Mathematics 1c. Practice Final Examination

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Print Your Name:

Your Section:

- This exam has ten questions.
- You may take four hours; there is no credit for overtime work.
- No aids (including notes, books, calculators etc.) are permitted.
- The exam **MUST** be turned in by **noon on Thursday, June 12.**
- All 10 questions should be answered on this exam, using the backs of the sheets or appended pages as needed. Each question is worth 20 points.
- Show all your work and justify all claims using plain English.
- **Good Luck !!**

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/200
1. (a) Suppose that an operator $T : V \to V$ on a vector space $V$ has at least one nonzero eigenvalue and that some integer power of $T$ is zero. Can $T$ be diagonalizable?

(b) Let $V$ be the vector space of real polynomials of degree 3.
   
   i. What is the dimension of $V$?
   
   ii. Is the operator $T : V \to V$ defined by $T(p) = x^2 p''$ diagonalizable?
   
   iii. Is the operator $S : V \to V$ defined by $S(p) = xp''$ diagonalizable?
2. Let $B$ be an $n \times n$ matrix that is symmetric, orthogonal, and has determinant equal to one.

(a) Show that $\mathbb{R}^n$ has a basis of eigenvectors of $B$ and that each eigenvalue of $B$ is either 1 or $-1$.

(b) Give a concrete example (other than the identity matrix) of a $3 \times 3$ matrix $B$ that is symmetric, orthogonal, and has determinant equal to one.

(c) Give a geometric interpretation of your example as a linear transformation of $\mathbb{R}^3$ to $\mathbb{R}^3$. 
3. Let a particle of mass $m$ move along the elliptical helix $\mathbf{c}(t) = (4 \cos t, \sin t, t)$. 

(a) Find the equation of the tangent line to the helix at $t = \pi/4$.

(b) Find the force acting on the particle at time $t = \pi/4$.

(c) Write an expression (in terms of an integral) for the arc length of the curve $\mathbf{c}(t)$ between $t = 0$ and $t = \pi/4$. 
4. (a) Let \( g(x, y, z) = x^3 + 5yz + z^2 \) and let \( h(u) \) be a function of one variable such that \( h'(1) = 1/2 \). Let \( f = h \circ g \). In what directions starting at \( (1, 0, 0) \) is \( f \) changing at 50% of its maximum rate?

(b) For \( g(x, y, z) = x^3 + 5yz + z^2 \), calculate \( F = \nabla g \), the gradient of \( g \) and verify directly that \( \nabla \times F = 0 \) at each point \((x, y, z)\).
5. Let $f(u,v,w)$ be a (smooth) function of three variables, let $h(r,s) = (r, r + s, r - s)$ and let $g = f \circ h$.

(a) Calculate $\frac{\partial^2 g}{\partial r \partial s}$ in terms of the derivatives of $f$.

(b) Consider the curve in the plane defined by $c(t) = (\cos t, \sin t)$ and the curve in space defined by $d(t) = (h \circ c)(t)$, where $h$ is as given above. Find the equation of the tangent line to $d(t)$ at $t = 0$.

(c) Find an expression as an integral for the arc length of the curve $d(t)$ between $t = 0$ and $t = \pi/4$. 
6. Let \( f(x,y,z) = x + z \) and let \( g(x,y,z) = x^2 + y^2 + z^2 \).

(a) Find the maximum point \((x_0, y_0, z_0)\) of \( f \) subject to the constraint \( g = 1 \).

(b) Let \( V \) be the plane containing all vectors in \( \mathbb{R}^3 \) tangent to the surface \( g = 1 \) at the point \((x_0, y_0, z_0)\) found in part (a).

(i) Find an equation for the plane \( V \).

(ii) Let \( A \) be a \( 3 \times 3 \) matrix whose transpose \( A^T \) is such that \( A^T v \) lies in \( V \) whenever \( v \) does. Show that

\[
A \nabla f(x_0, y_0, z_0) = \lambda \nabla f(x_0, y_0, z_0)
\]

for a constant \( \lambda \).
7. (a) Let $D$ be the parallelogram in the $xy$-plane with vertices 

$(0, 0), (1, 1), (1, 3), (0, 2)$.

Evaluate the integral

$$\int\int_D xy \, dxdy.$$ 

(b) Evaluate

$$\int\int\int_D (x^2 + y^2 + z^2)^{1/2} \exp[(x^2 + y^2 + z^2)^2] \, dx \, dy \, dz$$

where $D$ is the region defined by $1 \leq x^2 + y^2 + z^2 \leq 4$ and $z \geq \sqrt{x^2 + y^2}$. 
8. For each of the questions below, indicate if the statement is \textbf{true} or \textbf{false}. If true, justify (give a brief explanation or quote a relevant theorem from the course) and if false, give an explanation or a \textbf{counterexample}.

(a) If $P(x, y) = Q(x, y)$, then the vector field $\mathbf{F} = P\mathbf{i} + Q\mathbf{j}$ is a gradient.

(b) The flux of any gradient out of a closed surface is zero.

(c) There is a vector field $\mathbf{F}$ such that $\nabla \times \mathbf{F} = y\mathbf{j}$.

(d) If $f$ is a smooth function of $(x, y)$, $C$ is the circle $x^2 + y^2 = 1$ and $D$ is the unit disk $x^2 + y^2 \leq 1$, then

$$\int_C e^{xy} \frac{\partial f}{\partial x} \, dx + e^{xy} \frac{\partial f}{\partial y} \, dy = \int_D e^{xy} \left[ y \frac{\partial f}{\partial y} - x \frac{\partial f}{\partial x} \right] \, dx \, dy$$

(e) For any smooth function $f(x, y, z)$, we have

$$\int_0^1 \int_0^x \int_0^{x+y} f(x, y, z) \, dz \, dy \, dx = \int_0^1 \int_0^y \int_0^{x+y} f(x, y, z) \, dz \, dx \, dy$$
9. Let $W$ be the three dimensional region defined by

$$x^2 + y^2 \leq 1, \quad z \geq 0, \quad \text{and} \quad x^2 + y^2 + z^2 \leq 4.$$ 

(a) Find the volume of $W$.

(b) Find the flux of the vector field $\mathbf{F} = (2x - 3xy)i - yj + 3yzk$ out of the region $W$. 
10. Let $f(x, y, z) = x y z e^{xy}$.

(a) Compute the gradient vector field $\mathbf{F} = \nabla f$.

(b) Let $C$ be the curve obtained by intersecting the sphere $x^2 + y^2 + z^2 = 1$ with the plane $x = 1/2$ and let $S$ be the portion of the sphere with $x \geq 1/2$. Draw a figure including possible orientations for $C$ and $S$; state Stokes' theorem for this region.

(c) With $\mathbf{F}$ as in (a) and $S$ as in (b), let $\mathbf{G} = \mathbf{F} + (z - y)\mathbf{i} + y\mathbf{k}$, and evaluate the surface integral
\[
\iint_S (\nabla \times \mathbf{G}) \cdot d\mathbf{S}.
\]