Mathematics 1c. Practice Final Solutions

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Print Your Name:
Your Section:

- This exam has ten questions.
- You may take four hours; there is no credit for overtime work
- No aids (including notes, books, calculators etc.) are permitted.
- The exam **MUST** be turned in by ***noon on Thursday, June 12.***
- All 10 questions should be answered on this exam, using the backs of the sheets or appended pages as needed. Each question is worth 20 points.
- Show all your work and justify all claims using plain English.
- **Good Luck !!**

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1. (a) Suppose that an operator $T : V \to V$ on a vector space V has at least one nonzero eigenvalue and that some integer power of T is zero. Can T be diagonalizable?

   (b) Let V be the vector space of real polynomials of degree 3.

      i. What is the dimension of V?

      ii. Is the operator $T : V \to V$ defined by $T(p) = x^2p''$ diagonalizable?

      iii. Is the operator $S : V \to V$ defined by $S(p) = xp''$ diagonalizable?

   Solution.

   (a) Suppose that T were diagonalizable; say $T = QDQ^{-1}$, where $D = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$ is a diagonal operator (or if you wish, matrix). Then $T^N = QD^NQ^{-1}$. If this is zero for an integer N, then $D^N = 0$. But $D^N = \mathrm{diag}(\lambda_1^N, \ldots, \lambda_n^N)$ and so if this is zero, then D itself and hence T is zero, contradicting the fact that T has at least one nonzero eigenvalue.

   (b) i. The dimension is 4 with a basis given by the standard polynomials: $1, x, x^2, x^3$.

      ii. Note that each of the polynomials in the preceding basis is an eigenvector of T; for instance, $T(x^2) = 2x^2$. Thus, we have a basis of eigenvectors, so T is diagonalizable. Note that T does *not* have distinct eigenvalues (there are two zero eigenvalues), but it still is diagonalizable.

      iii. We claim that for this operator, $S^3 = 0$. For instance, acting on x^3, we have $S(x^3) = 6x^2$, $S(x^2) = 2x$ and $S(x) = 0$. Thus, $S^3(x^3) = S^2(6x^2) = S(12x) = 0$. Thus, by part (a), S is not diagonalizable.

2. Let B be an $n \times n$ matrix that is symmetric, orthogonal, *and* has determinant equal to one.

   (a) Show that $\mathbb{R}^n$ has a basis of eigenvectors of B and that each eigenvalue of B is either 1 or -1.

   (b) Give a concrete example (other than the identity matrix) of a 3×3 matrix B that is symmetric, orthogonal, and has determinant equal to one.

   (c) Give a geometric interpretation of your example as a linear transformation of $\mathbb{R}^3$ to $\mathbb{R}^3$.

Solution.

(a) B has an orthonormal basis of eigenvectors since B is symmetric. Suppose that v is an eigenvector, say $Bv = \lambda v$. Since B is orthogonal, that is, $B^T B = \mathrm{Id}$, it preserves length (or use the property $\|v\|^2 = v \cdot v = (B^T B)v \cdot v = Bv \cdot Bv = \|Bv\|^2$). Thus, taking the length of each side of the equation $Bv = \lambda v$, we see that $|\lambda| = 1$; since λ is real (because B is symmetric), we get $\lambda = \pm 1$.

(b) Using what we found in (i) and checking symmetry, orthogonality and the determinant, we see that an example of such a matrix is $\mathrm{diag}(-1, -1, 1)$.

(c) As a linear operator, this matrix has the interpretation as a rotation through π —that is, $180°$, about the z-axis.

3. Let a particle of mass m move along the elliptical helix $\mathbf{c}(t) = (4\cos t, \sin t, t)$.

(a) Find the equation of the tangent line to the helix at $t = \pi/4$.

(b) Find the force acting on the particle at time $t = \pi/4$.

(c) Write an expression (in terms of an integral) for the arc length of the curve $\mathbf{c}(t)$ between $t = 0$ and $t = \pi/4$.

Solution.

(a) Using the notation $\mathbf{r} = \mathbf{r}_0 + s\mathbf{v}$ for the equation of a straight line passing through the point $\mathbf{r}_0$ at $s = 0$ and having the direction $\mathbf{v}$, the equation of the tangent line to $\mathbf{c}$ is

$$\mathbf{r} = \mathbf{c}(\pi/4) + s\mathbf{c}'(\pi/4),$$

that is,

$$\mathbf{r} = (2\sqrt{2}, \sqrt{2}/2, \pi/4) + s(-2\sqrt{2}, \sqrt{2}/2, 1),$$

which can also be written as

$$\mathbf{r} = \left(2\sqrt{2}(1-s), \frac{\sqrt{2}}{2}(1+s), \frac{\pi}{4} + s\right).$$

Comment. One can also use the parametrization

$$\mathbf{r} = \mathbf{c}(\pi/4) + \mathbf{c}'(\pi/4)(t - \pi/4),$$

so that the line passes through the point $\mathbf{c}(\pi/4)$ when $t = \pi/4$; that is,

$$\mathbf{r} = (2\sqrt{2}, \sqrt{2}/2, \pi/4) + (-2\sqrt{2}, \sqrt{2}/2, 1)(t - \pi/4),$$

which can also be written as

$$\mathbf{r} = \left(2\sqrt{2}\left(1 + \frac{\pi}{4} - t\right), \frac{\sqrt{2}}{2}\left(1 - \frac{\pi}{4} + t\right), t\right).$$

(b) The second derivative is

$$\mathbf{c}''(t) = (-4\cos t, -\sin t, 0)$$

which at $t = \pi/4$ is $(-2\sqrt{2}, -\sqrt{2}/2, 0)$ and so the force is, from $\mathbf{F} = m\mathbf{a}$,

$$\mathbf{F} = (-2m\sqrt{2}, -\sqrt{2}m/2, 0).$$

(c) By the formula for arc length, this is

$$\begin{aligned} L &= \int_0^{\pi/4} \sqrt{(-4\sin t)^2 + \cos^2 t + 1}\, dt \\ &= \int_0^{\pi/4} \sqrt{2 + 15\sin^2 t}\, dt. \end{aligned}$$

4. (a) Let $g(x, y, z) = x^3 + 5yz + z^2$ and let $h(u)$ be a function of one variable such that $h'(1) = 1/2$. Let $f = h \circ g$. In what directions starting at $(1, 0, 0)$ is f changing at 50% of its maximum rate?

(b) For $g(x, y, z) = x^3 + 5yz + z^2$, calculate $\mathbf{F} = \nabla g$, the gradient of g and verify directly that $\nabla \times \mathbf{F} = \mathbf{0}$ at each point (x, y, z).

Solution.

(a) Using the chain rule and noting that $g(1, 0, 0) = 1$, the gradient of f at the point $(1, 0, 0)$ is given by

$$\nabla f(1, 0, 0) = h'(1)\nabla g(1, 0, 0).$$

However,

$$h'(1) = 1/2 \quad \text{and} \quad \nabla g(x, y, z) = (3x^2, 5z, 5y + 2z),$$

so $\nabla g(1, 0, 0) = 3\mathbf{i}$, and so $\nabla f(1, 0, 0) = 3\mathbf{i}/2$. Therefore, the direction in which f is increasing the fastest is $3\mathbf{i}/2$, or after normalization, the vector $\mathbf{i}$.

The directions $\mathbf{n}$ in which f is changing at 50% of its maximum rate satisfy

$$\nabla f(1, 0, 0) \cdot \mathbf{n} = \frac{1}{2}\nabla f(1, 0, 0) \cdot \mathbf{i}$$

that is,

$$\frac{3}{2}\mathbf{i}\cdot\mathbf{n} = \frac{1}{2}\frac{3}{2}\mathbf{i}\cdot\mathbf{i}$$

that is,

$$\mathbf{i}\cdot\mathbf{n} = \frac{1}{2}.$$

Thus, if the angle between the vectors $\mathbf{i}$ and $\mathbf{n}$ is denoted θ, then $\cos\theta = 1/2$, so $\theta = \pi/3$. These directions form a cone about the direction $\mathbf{i}$ that make an angle of $\pi/3$ with $\mathbf{i}$.

(b) We first calculate the gradient:

$$\nabla g(x, y, z) = (3x^2, 5z, 5y + 2z)$$

and then we calculate the curl of F taking the cross product of ∇ and $\mathbf{F}$:

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 3x^2 & 5z & 5y + 2z \end{vmatrix} = (5 - 5, 0, 0) = (0, 0, 0)$$

5. Let $f(u, v, w)$ be a (smooth) function of three variables, let $h(r, s) = (r, r + s, r - s)$ and let $g = f \circ h$.

(a) Calculate $\dfrac{\partial^2 g}{\partial r \partial s}$ in terms of the derivatives of f.

(b) Consider the curve in the plane defined by $\mathbf{c}(t) = (\cos t, \sin t)$ and the curve in space defined by $\mathbf{d}(t) = (h \circ \mathbf{c})(t)$, where h is as given above. Find the equation of the tangent line to $\mathbf{d}(t)$ at $t = 0$.

(c) Find an expression as an integral for the arc length of the curve $\mathbf{d}(t)$ between $t = 0$ and $t = \pi/4$.

Solution.

(a) First of all, we calculate the partial derivative of g with respect to r using the chain rule with $u = r$, $v = r + s$ and $w = r - s$:

$$\frac{\partial g}{\partial r} = \frac{\partial f}{\partial u}\cdot 1 + \frac{\partial f}{\partial v}\cdot 1 + \frac{\partial f}{\partial w}\cdot 1 = \frac{\partial f}{\partial u} + \frac{\partial f}{\partial v} + \frac{\partial f}{\partial w}$$

since the partial derivatives of u, v, and w with respect to r are all 1. Now we use this to calculate the derivative with respect to s, again using the chain rule:

$$\begin{aligned}\frac{\partial}{\partial s}\left(\frac{\partial g}{\partial r}\right) &= \frac{\partial}{\partial u}\left(\frac{\partial f}{\partial u}+\frac{\partial f}{\partial v}+\frac{\partial f}{\partial w}\right)\frac{\partial u}{\partial s} \\ &+ \frac{\partial}{\partial v}\left(\frac{\partial f}{\partial u}+\frac{\partial f}{\partial v}+\frac{\partial f}{\partial w}\right)\frac{\partial v}{\partial s} \\ &+ \frac{\partial}{\partial w}\left(\frac{\partial f}{\partial u}+\frac{\partial f}{\partial v}+\frac{\partial f}{\partial w}\right)\frac{\partial w}{\partial s}.\end{aligned}$$

However,

$$\frac{\partial u}{\partial s}=0 \quad \frac{\partial v}{\partial s}=1 \quad \text{and} \quad \frac{\partial w}{\partial s}=-1$$

and so

$$\begin{aligned}\frac{\partial^2 g}{\partial s\partial r} = \frac{\partial^2 g}{\partial r\partial s} &= \frac{\partial^2 f}{\partial v\partial u}+\frac{\partial^2 f}{\partial^2 v}+\frac{\partial^2 f}{\partial v\partial w}-\frac{\partial^2 f}{\partial w\partial u}-\frac{\partial^2 f}{\partial w\partial v}-\frac{\partial^2 f}{\partial w^2} \\ &= \frac{\partial^2 f}{\partial v\partial u}+\frac{\partial^2 f}{\partial^2 v}-\frac{\partial^2 f}{\partial w\partial u}-\frac{\partial f^2}{\partial w^2}\end{aligned}$$

(b) The general equation of the tangent line to a curve $\mathbf{d}(t)$ at t_0 is

$$\begin{bmatrix}x\\y\\z\end{bmatrix} = \mathbf{d}(t_0)+s\mathbf{d}'(t_0).$$

In our case, we have

$$\mathbf{d}(t) = (\cos t, \sin t+\cos t, \cos t-\sin t)$$

and so

$$\mathbf{d}'(t) = (-\sin t, -\sin t+\cos t, -\sin t-\cos t)$$

and also

$$\mathbf{d}(0) = (1,1,1) \quad \text{and} \quad \mathbf{d}'(0) = (0,1,-1)$$

Therefore, our tangent line is given by the equation

$$\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}1\\1\\1\end{bmatrix} + s\begin{bmatrix}0\\1\\-1\end{bmatrix} = \begin{bmatrix}1\\1+s\\1-s\end{bmatrix}$$

that is,

$$\begin{aligned}x &= 1\\ y &= 1+s\\ z &= 1-s\end{aligned}$$

(c) By the formula for the arc length of a curve, we get

$$\begin{aligned} L &= \int_C ds = \int_0^{\frac{\pi}{4}} \|\mathbf{d}'(t)\|\, dt \\ &= \int_0^{\frac{\pi}{4}} \sqrt{\sin^2 t + (\cos t - \sin t)^2 + (-\sin t - \cos t)^2}\, dt \\ &= \int_0^{\frac{\pi}{4}} \sqrt{2 + \sin^2 t}\, dt. \end{aligned}$$

Another Exposition of the Solution to Problem 5.

(a) We have

$$\begin{aligned} u &= r \\ v &= r + s \\ w &= r - s \end{aligned}$$

Thus, the derivative matrix of the mapping h from (r, s) to (u, v, w) is given by

$$\mathbf{D}h(r, s) = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & -1 \end{bmatrix}.$$

By the chain rule,

$$\begin{aligned} \frac{\partial g}{\partial s} &= \frac{\partial f}{\partial u}\frac{\partial u}{\partial s} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial s} + \frac{\partial f}{\partial w}\frac{\partial w}{\partial s} \\ &= \frac{\partial f}{\partial v} - \frac{\partial f}{\partial w} \end{aligned}$$

Hence,

$$\begin{aligned} \frac{\partial^2 g}{\partial r \partial s} &= \frac{\partial}{\partial r}\left(\frac{\partial f}{\partial v} - \frac{\partial f}{\partial w}\right) \\ &= \left(\frac{\partial^2 f}{\partial u \partial v} - \frac{\partial^2 f}{\partial u \partial w}\right)\frac{\partial u}{\partial r} + \left(\frac{\partial^2 f}{\partial v^2} - \frac{\partial^2 f}{\partial v \partial w}\right)\frac{\partial v}{\partial r} + \left(\frac{\partial^2 f}{\partial w \partial v} - \frac{\partial^2 f}{\partial w^2}\right)\frac{\partial w}{\partial r} \\ &= \frac{\partial^2 f}{\partial u \partial v} - \frac{\partial^2 f}{\partial u \partial w} + \frac{\partial^2 f}{\partial v^2} - \frac{\partial^2 f}{\partial v \partial w} + \frac{\partial^2 f}{\partial w \partial v} - \frac{\partial^2 f}{\partial w^2} \\ &= \frac{\partial^2 f}{\partial u \partial v} - \frac{\partial^2 f}{\partial u \partial w} + \frac{\partial^2 f}{\partial v^2} - \frac{\partial^2 f}{\partial w^2} \end{aligned}$$

Note that the final step is a consequence of the equality of mixed partials for smooth functions.

(b) By the chain rule,

$$\begin{aligned}\mathbf{d}'(t)^T &= \mathbf{D}h(\mathbf{c}(t))\mathbf{c}'(t)^T \\ &= \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} -\sin t \\ \cos t \end{bmatrix} \\ &= \begin{bmatrix} -\sin t \\ \cos t - \sin t \\ -\cos t - \sin t \end{bmatrix}\end{aligned}$$

The tangent line to $\mathbf{d}(t)$ at $t = 0$ consists of all points of the form $\mathbf{d}(0) + s\mathbf{d}'(0) = (1, 1, 1) + s(0, 1, -1) = (1, 1 + s, 1 - s)$, where $s \in \mathbb{R}$.

(c) The arc length of the curve $\mathbf{d}(t)$ from $t = 0$ to $t = \pi/4$ is given by

$$\int_0^{\pi/4} \|\mathbf{d}'(t)\| \, dt.$$

From part (b),

$$\mathbf{d}'(t) = (-\sin t, \cos t - \sin t, -\cos t - \sin t).$$

Thus,

$$\begin{aligned}\|\mathbf{d}'(t)\| &= \sqrt{(-\sin t)^2 + (\cos t - \sin t)^2 + (-\cos t - \sin t)^2} \\ &= \sqrt{2 + \sin^2 t},\end{aligned}$$

and the arc length is given by

$$\int_0^{\pi/4} \sqrt{2 + \sin^2 t} \, dt.$$

6. Let $f(x, y, z) = x + z$ and let $g(x, y, z) = x^2 + y^2 + z^2$.

   (a) Find the maximum point (x_0, y_0, z_0) of f subject to the constraint $g = 1$.

   (b) Let V be the plane containing all vectors in $\mathbb{R}^3$ tangent to the surface $g = 1$ at the point (x_0, y_0, z_0) found in part (a).

      (i) Find an equation for the plane V.

      (ii) Let A be a 3×3 matrix whose transpose A^T is such that $A^T\mathbf{v}$ lies in V whenever $\mathbf{v}$ does. Show that

      $$A\nabla f(x_0, y_0, z_0) = \lambda \nabla f(x_0, y_0, z_0)$$

      for a constant λ.

Solution.

(a) We use the method of Lagrange multipliers. We have $\nabla g(x, y, z) = (2x, 2y, 2z)$ and $\nabla f(x, y, z) = (1, 0, 1)$, so the Lagrange multiplier equations, namely $\nabla f = \lambda \nabla g$ become

$$\begin{aligned} 1 &= \lambda 2x \\ 0 &= \lambda 2y \\ 1 &= \lambda 2z \\ x^2 + y^2 + z^2 &= 1 \end{aligned}$$

From the first equation, $\lambda \neq 0$ and so from the second equation we get $y = 0$. Comparing the first and third equations, $x = z$ and so from the last equation, $x = z = \pm 1/\sqrt{2}$. Thus, the two possible points are

$$\left(\frac{\sqrt{2}}{2}, 0, \frac{\sqrt{2}}{2}\right) \quad \text{and} \quad \left(-\frac{\sqrt{2}}{2}, 0, -\frac{\sqrt{2}}{2}\right).$$

Evaluating f at these two points, we see that the first is the maximum and the second is the minimum.

(b) i. The gradient of g is $\nabla g(x, y, z) = (2x, 2y, 2z)$, which becomes, at the maximum point,

$$\nabla g\left(\frac{\sqrt{2}}{2}, 0, \frac{\sqrt{2}}{2}\right) = (\sqrt{2}, 0, \sqrt{2}).$$

Thus, the plane V is described by the equation

$$\left(x - \frac{\sqrt{2}}{2}\right) \cdot \sqrt{2} + \left(z - \frac{\sqrt{2}}{2}\right) \cdot \sqrt{2} = 0,$$

which simplifies to

$$x + z = \sqrt{2}.$$

One could conceivably interpret the question to ask for the plane through the origin that contains the tangent vectors. In this case the equation for V would be $x + z = 0$.

ii. The space V is the plane that is perpendicular to each of the vectors $\nabla f(x_0, y_0, z_0)$ and $\nabla g(x_0, y_0, z_0)$. Keep in mind that in view of the Lagrange multiplier equations, these two vectors are parallel. Therefore, for any $\mathbf{v}$ in V, $\nabla f \cdot \mathbf{v} = 0$, so for any $\mathbf{v}$ in V, we have

$$\nabla f(x_0, y_0, z_0) \cdot A^T \mathbf{v} = 0.$$

Therefore, by (b) for any $\mathbf{v}$ in V, $A\nabla f(x_0, y_0, z_0) \cdot \mathbf{v} = 0$, hence $A\nabla f(x_0, y_0, z_0)$ is (the zero vector or) is perpendicular to every vector in V. It must therefore be a constant multiple of $\nabla f(x_0, y_0, z_0)$.

7. (a) Let D be the parallelogram in the xy-plane with vertices

$$(0,0), (1,1), (1,3), (0,2).$$

Evaluate the integral

$$\iint_D xy\, dxdy.$$

(b) Evaluate

$$\iiint_D (x^2+y^2+z^2)^{1/2} \exp[(x^2+y^2+z^2)^2]\, dx\, dy\, dz$$

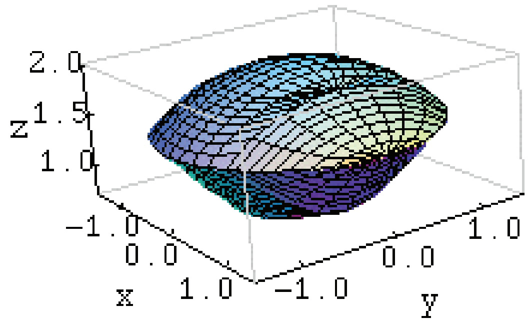
where D is the region defined by $1 \le x^2+y^2+z^2 \le 4$ and $z \ge \sqrt{x^2+y^2}$.

Solution.

(a) This is an elementary region with sides given by the lines $x = 0$ and $x = 1$ and top and bottom by the lines $y = x$ and $y = x + 2$. Therefore, the integral is

$$\begin{aligned}
\iint_D xy\, dx\, dy &= \int_0^1 \int_x^{x+2} xy\, dy\, dx \\
&= \int_0^1 x\left(\frac{(x+2)^2}{2} - \frac{x^2}{2}\right) dx \\
&= \int_0^1 2(x^2+x)\, dx \\
&= \frac{2}{3}x^3 + x^2 \Big|_0^1 = \frac{5}{3}.
\end{aligned}$$

(b) The region in question is that between two spheres and inside a cone centered around the z-axis. A rough sketch is given in the Figure.



The region for Problem 7 is the region between spheres
of radii 1 and 2 and lying inside the cone $z^2 = x^2 + y^2$.

By drawing a careful figure and using a little trigonometry, one finds that a side of the cone makes an angle of $\pi/4$ with the z-axis.

Denoting the integral required by I, we get

$$\begin{aligned} I &= \int_0^{2\pi} \int_0^{\frac{\pi}{4}} \int_1^2 \rho \left(e^{\rho^4}\right) \rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta \\ &= 2\pi \left(1 - \frac{1}{\sqrt{2}}\right) \int_1^2 \exp\left(\rho^4\right) \rho^3 \, d\rho \\ &= \frac{\pi}{2} \left(1 - \frac{1}{\sqrt{2}}\right) e^{\rho^4} \Big|_1^2 \\ &= \frac{\pi}{2} \left(1 - \frac{1}{\sqrt{2}}\right) \left(e^{16} - e\right) \end{aligned}$$

8. For each of the questions below, indicate if the statement is **true** or **false**. If true, **justify** (give a brief explanation or quote a relevant theorem from the course) and if false, give an explanation or a **counterexample**.

   (a) If $P(x, y) = Q(x, y)$, then the vector field $\mathbf{F} = P\mathbf{i} + Q\mathbf{j}$ is a gradient.

   (b) The flux of any gradient out of a closed surface is zero.

   (c) There is a vector field $\mathbf{F}$ such that $\nabla \times \mathbf{F} = y\mathbf{j}$.

   (d) If f is a smooth function of (x, y), C is the circle $x^2 + y^2 = 1$ and D is the unit disk $x^2 + y^2 \leq 1$, then
   $$\int_C e^{xy} \frac{\partial f}{\partial x} dx + e^{xy} \frac{\partial f}{\partial y} dy = \iint_D e^{xy} \left[y \frac{\partial f}{\partial y} - x \frac{\partial f}{\partial x} \right] dxdy$$

   (e) For any smooth function $f(x, y, z)$, we have
   $$\int_0^1 \int_0^x \int_0^{x+y} f(x, y, z) dz \, dy \, dx = \int_0^1 \int_0^y \int_0^{x+y} f(x, y, z) dz \, dx \, dy$$

Solution.

(a) This is FALSE. For example, the vector field $\mathbf{F} = x\mathbf{i}+x\mathbf{j}$ is not a gradient because it fails to satisfy the cross derivative test.

(b) This is FALSE. For example, the vector field $\mathbf{F}(\mathbf{r}) = \mathbf{r}$ is the gradient of $f(\mathbf{r}) = \|\mathbf{r}\|^2/2$ yet its flux out of the unit sphere is, by the divergence theorem, 4π.

(c) This is FALSE. We know that div curl $\mathbf{F} = 0$ for all vector fields $\mathbf{F}$, but $\mathrm{div}(y\mathbf{j}) = 1$ and so no such $\mathbf{F}$ can exist.

(d) This is TRUE. To see why, let

$$P(x, y) = e^{xy}\frac{\partial f}{\partial x} \quad \text{and} \quad Q(x, y) = e^{xy}\frac{\partial f}{\partial y}$$

so that the left hand side of the expression in the problem is the line integral of $Pdx + Qdy$. By Green's theorem we have

$$\int_C Pdx + Qdy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial p}{\partial y}\right) dx\, dy$$

To work out the right hand side in Green's theorem, we calculate the partial derivatives:

$$\frac{\partial Q}{\partial x} = ye^{xy}\partial f \partial y + e^{xy}\frac{\partial^2 f}{\partial x \partial y}$$
$$\frac{\partial P}{\partial y} = xe^{xy}\frac{\partial f}{\partial x} + e^{xy}\frac{\partial^2 f}{\partial y \partial x}$$

By the equality of mixed partials for f, we get the desired equality from Green's theorem.

(e) This is TRUE. Each side of the equality is the integral of f over the region that lies between the graphs $z = 0$ and $z = x + y$ and lying over the triangle in the xy-plane with vertices $(0, 0), (1, 0), (0, 1)$.

9. Let W be the three dimensional region defined by

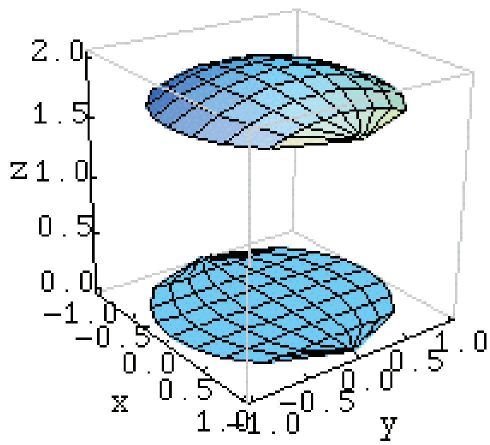
$$x^2 + y^2 \le 1, \quad z \ge 0, \quad \text{and} \quad x^2 + y^2 + z^2 \le 4.$$

(a) Find the volume of W.

(b) Find the flux of the vector field $\mathbf{F} = (2x - 3xy)\mathbf{i} - y\mathbf{j} + 3yz\mathbf{k}$ out of the region W.

Solution.

(a) The region is sketched in the accompanying figure.



The region for Problem 9 is the region between the xy-plane, the sphere of radius 2 and inside the cylinder $x^2 + y^2 = 1$ (the vertical cylindrical side of the region is not shown).

Setting up the volume as a triple integral, one gets

$$\begin{aligned}
\text{Volume} = V &= \iiint_W dx\,dy\,dz \\
&= \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_0^{\sqrt{4-x^2-y^2}} dz\,dy\,dx \\
&= \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \sqrt{4-x^2-y^2}\,dy\,dx
\end{aligned}$$

Now one switch to polar coordinates to give

$$\begin{aligned}
V &= \int_0^1 \int_0^{2\pi} r\sqrt{4-r^2}\,dr\,d\theta \\
&= 2\pi \int_0^1 r\sqrt{4-r^2}\,dr
\end{aligned}$$

This is now integrated using substitution and after a little computation, one gets the answer

$$V = \frac{2\pi}{3}\left(8 - 3\sqrt{3}\right).$$

Alternative Solution. See the sketch of the region above. Using evident notation, we have

$$V(W) = V(\text{cap}) + V(\text{cylinder of height } h)$$

where the region is divided into a cylinder with a certain height h and the portion of the sphere above it. Now

$$V(\text{cylinder}) = A(\text{base}) \cdot \text{height} = \pi \cdot h.$$

To determine h, note that h is the value of z where the cylinder $x^2+y^2 = 1$ and the sphere $x^2 + y^2 + z^2 = 4$ intersect. At such an intersection, $1+z^2 = 4$ and so $z^2 = 3$ i.e., $z = \sqrt{3} = h$. Therefore, $V(\text{cylinder}) = \pi\sqrt{3}$. To find the volume of the cap we think of it as a capped cone minus a cone with a flat top. We can find the volume of the capped cone using spherical coordinates, and we know the volume of the cone with the flat top (the base) is $(\frac{1}{3}A(\text{base} \cdot \text{height})$. First of all, using spherical coordinates,

$$V(\text{capped cone}) = \int_0^{2\pi} \int_0^{\varphi_0} \int_0^2 \rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta$$

To find φ_0 refer to the figure that we suggested be drawn for this problem and one sees that for the relevant right triangle,

$$\cos\varphi_0 = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{\sqrt{3}}{2} \quad \text{and so} \quad \varphi_0 = \frac{\pi}{6}.$$

Therefore,

$$V(\text{capped cone}) = \int_0^{2\pi} \int_0^{\frac{\pi}{6}} \int_0^z \rho^2 \sin\varphi d\rho d\varphi d\theta = \frac{16\pi}{3}\left(1 - \frac{\sqrt{3}}{2}\right)$$

However,

$$\begin{aligned} V(\text{cone with the flat top}) &= \frac{1}{3}\text{Area of base times height} \\ &= \frac{1}{3}\pi\sqrt{3} = \frac{\pi\sqrt{3}}{3} \end{aligned}$$

Therefore,

$$V(\text{cap}) = \frac{16\pi}{3} - \frac{8\pi\sqrt{3}}{3} - \frac{\pi\sqrt{3}}{3}$$

and finally,

$$\begin{aligned} V(W) &= \frac{16\pi}{3} - \frac{8\pi\sqrt{3}}{3} - \frac{\pi\sqrt{3}}{3} + \frac{3\pi\sqrt{3}}{3} \\ &= \frac{16\pi}{3} - \frac{6\pi\sqrt{3}}{3} = \frac{2\pi}{3}(8 - 3\sqrt{3}) \end{aligned}$$

(b) The flux is given by

$$\iint_{\partial W} \mathbf{F} \cdot d\mathbf{S}.$$

To compute this, we use the divergence theorem, which gives

$$\iint_{\partial W} \mathbf{F} \cdot d\mathbf{S} = \iiint_W \operatorname{div} \mathbf{F}\, dV = \iiint_W (2 - 3y - 1 + 3y)\, dV$$
$$= V(W) = \frac{2\pi}{3}(8 - 3\sqrt{3})$$

10. Let $f(x, y, z) = xyze^{xy}$.

   (a) Compute the gradient vector field $\mathbf{F} = \nabla f$.

   (b) Let C be the curve obtained by intersecting the sphere $x^2 + y^2 + z^2 = 1$ with the plane $x = 1/2$ and let S be the portion of the sphere with $x \geq 1/2$. Draw a figure including possible orientations for C and S; state Stokes' theorem for this region.

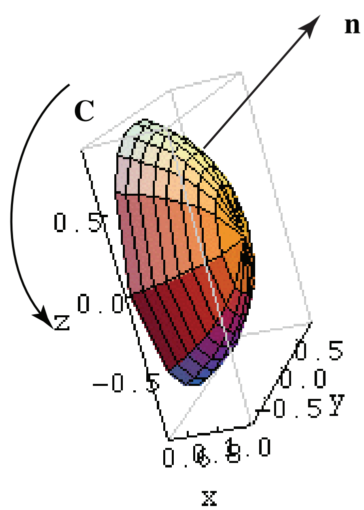
   (c) With $\mathbf{F}$ as in (a) and S as in (b), let $\mathbf{G} = \mathbf{F} + (z - y)\mathbf{i} + y\mathbf{k}$, and evaluate the surface integral
$$\iint_S (\nabla \times \mathbf{G}) \cdot d\mathbf{S}.$$

Solution.

   (a) Using the definition of the gradient one gets

$$\mathbf{F}(x, y, z) = e^{xy}[(yz + xy^2z)\mathbf{i} + (xz + x^2yz)\mathbf{j} + xy\mathbf{k}].$$

   (b) The region is shown in the accompanying figure.



The surface for Problem 10(b) is the portion of the sphere of radius 1 lying to the right of the plane $x = 1/2$.

One has to choose an orientation for the surface and the curve. For example, if the normal points to the right, then the curve should be marked with an arrowhead indicating a counter clockwise orientation when the curve is viewed from the positive x-axis—as in the figure. Stokes' Theorem for this region states that for a vector field $\mathbf{K}$,

$$\int_C \mathbf{K} \cdot d\mathbf{s} = \iint_S \nabla \times \mathbf{K} \cdot d\mathbf{S}$$

(c) Write $\mathbf{G} = \mathbf{F} + \mathbf{H}$, where $\mathbf{H} = (z - y, 0, y)$. Thus, we may evaluate the given integral as follows:

$$\begin{aligned}\iint_S (\nabla \times \mathbf{G}) \cdot d\mathbf{S} &= \iint_S \nabla \times (\mathbf{F} + \mathbf{H}) \cdot d\mathbf{S} \\ &= \iint_S \nabla \times \mathbf{F} \cdot d\mathbf{S} + \iint_S \nabla \times \mathbf{H} \cdot d\mathbf{S}\end{aligned}$$

As we calculated above, $\nabla \times \mathbf{F} = \mathbf{0}$, and so

$$\iint_S \nabla \times \mathbf{G} \cdot d\mathbf{S} = \iint_S \nabla \times \mathbf{H} \cdot d\mathbf{S} = \int_C \mathbf{H} \cdot d\mathbf{S}$$

by Stokes' theorem. To evaluate the line integral, we shall parameterize C; we do this by letting

$$y = \frac{\sqrt{3}}{2} \cos t \qquad x = \frac{1}{2} \quad \text{and} \quad z = \frac{\sqrt{3}}{2} \sin t$$

for $t \in [0, 2\pi]$. Thus, the line integral of $\mathbf{H}$ is

$$\begin{aligned}\int_C \mathbf{H} \cdot d\mathbf{S} &= \int_0^{2\pi} \mathbf{H}(\mathbf{c}(t)) \cdot \mathbf{c}'(t) dt \\ &= \frac{3}{4} \int_0^{2\pi} (\sin t - \cos t, 0, \cos t) \cdot (0, -\sin t, \cos t) dt \\ &= \frac{3}{4} \int_0^{2\pi} \cos^2 t \, dt\end{aligned}$$

Remembering that the average of $\cos^2 \theta$ over the interval from 0 to π is $1/2$, we get $3\pi/4$. Thus, the required surface integral of $\nabla \times \mathbf{G}$ is $3\pi/4$.