of $\frac{1}{2}\begin{bmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{bmatrix}$
(b) Find the area of the triangle with vertices $(1, 2)$, $(0, 1)$,$(-1, 1)$.

**Solution** (a)Let $a = (x_1, y_1)$, $b = (x_2, y_2)$, $c = (x_3, y_3)$. By Example 8, the

area of the given triangle is $\frac{1}{2}||\mathbf{b} - \mathbf{a} \times \mathbf{c} - \mathbf{a}|| = \left| \frac{1}{2} \det( \begin{bmatrix} 1 & 0 & 0 \\ 0 & x_2 - x_1 & y_2 - y_1 \\ 0 & x_3 - x_1 & y_3 - y_1 \end{bmatrix} ) \right|$.

On the other hand, $\left|\frac{1}{2}\det(\begin{bmatrix}1 & 1 & 1\\ x_1 & x_2 & x_3\\ y_1 & y_2 & y_3\end{bmatrix})\right| = \left|\frac{1}{2}\det(\begin{bmatrix}1 & 0 & 0\\ x_1 & x_2-x_1 & x_3-x_1\\ y_1 & y_2-y_1 & y_3-y_1\end{bmatrix})\right| = \left|\frac{1}{2}\det(\begin{bmatrix}1 & x_1 & y_1\\ 0 & x_2-x_1 & y_2-y_1\\ 0 & x_3-x_1 & y_3-y_1\end{bmatrix})\right| = \left|\frac{1}{2}\det(\begin{bmatrix}1 & 0 & 0\\ 0 & x_2-x_1 & y_2-y_1\\ 0 & x_3-x_1 & y_3-y_1\end{bmatrix})\right|$

(b)By (a), the area of the given triangle is $\left|\frac{1}{2}\begin{bmatrix}1 & 1 & 1\\ 1 & 0 & -1\\ 2 & 1 & 1\end{bmatrix}\right| = \left|\frac{1}{2}(1-(1-(-2))+1)\right| = \frac{1}{2}$ by expanding the first row.

**Question 8: P90.34** (a)If a particle with mass $m$ moves with velocity $v$, its momentum is $p = mv$. In a game of marbles, a marble with mass 2 grams is shot with velocity 2 meters per second, hits two marbles with mass $1g$ each, and comes to a dead halt. One of the marbles flies off with a velocity of $3m/s$ at an angle of 45 to the incident direction of the larger marble as in Figure. Assuming that the total momentum before and after the collision is the same, at what angle and speed does the second marble move?

**Solution** Before the collision, the total momentum $p_b$ just consists of the momentum of the larger marble, and $p_b = 2(2\mathbf{i}) = 4\mathbf{i}mg/s$. After the collision, the known velocity of a marble is $v_1 = 3\cos(\frac{\pi}{4})\mathbf{i} + 3\cos(\frac{\pi}{4})\mathbf{j}m/s$. Let $v_2$ denote the velocity of the other marble, and $p_f$ the total momentum after the collision. By conservation of momentum, $p_b = p_f$, and $4\mathbf{i} = 1(v_1+v_2) = 3\cos(\frac{\pi}{4})\mathbf{i}+3\cos(\frac{\pi}{4})\mathbf{j}+v_2$. Therefore, $v_2 = (4-\frac{3\sqrt{2}}{2})\mathbf{i} - (\frac{3\sqrt{2}}{2})\mathbf{j}$. $|v_2| = \sqrt{(4-\frac{3\sqrt{2}}{2})^2 + (\frac{3\sqrt{2}}{2})^2} = \sqrt{25-12\sqrt{2}}$. The angle is $\theta = -\tan^{-1}(3\sqrt{2}/(8-3\sqrt{2}))$

**Question 9: 2.1.16** Describe the graph of each function by computing some level sets and sections. $f(x,y,z) = xy + z^2$.

**Solution** The level set with value $c$ is the set $(x,y,z)|xy + z^2 = c$. When $x$ or $y$ is a constant $t \neq 0$, we are looking at equation of the form $tw + z^2 = c$, which is a parabola. If $t = 0$, $z^2 = c$, which gives 2 lines. If $z$ is a constant $t$, we have $xy + t = c$, which is a hyperbola.

**Question 10: 2.1.24** Sketch or describe the surface of the equation $\frac{y^2}{9} + \frac{z^2}{4} = 1 + \frac{x^2}{16}$.

**Solution** If $x$ is a constant $k$, we're looking at $\frac{y^2}{9} + \frac{z^2}{4} = 1 + \frac{k^2}{16}$, which defines an ellipse. If $y$ or $z$ is a constant $k$, we have $\frac{z^2}{4} - \frac{x^2}{16} = 1 - \frac{k^2}{9}$, or $\frac{y^2}{9} - \frac{x^2}{16} = 1 - \frac{k^2}{4}$. In the first equation, if $y = k = 3$, we get two lines with slopes additive inverses of each other. Similarly for $z = k = 2$. Otherwise, we get hyperbolas.