Mathematics 1c. Practice Final Examination

J. Marsden

Print Your Name:
Your Section:

- This exam has ten questions.
- You may take four hours; there is no credit for overtime work
- No aids (including notes, books, calculators etc.) are permitted.
- The exam **MUST** be turned in by ***noon on Thursday, June 12.***
- All 10 questions should be answered on this exam, using the backs of the sheets or appended pages as needed. Each question is worth 20 points.
- Show all your work and justify all claims using plain English.
- **Good Luck !!**

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1. (a) Suppose that an operator $T : V \to V$ on a vector space V has at least one nonzero eigenvalue and that some integer power of T is zero. Can T be diagonalizable?

   (b) Let V be the vector space of real polynomials of degree 3.

      i. What is the dimension of V?

      ii. Is the operator $T : V \to V$ defined by $T(p) = x^2 p''$ diagonalizable?

      iii. Is the operator $S : V \to V$ defined by $S(p) = xp''$ diagonalizable?

2. Let B be an $n \times n$ matrix that is symmetric, orthogonal, *and* has determinant equal to one.

   (a) Show that $\mathbb{R}^n$ has a basis of eigenvectors of B and that each eigenvalue of B is either 1 or -1.

   (b) Give a concrete example (other than the identity matrix) of a 3×3 matrix B that is symmetric, orthogonal, and has determinant equal to one.

   (c) Give a geometric interpretation of your example as a linear transformation of $\mathbb{R}^3$ to $\mathbb{R}^3$.

3. Let a particle of mass m move along the elliptical helix $\mathbf{c}(t) = (4\cos t, \sin t, t)$.

   (a) Find the equation of the tangent line to the helix at $t = \pi/4$.

   (b) Find the force acting on the particle at time $t = \pi/4$.

   (c) Write an expression (in terms of an integral) for the arc length of the curve $\mathbf{c}(t)$ between $t = 0$ and $t = \pi/4$.

4. (a) Let $g(x, y, z) = x^3 + 5yz + z^2$ and let $h(u)$ be a function of one variable such that $h'(1) = 1/2$. Let $f = h \circ g$. In what directions starting at $(1, 0, 0)$ is f changing at 50% of its maximum rate?

(b) For $g(x, y, z) = x^3 + 5yz + z^2$, calculate $\mathbf{F} = \nabla g$, the gradient of g and verify directly that $\nabla \times \mathbf{F} = \mathbf{0}$ at each point (x, y, z).

5. Let $f(u, v, w)$ be a (smooth) function of three variables, let $h(r, s) = (r, r + s, r - s)$ and let $g = f \circ h$.

   (a) Calculate $\dfrac{\partial^2 g}{\partial r \partial s}$ in terms of the derivatives of f.

   (b) Consider the curve in the plane defined by $\mathbf{c}(t) = (\cos t, \sin t)$ and the curve in space defined by $\mathbf{d}(t) = (h \circ \mathbf{c})(t)$, where h is as given above. Find the equation of the tangent line to $\mathbf{d}(t)$ at $t = 0$.

   (c) Find an expression as an integral for the arc length of the curve $\mathbf{d}(t)$ between $t = 0$ and $t = \pi/4$.

6. Let $f(x, y, z) = x + z$ and let $g(x, y, z) = x^2 + y^2 + z^2$.

   (a) Find the maximum point (x_0, y_0, z_0) of f subject to the constraint $g = 1$.

   (b) Let V be the plane containing all vectors in $\mathbb{R}^3$ tangent to the surface $g = 1$ at the point (x_0, y_0, z_0) found in part (a).

      (i) Find an equation for the plane V.

      (ii) Let A be a 3×3 matrix whose transpose A^T is such that $A^T\mathbf{v}$ lies in V whenever $\mathbf{v}$ does. Show that

$$A\nabla f(x_0, y_0, z_0) = \lambda \nabla f(x_0, y_0, z_0)$$

      for a constant λ.

7. (a) Let D be the parallelogram in the xy-plane with vertices

$$(0,0),(1,1),(1,3),(0,2).$$

Evaluate the integral

$$\iint_D xy\,dxdy.$$

(b) Evaluate

$$\iiint_D (x^2+y^2+z^2)^{1/2}\exp[(x^2+y^2+z^2)^2]\,dx\,dy\,dz$$

where D is the region defined by $1 \leq x^2+y^2+z^2 \leq 4$ and $z \geq \sqrt{x^2+y^2}$.

8. For each of the questions below, indicate if the statement is **true** or **false**. If true, **justify** (give a brief explanation or quote a relevant theorem from the course) and if false, give an explanation or a **counterexample**.

   (a) If $P(x, y) = Q(x, y)$, then the vector field $\mathbf{F} = P\mathbf{i} + Q\mathbf{j}$ is a gradient.

   (b) The flux of any gradient out of a closed surface is zero.

   (c) There is a vector field $\mathbf{F}$ such that $\nabla \times \mathbf{F} = y\mathbf{j}$.

   (d) If f is a smooth function of (x, y), C is the circle $x^2 + y^2 = 1$ and D is the unit disk $x^2 + y^2 \leq 1$, then

$$\int_C e^{xy}\frac{\partial f}{\partial x}dx + e^{xy}\frac{\partial f}{\partial y}dy = \iint_D e^{xy}\left[y\frac{\partial f}{\partial y} - x\frac{\partial f}{\partial x}\right] dxdy$$

   (e) For any smooth function $f(x, y, z)$, we have

$$\int_0^1\int_0^x\int_0^{x+y} f(x, y, z)dz\, dy\, dx = \int_0^1\int_0^y\int_0^{x+y} f(x, y, z)dz\, dx\, dy$$

9. Let W be the three dimensional region defined by

$$x^2 + y^2 \leq 1, \quad z \geq 0, \quad \text{and} \quad x^2 + y^2 + z^2 \leq 4.$$

(a) Find the volume of W.

(b) Find the flux of the vector field $\mathbf{F} = (2x - 3xy)\mathbf{i} - y\mathbf{j} + 3yz\mathbf{k}$ out of the region W.

10. Let $f(x, y, z) = xyze^{xy}$.

   (a) Compute the gradient vector field $\mathbf{F} = \nabla f$.

   (b) Let C be the curve obtained by intersecting the sphere $x^2 + y^2 + z^2 = 1$ with the plane $x = 1/2$ and let S be the portion of the sphere with $x \geq 1/2$. Draw a figure including possible orientations for C and S; state Stokes' theorem for this region.

   (c) With $\mathbf{F}$ as in (a) and S as in (b), let $\mathbf{G} = \mathbf{F} + (z - y)\mathbf{i} + y\mathbf{k}$, and evaluate the surface integral
$$\iint_S (\nabla \times \mathbf{G}) \cdot d\mathbf{S}.$$