Mathematics 1c, Spring 2008
Solutions to the Midterm Exam
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Print Your Name:
Your Section:

• This exam has five questions.
• You may take three hours; there is no credit for overtime work
• No aids (including notes, books, calculators etc.) are permitted.
• The exam must be turned in by noon on Wednesday, May 7. Please turn in your exam to the Math 1C Exam Drop Slot outside Room 255 Sloan.
• All 5 questions should be answered on this exam, using the backs of the sheets or appended pages if necessary.
• Show all your work and justify all claims using plain English.
• Each question is worth 20 points.
• The exam has pages numbered 1–6, including this cover sheet.
• Good Luck !!

/100
1.

(a) Compute the orthogonal projection of the vector \( \mathbf{i} + \mathbf{j} + \mathbf{k} \) onto the subspace \( W \subset \mathbb{R}^3 \) that is spanned by the two vectors \( \mathbf{i} \) and \( \mathbf{i} + \mathbf{j} - \mathbf{k} \).

(b) Let \( V \) be the vector space of polynomials of degree 3. Define the operator \( T : V \to V \) by \( T(f) = xf' \). Show \( T \) is diagonalizable.

(c) Suppose that \( A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \) is a \( 2 \times 2 \) matrix. Assemble a \( 3 \times 3 \) matrix \( B \) using the following notation:

\[
B = \begin{bmatrix} A & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & 1 \end{bmatrix}
\]

Show that \( B^2 = \begin{bmatrix} A^2 & 0 \\ 0 & 1 \end{bmatrix} \)

(d) Compute

\[
\begin{bmatrix} 2 & 3 & 0 \\ -1 & -2 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{100}
\]

and

\[
\begin{bmatrix} 2 & 3 & 0 \\ -1 & -2 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{101}
\]

Solution.

(a) First of all, orthonormalize the two vectors \( \mathbf{i} \) and \( \mathbf{i} + \mathbf{j} - \mathbf{k} \) to produce the orthonormal basis of \( W \) given by \( \mathbf{v}_1 = \mathbf{i} \) and \( \mathbf{v}_2 = (\mathbf{j} - \mathbf{k})/\sqrt{2} \). Letting \( \mathbf{u} = \mathbf{i} + \mathbf{j} + \mathbf{k} \), the orthogonal projection is then given by

\[
(\mathbf{u} \cdot \mathbf{v}_1)\mathbf{v}_1 + (\mathbf{u} \cdot \mathbf{v}_2)\mathbf{v}_2 = \mathbf{i} + 0(\mathbf{j} - \mathbf{k})/\sqrt{2} = \mathbf{i}
\]
(b) The space $V$ has dimension 4 and by inspection, there are four distinct eigenvalues $0, 1, 2, 3$ with associated eigenvectors given by the polynomials $1, x, x^2, x^3$. Since their are 4 distinct eigenvalues (or directly the eigenvectors form a basis), the operator is diagonalizable.

(c) One verifies this by a simple direct calculation.

(d) If we let $A = \begin{bmatrix} 2 & 3 \\ -1 & -2 \end{bmatrix}$ and construct $B$ as in part (c), we get the given matrix. As in (c), raised to the 100th power, it is

$$B^{100} = \begin{bmatrix} A^{100} & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and similarly

$$B^{101} = B$$

The computation of $A^{100}$ and $A^{101}$ was done explicitly in lecture. This can be done in a couple of ways.

**Method 1.** Perhaps the simplest way is to directly compute that $A^2 = \text{Identity}$ and thus, $A^{100} = \text{Identity}$ and $A^{101} = A$.

**Method 2—the one done in lecture.** Another, more generally applicable method, is as follows. First, compute that the eigenvalues of $A$ are $1$ and $-1$. Since the eigenvalues are distinct, $A$ is diagonalizable. Thus, $A = QDQ^{-1}$, where $D$ is the diagonal matrix with diagonal entries $1, -1$ and $Q$ is the matrix whose columns are the two eigenvectors. Clearly $D^{100} = \text{Identity}$ and so (as was explained in lecture), $A^{100} = QD^{100}Q^{-1} = \text{Identity}$. Then $A^{101} = A$. 

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2. Let $A_\epsilon$ be the matrix $A_\epsilon = \begin{bmatrix} 1 & \epsilon & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, where $\epsilon$ is a real number.

(a) For what values of $\epsilon$ is $A_\epsilon$ orthogonal?

(b) Assume $\epsilon \neq 0$. Find the eigenvalues and corresponding eigenspaces of $A_\epsilon$.

(c) Assume $\epsilon \neq 0$. Show that $A_\epsilon$ is not diagonalizable.

(d) Let $\epsilon$ be any real number. Define the function $F : \mathbb{R} \to \mathbb{R}$ by letting $F(\epsilon)$ equal the dimension of the eigenspace of $A_\epsilon$ corresponding to eigenvalue 1. Is $F$ continuous at $\epsilon = 0$? Hint: use (b) to find $\lim_{\epsilon \to 0} F(\epsilon)$, and compare this to $F(0)$.

Solution.

(a) There are a couple of ways one can do the first part.

**Method 1.** The columns of an orthogonal matrix are orthonormal vectors. Note that the second column of $A_\epsilon$ has square length $1 + \epsilon^2$, so this equals one if and only if $\epsilon = 0$, in which case $A_\epsilon$ is the identity, an orthogonal matrix. Thus, $A_\epsilon$ is orthogonal if and only if $\epsilon = 0$.

**Method 2.** $A_\epsilon$ is orthogonal when $A_\epsilon^T A_\epsilon = I$. Calculate

$$A_\epsilon^T A_\epsilon = \begin{bmatrix} 1 & \epsilon & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ \epsilon & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 + \epsilon^2 & \epsilon & 0 \\ \epsilon & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$ 

Hence $A_\epsilon$ is orthogonal if and only if $\epsilon$ is zero.

(b) $\det(\lambda I - A_\epsilon) = (\lambda - 1)^3$, so $A_\epsilon$ has the single eigenvalue $\lambda = 1$. The corresponding eigenspace is

$$\ker(I - A_\epsilon) = \ker \begin{bmatrix} 0 & -\epsilon & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \operatorname{span} \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

since $\epsilon \neq 0$.  

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(c) We see in (b) that for $\epsilon \neq 0$, $A_\epsilon$ does not have a basis of eigenvectors. Therefore $A_\epsilon$ is not diagonalizable.

(d) From (b) we know that the eigenspace corresponding to eigenvalue 1 is span\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ for $\epsilon \neq 0$; hence $F(\epsilon) = 2$ for $\epsilon \neq 0$. So $\lim_{\epsilon \to 0} F(\epsilon) = 2$. To calculate $F(0)$, note that $A_0$ is just the identity matrix which has a full basis of eigenvectors (for instance, $\{e_1, e_2, e_3\}$ is a basis of eigenvectors). So $F(0) = 3$. Since $\lim_{\epsilon \to 0} F(\epsilon) = 2 \neq 3 = F(0)$, $F$ is not continuous at $\epsilon = 0$.

3. Let $g(x, y, z)$ be a smooth function defined on the whole of $\mathbb{R}^3$.

(a) If $\nabla g$ has negative $x$ component in the half space $x \geq 0$, must $g(1, 3, 8)$ be bigger than $g(2, 3, 8)$, and must $g(0, 2, 8)$ be bigger than $g(3, 3, 8)$? In each case prove or give a counter-example.

Let $f(x, y) = x^3 + 3y^2 - 2x^2y + 6$.

(b) Find $\nabla f$ at $(2, 3)$.

(c) Find the equation of the tangent plane to the graph of $f$ at the point $(x, y) = (2, 3)$.

(d) In what direction is the function $f$ decreasing the most, and what is the rate of decrease in that direction?

(e) What is the rate of decrease of $f$ in the direction $\left( \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \right)$?

Solution.

(a) i. If $\nabla g$ has negative $x$ component for $x \geq 0$ then the one-dimensional function $h(x) = g(x, 3, 8)$ has negative derivative for $x \geq 0$ and thus is decreasing. [Extra: recall from one variable calculus that one proves this by the mean value theorem (e.g. by assuming $h(1) \leq h(2)$ and getting a contradiction)]. Thus, as $g$ is decreasing, $g(1, 3, 8) > g(2, 3, 8)$. 
ii. Let \( g = (-x, 10y, 0) \), then \( \nabla g = (-1, 10, 0) \). Note, however, that \( g(0, 2, 8) = 20 < g(3, 3, 8) = 27 \), and so we have a counterexample. There are many other possible counterexamples.

(b) \( \nabla f = (3x^2 - 4xy, 6y - 2x^2) \) and thus \( \nabla f(2, 3) = (-12, 10) \).

(c) The equation for the tangent plane at \((x_0, y_0)\) is

\[
z = f(x_0, y_0) + \frac{\partial f}{\partial x}(x - x_0) + \frac{\partial f}{\partial y}(y - y_0)
\]

So given that \( \nabla f(2, 3) = (\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}) \), we have

\[
z = 17 - 12(x - 2) + 10(y - 3)
\]

So the tangent plane equation is

\[
z = 11 - 12x + 10y
\]

(d) \( f \) is decreasing the fastest in the direction \(-\nabla f(2, 3) = (12, -10)\).

The rate of decrease is \( ||-\nabla f(2, 3)|| = \sqrt{12^2 + (-10)^2} = \sqrt{244} = 2\sqrt{61} \).

(e) Since the vector is already normalized, this is determined by computing

\[
\nabla f(2, 3) \cdot \left( \frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}} \right) = \frac{-12 - 10}{\sqrt{2}} = -11\sqrt{2}
\]

So \( 11\sqrt{2} \) is the rate of decrease.

4. Consider the functions \( A \) and \( B \) defined on \( \mathbb{R}^3 \) as follows:

\[
A(x, y, z) = x^2 + xy - \sin(xy), \quad B(x, y, z) = x^2y^2\cos(z^2).
\]

(a) Suppose \( x \) and \( y \) are functions of two other variables, \( r \) and \( s \). Write down the partials of \( A \) and \( B \) with respect to \( r \) and \( s \) in matrix notation.
(b) Simplify the your answer in (a) if \( x = s + r, \ y = sr, \ z = s - r. \)

Let \( f(x, y) = x^2 - 3xy + 5x - 2y + 6y^2 - 8. \)

(c) Find the critical points of \( f. \)

(d) Characterize the critical points in terms of maxima, minima, or saddle points.

**Solution.**

(a) Let \( F \) be the mapping \((r,s) \rightarrow (A,B).\) Then by the chain rule,

\[
D F = \begin{bmatrix}
\frac{\partial A}{\partial s} & \frac{\partial A}{\partial r} \\
\frac{\partial B}{\partial s} & \frac{\partial B}{\partial r}
\end{bmatrix}
\begin{bmatrix}
\frac{\partial x}{\partial s} & \frac{\partial x}{\partial r} \\
\frac{\partial y}{\partial s} & \frac{\partial y}{\partial r} \\
\frac{\partial z}{\partial s} & \frac{\partial z}{\partial r}
\end{bmatrix}
\]

\[
= \begin{bmatrix}
2x + y - y\cos xy & x - x\cos xy & 0 \\
2xy^2\cos z^2 & 2x^2y\cos z^2 & -2x^2y^2z\sin z^2
\end{bmatrix}
\begin{bmatrix}
\frac{\partial x}{\partial s} & \frac{\partial x}{\partial r} \\
\frac{\partial y}{\partial s} & \frac{\partial y}{\partial r} \\
\frac{\partial z}{\partial s} & \frac{\partial z}{\partial r}
\end{bmatrix}
\]

(b) Putting \( x = s + r, \ y = sr, \) and \( z = s - r, \) we see that \( D F \) equals \( AB, \) where \( A \) is

\[
\begin{bmatrix}
2(s + r) + sr - sr\cos((s + r)sr) & (s + r) - (s + r)\cos((s + r)sr) & 0 \\
2(s + r)s^2\cos(s - r)^2 & 2(s + r)^2sr\cos(s - r)^2 & -2(s + r)^2s^2r^2(s - r)\sin(s - r)^2
\end{bmatrix}
\]

and \( B \) is

\[
\begin{bmatrix}
1 & 1 \\
r & s \\
1 & -1
\end{bmatrix}
\]

If one were to multiply this out, one would get, for example,

\[
\frac{\partial A}{\partial s} = 2(s + r) + sr - sr\cos((s + r)sr) + r[(s + r) - (s + r)\cos((s + r)sr)]
\]

\[
= 2s + 2r + 2sr + r^2 - r^2\cos((s + r)sr).
\]
(c) The critical points are found by setting the partial derivatives of $f$ equal to zero; this gives the system

\[
\begin{align*}
2x - 3y + 5 &= 0 \\
-3x + 12y - 2 &= 0
\end{align*}
\]

Solving this system, we see that the only critical point is $(-18/5, -11/15)$.

(d) The above critical point is a local (in fact, global) minimum, as can be seen from the fact that the Hessian is positive definite. The Hessian is

\[
\frac{1}{2} \begin{bmatrix} 2 & -3 \\ -3 & 12 \end{bmatrix}
\]

The top left entry is $\partial^2 f/\partial x^2 = 2 > 0$ and the determinant is $15 > 0$, so the Hessian is indeed positive definite.

5. (a) Find the maxima and minima of the function

\[f(x, y) = 2x^2 y - x^2 - y^2.
\]
on the region $x^2 + y^2 \leq 1$.

(b) Find the extrema of $f(x, y) = xy$ subject to the three conditions $2x + 3y \leq 10, 0 \leq x, 0 \leq y$.

(c) Consider the function $f(x, y) = ax^2 + 2bxy + cy^2$. Suppose that the eigenvalues of the matrix

\[
\begin{bmatrix} a & b \\ b & c \end{bmatrix}
\]

are both positive. Must the origin be a minimum of $f$?

Solution.

(a) The critical points of $f$ are found by solving the system

\[
\begin{align*}
\frac{\partial f}{\partial x} &= 4xy - 2x = 0 \\
\frac{\partial f}{\partial y} &= 2x^2 - 2y = 0,
\end{align*}
\]
which gives the critical points (0, 0) and \((\pm 1/\sqrt{2}, 1/2)\). Both points are inside the unit circle. The second derivative matrix is

\[
\begin{bmatrix}
4y - 2 & 4x \\
4x & -2
\end{bmatrix}
\]

which is negative definite at the origin (so the origin is a local maximum, with \(f(0, 0) = 0\) and it is indefinite at the points \((\pm 1/\sqrt{2}, 1/2)\), which are therefore saddle points.

One must now examine the behavior of \(f\) on the boundary using Lagrange multipliers. The Lagrange multiplier conditions \((\nabla f = \lambda \nabla g\) and \(g = 0\), give (after canceling factors of 2):

\[
\begin{align*}
2xy - x &= \lambda x \\
x^2 - y &= \lambda y \\
x^2 + y^2 - 1 &= 0
\end{align*}
\]

If \(x = 0\), the solution is \((0, \pm 1)\) and if \(x \neq 0\), the solution is found by canceling \(x\) in the first equation, solving for \(y\) and substituting into the remaining equations. This gives \(\lambda = \pm 2/\sqrt{3} - 1\), \(x = \pm \sqrt{2/3}\) and \(y = \pm 1/\sqrt{3}\). Evaluating \(f\) at these points gives the answer: \((0, 0)\) is a maximum and the minimum value of \(f\) is \((-9 - 4\sqrt{3})/9\), which occurs at the points \((\pm \sqrt{2/3}, -1/\sqrt{3})\) on the boundary.

(b) The region on which \(f\) is defined is the triangle in the first quadrant of the \(xy\)-plane bounded by the axes and the line \(2x + 3y = 10\). Since \(f_x = y\), \(f_y = x\), we see there is no critical point strictly inside this region. Thus, the extrema must be on the boundary.

The boundary of this triangular region consists of 3 straight line segments, and we need to find the absolute maximum and minimum of \(f\) on this boundary. Clearly \(f = 0\) on both \(x = 0\) and \(y = 0\); since \(f \geq 0\) in the region, the minimum value of 0 occurs
on the two axes. To find the maximum, we only need to solve the following Lagrange multiplier system with \( x > 0 \) and \( y > 0 \):

\[
\begin{align*}
y &= 2\lambda \\
x &= 3\lambda \\
2x + 3y &= 10
\end{align*}
\]

Solving these gives \( \lambda = \frac{5}{6}, x = \frac{5}{2}, y = \frac{5}{3} \). Also, \( f\left(\frac{5}{2}, \frac{5}{3}\right) = \frac{25}{6} \).

In summary, the absolute maximum value is \( \frac{25}{6} \) which occurs at the point \( (5/2, 5/3) \) and the absolute minimum value of 0 occurs at points on the two axes.

(c) Yes. Call the eigenvalues \( \lambda \) and \( \mu \). The condition that \( \lambda > 0 \) and \( \mu > 0 \) is equivalent to the positive definiteness of the Hessian, which is half of the second derivative matrix,

\[
A = \begin{bmatrix} a & b \\ b & c \end{bmatrix}.
\]

One should relate this to the second derivative conditions given in the book, namely that \( a > 0 \) and \( ac - b^2 > 0 \). One way to do this is as follows. Taking the trace and determinant of \( A \) (recall that the trace is the sum of the eigenvalues and the determinant is their product), we get

\[
\lambda + \mu = a + c \\
\lambda\mu = ac - b^2
\]

From this, it is clear that the conditions \( a > 0 \) and \( ac - b^2 > 0 \) are equivalent to the conditions \( \lambda > 0 \) and \( \mu > 0 \).